Mémoire

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Promoteur(s) : Bastin, Thierry
Faculté : Faculté des Sciences
Diplôme : Master en sciences physiques, à finalité approfondie
Année académique : 2020-2021
URI/URL : http://hdl.handle.net/2268.2/12230

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Entanglement Classification by Algebraic Geometry

Master Thesis

BY

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UNDER THE GUIDANCE OF

PROF. BASTIN THIERRY

Academic Year 2020-2021

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Acknowledgements

I would like to warmly thank Professor Thierry Bastin for many reasons. His support, advice and kindness allowed me to thrive on the writing of this thesis. It has been my best academic experience so far and I am sincerely grateful to him. I hope that I have been able to fulfil his expectations, and I hope that this work will be a credit to him. I would like to thank Professors S. Dorbolo, E. Bousquet, J. Martin and P. Mathonet for the honour they have accorded me in constituting the reading jury for my thesis. I hope you will find this thesis interesting and instructive. I would also thank the Department of Physics of the University of Liege for giving me this opportunity to pursue my passion and grow in physics.

Without some people, the journey of writing this thesis would have been much more complicated. Thank you Naïm Zenaïdi and Pierre Mathonet for your incredible support and constant help in the world of mathematics. Naïm, our discussions on mathematics as well as on philosophy, ethics, teaching, ecology, and many other subjects have been a great pleasure for me, and I look forward to our future discussions.

It is also important for me to thank my closest friends, in random order. Thank you Jerome D. for such a brotherly friendship and my personal bedroom. Thank you Celik T. for your continuous support and interest in my work, and also for your swimming pool. Thank you Romain P. for your support and for our moments of relaxation-horror which were important to me, and also for our golf games which I like so much. Thank you Robin D. for your constant friendship and support over the years, and your bad jokes. Thank you Nathan L. for your unfailing support, for your presence so important to me, and for many other things you know, like the pleasure of suffering your negative karma. Thank you Jean-Michel P. for being the parachute one needs when jumping into the unknown, and for guiding my steps into the Empire, Master Sith.

I would like to thank those people who have made me happy and proud for so many years, my dear Scouts for whom I would sacrifice many things. Chico, Malamute, Tamatia, Choucas, Maxime, Yannis, Saki, Isatis, Callimico, Hugo L, Nicolas, Aïdi, Douc, Havana, Casoar, Clément, Hugo V, Capucin, Cumatanga, Donskoy, Julia, Gregory, Charline, Chaoui, Colley, Sotalia, Gwendoline, Martin, Yaniss, Ocicat, Pika, Chickaree, Emy, and of course Boomslang, Shiba, Zokor, Grison and Beira. Thank you to all those who make this Scout unit so beautiful, dear animators of all sections.

What would this acknowledgement section be without thanking those who made up my years of study at university? Thank you Youri N., Thibaud C., Matteo L. and Pierrick V. for having shared with me these moments of pleasure, suffering, joy, sorrow, but above all, friendship and support. I hope that years and distance will not in any way damage our beautiful friendship forged during this very special journey. I hope the best for you guys.

Finally, an unconditional thanks to my blood family. Thank you mum for your comforting presence, your support, and your courage that inspires me every day. Thank you dad for your constant support and the pride that shines in your eyes when I tell you about my academic adventures. Thank you cousin for your presence and for the example of success in life that you represent in my eyes.

Introduction

Whether it be the Egyptians developing an algorithm for multiplying two numbers, in 2000BC, or the Babylonians and their algorithm for factorisation in 1600BC, humans have always been trying to make their life easier through algorithms that make certain tasks systematic. Since the very first beginning of electrical classical computation with the construction of the *Automatic Sequence Controlled Calculator* (ASCC, also known as *Mark 1*), an IBM calculator of 1944, a long road has been traced, constantly pushing back the limits encountered along the way. But these limits can not be pushed out to infinity, and that is a result state by a famous mathematician and computer scientist, Alan Turing. By introducing its well-known Turing machine in 1936, a mathematical model of computation defining an abstract machine manipulating symbols on a strip of tape according to a table of rules, he proved the existence of fundamental limitations on the power of mechanical computation. For instance, finite memory limitation is one of the most common limitations a computer designer has to face. But nowadays, these limitations have been pushed a lot further, and are still pushed back by the arrival of significantly faster computational systems, quantum computers.

In the early 1980s, a first quantum mechanical model of a Turing machine was proposed by Paul Benioff, an American physicist and pioneer of quantum computing. In 1982, Richard Feynman suggested in [1] that computers based on quantum mechanics would have the potential to simulate physics, which a classical computer could not. This suggestion has also been stated by Yuri Manin in [2], a mathematician known for work in algebraic geometry. A decade later in 1994, this suggestion has been taken into account by Peter Shor who developed in [3] a quantum algorithm for factoring integers. Its computational power had the potential (see [4]) to decrypt RSA-encrypted communication, a public-key cryptosystem that found its application in secure data transmission.

The true power of quantum computers would not just to perform computations faster than classical computers could. Otherwise, quantum computers would not provide any additional advantages over classical computers in term of computability. Quantum computers have the advantage to perform computations that any classical computer could not perform in a feasible amount of time. It is called *quantum supremacy*, and this feature is not formally proved. However, in December 2020, a group based in the University of Science and Technology of China reached what we would call quantum supremacy by making a quantum computation in 20 seconds, while a classical supercomputer would require 600 million years of computation. (See [5]).

Under these circumstances, it would be difficult to argue against the fact that quantum information and quantum computation are clearly a cornerstone in the future of technology. Hence, since it is based on quantum mechanics, and moreover on quantum entanglement[1], an enormous quantity of work has been carried in the past decades to answer some natural questions about entanglement. How to know if a state is entangled or not (Separability problem) ? If a state is entangled, how much is it, and how far from a separable[2] state is it (Entanglement measure) ? Can an entangled state

[1] Some quantum computations could be achieved without entanglement, but it would not be far from classical computations. The role of entanglement is to substantially enhance the speed of computations by a process that classical computers could not achieve. Indeed, entanglement is something mysterious from quantum mechanics that has no equivalent in classical mechanics. A better definition of entanglement will be given in the following

[2] A separable state could be defined here as a not entangled state. A better definition will be given in the following.

perform the same tasks as another entangled state ? Would it be possible to classify entangled states in such a way that states from a given equivalence class would perform the same tasks (Entanglement classification) ? These questions are increasingly mastered over time and are the keystone of progress in quantum computation, in conjunction with technical progress.
In the first chapter, we will give in Section 1.1 some insight about entanglement, before introducing formal definitions and generalisations. Separability problem and quantum measure will be developed in Sections 1.2 and 1.3 of this thesis. As one will see in Section 1.4, entanglement classification fails to be finite when we consider something greater than a 4-qubit system. This issue has been analysed and solved by Masoud Gharahi, Stefano Mancini and Giorgio Ottaviani in their paper [6] with help of algebraic geometry, which is roughly the mathematical frame of study of zero sets of polynomials. More abstractly, it could be defined as the mathematical frame of study of projective spaces, these latter being spaces whose points are equivalence classes. The purpose of this thesis is to make a detailed overview of the notions this article needs to be understood.
In the second chapter, these latter notions will be developed through algebraic geometry, to give rise to tools such as projective Hilbert space, proper k-secant varieties and tangent varieties, which play a great role in this classification. For a clarity purpose, we will assume that the reader is comfortable with notions of linear algebra and topology, and a brief list of used definitions and theorems is proposed in the appendix. However, the reader not comfortable with these notions is invited to see Section 2.5 to understand the tools we are using without the mathematical background.
Finally, in chapter three, after introducing l-multilinear ranks, a notion of multilinear algebra, we will present the main result of [6], which is the finite classification of SLOCC classes by linking them to a finite number of projective subvarieties. We will work out some examples to show how it works practically.

1 General concepts

1.1 Entanglement

The purpose of this present chapter is to give the reader the basic notions about entanglement, not only from a mathematical point of view, but also from a physical point of view. To acquire a physical insight about entanglement, we will explore the EPR paradox, its context and the conclusion to this paradox brought by the famous Aspect's experiment. After that, we will shed light on the mathematical definition of entanglement. To end this chapter, we will introduce the notion of (Stochastic) Local Operations with Classical Communication.

1.1.1 Physical insight about entanglement

The following discussion can be found in a lot of great vulgarisation books. First, a large overview of the history of quantum mechanics can be found in [7]. The discussion about the deep nature of the reality can be also found in [8, 9, 10, 11]. Great insight can also be acquired with [12].

Polarization of two photons entangled

Given a pair of photons entangled[1], distant enough to prevent direct interaction between them, we may be interested to know how the entanglement will affect this pair. To answer this, we will consider the polarization of the photon to be either vertically, or horizontally, and the two photons to be perpendicularly polarised to each other, representing the entanglement.
Two possibilities arise : either the first photon is polarised vertically, and then the second is polarised horizontally, or the first photon is polarised horizontally, and then the second is polarised vertically. At first sight, it is trivial and not interesting, but this is our classical mind speaking.

[1] For instance, when an electron meets a positron, they annihilate together to produce an entangled pair of photons. This is called *pair production*

Since quantum mechanics have a probabilistic nature, the polarization state of each photon is random between horizontal and vertical at the exact time of the measurement. Although one is measured vertical, the second should be randomly measured horizontal or vertical[2].
Thus, how can we explain that these two polarizations are always perpendicular ? Why the probabilities do not affect the second photon polarization measurement once the first is measured ? This is anything but trivial.
We can explain that the way Niels Bohr did in 1930 : before measurement, both polarization states are indeterminate. Then, measuring the first photon polarization instantaneously[3] sets the second photon polarization perpendicularly to the first.

Here is our first consequence and physical insight of pair entanglement : the measurement of one element of a pair leads simultaneously to the determination of the second element. In other words, when a measurement is done on the first photon to determine its polarization, the information about this latter is sent with an infinite speed to the second photon, whose polarization becomes perpendicular to the first. This infinite speed is characteristic of the simultaneity we are talking about here.

But this insight we have acquired about entanglement must be interrogated, as it is the heart of a paradox, the EPR paradox.

EPR paradox

"Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?" is the question that Einstein, Podolsky and Rosen had asked the quantum theory community via an experiment of measurement. The paper [13], redacted and published in 1935 by Podolsky alone, relate the discussion they had about the inconsistencies of quantum mechanics, more precisely about the physical interpretation of entangled systems. It involves the contradiction of two fundamental elements of quantum theory : the uncertainty principle, and the locality principle.

Violation of the Locality Principle

Special Relativity forbids information to travel faster than light, which velocity is finite, hence the *locality principle* states that an object can not be directly influenced by another distant object because its influence is mediated either by waves from a field or by particles, which both can not travel faster than light. The locality principle is thus violated by entanglement, as seen in the previous example

[2]Locality requires that any disturbance triggered by the measurement on one photon can not influence the result of the measurement on the other photon.

[3]This notion of simultaneity plays a central role in entanglement. Indeed, simultaneity prevents interaction between the two photons, because the speed of any interaction is restricted to a maximum, which is the speed of light. Therefore, this kind of correlations is highly surprising, in opposition to the case of correlations where simultaneity is not required.

of two photons. This is the first paradox.

Violation of the Uncertainty Principle

This Uncertainty Principle states[4] that we can not measure the position and momentum observables, denoted r and p, for the same state, that is for the same system at the same time, without introducing an error in the precision of measurements. This is not a familiar case in the classical theory, because measuring the position of a moving car with high precision (i.e. with a lot of data) does not affect the precision of the momentum measurement.

Let's now consider two electrons entangled and distant enough to prevent direct interaction between them. Say that the first electron is in Einstein's office in Princeton in the USA, and the other one is in Schrödinger's office in Dublin in Ireland. They will both make a measurement, at the same time. Einstein will measure the position of his electron, and Schrödinger will measure the momentum of his electron, both at the same time, i.e. simultaneously[5].

Two consequences arise from the Einstein measurement. On one hand, he affects its electron further positions because of the interaction, and on the other hand, he fixes the position of the Schrödinger's electron. Hence this latter is known without any interaction with Schröndinger's one. Then, the Schrödinger's electron momentum measurement is not affected by an error introduced by the position measurement. Therefore, position and momentum can be measured at the same time without lack of precision, which is contrary to the Uncertainty Principle.

Einstein's conclusion

Einstein's conclusion is the following : quantum theory is an incomplete theory since it can not explain these two inconsistencies. There must be local hidden variables, not taken into account by the fundamental postulates of quantum mechanics, which can explain these inconsistencies. Einstein's ambitions in 1935 were not to determine the veracity of quantum mechanics, but its incompleteness.

[4]The following statement is a particular case of the uncertainty principle. The general principle is

$$\Delta_\psi A \Delta_\psi B \ \geq \ \frac{1}{2}|\langle\psi|[\widehat{A},\widehat{B}]|\psi\rangle|, \tag{1.1}$$

and this particular case is

$$\Delta r \Delta p \ \geq \ \frac{\hbar}{2\pi} \tag{1.2}$$

[5]Even if the simultaneity is not friendly to Einstein, they agree together to have their measurements tools at rest with respect to their office, avoiding then the effect of space-time contractions.

Bell's theorem and Aspect's experiment

Bell's theorem

The Irish physicist John Stuart Bell enunciated in 1964 a theorem based on three hypotheses. Without going into details, this theorem is about an inequality that remains true if and only if all these three hypotheses are true :

- A particle state is entirely determined by the past of the particle and its various interactions.
- Locality principle, which states that distant particles can not interact directly with each other, but via waves in a field or via other particles, moving slower or at the same speed as the speed of light.
- Particles carry their own properties without the influence of measurements. In other words, particles are objectively defined without the need of a measurement.

The two first hypothesis are recognized to be fundamentals and true, both by Einstein than the quantum theorists. The last hypothesis, also called the *realism hypothesis*, is Einstein's point of view, in contrast to Bohr's point of view. This latter consider that the properties of particles are materialised with them by measurements. Bell urged the experimental community to make the first experiment to prove whether or not this inequality is true, and then to decide the winner of this roaring debate.

Aspect's experiment

In 1976, a great idea about an experiment has been published by a French physicist, Alain Aspect, but nobody paid attention. He decided to realize its idea in 1981 and 1982 and was the first to prove whether or not Bell's inequality is true. And the result is clear and irrefutable : Bell's inequality is violated, hence Einstein's realism hypothesis is wrong. Particles are not objectively defined without the action of measurement, they do not carry their own intrinsic properties.

Therefore, our insight we get about entanglement is still appropriate, but we need to consider that particles are not objectively defined, that is they do not objectively exist without measurement, because particle properties do not exist until one measures them. That seems surprising, and this is still subject to many philosophical discussion about quantum theory.

1.1.2 Mathematical definition of entangled systems

Here are presented basic notions about entanglement. If the reader is not comfortable with those, a great reference is [14]

Tensorial product space of pure states

Let us first consider a system of one particle. It is fully described by its wave function $|\phi\rangle$ from a Hilbert space $\mathcal{H}_1$ of dimension d_1. Then, let us consider another system of one particle, described by $|\chi\rangle$ from a Hilbert space $\mathcal{H}_2$ of dimension d_2. This bipartite system is described by a wave function from $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ which is called the tensorial product space.

Definition 1.1. (Tensorial product space) Hilbert spaces whose elements describe a system of two subsystems are *tensorial product spaces*

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \tag{1.3}$$

of dimension $d = d_1 \times d_2$ where $\mathcal{H}_1$ of dimension d_1 and $\mathcal{H}_2$ of dimension d_2 are Hilbert spaces whose elements respectively describe subsystem 1 and subsystem 2 and where $\otimes$ denotes the tensorial product.

Theorem 1.1. *(Tensorial product space basis) Let* $\mathcal{B}_1 = \{|\psi_i^1\rangle\}_{i\in\{1,..,d_1\}}$ *and* $\mathcal{B}_2 = \{|\psi_j^2\rangle\}_{j\in\{1,..,d_2\}}$ *be respectively a basis from Hilbert spaces* $\mathcal{H}_1$ *of dimension* d_1 *and* $\mathcal{H}_2$ *of dimension* d_2*. A basis of* $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ *of dimension* $d = d_1 \times d_2$ *is given by* $\mathcal{B} = \{|\psi_i^1\rangle \otimes |\psi_j^2\rangle\}_{i\in\{1,..,d_1\},j\in\{1,..,d_2\}}$.

Remark 1.1. Considering N particles is similar and Hilbert space of such a system is simply

$$\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i. \tag{1.4}$$

If $\mathcal{B}_i = \{|\psi_j^i\rangle\}_{j\in\{1,..,d_i\},i\in\{1,...,N\}}$ is a basis of $\mathcal{H}_i$, then

$$\mathcal{B} = \bigotimes_{i=1}^{N} \mathcal{B}_i \tag{1.5}$$

is a basis of $\mathcal{H}$.

Separated and entangled pure states

Investigating $\mathcal{H}$, the Hilbert space of a bipartite system, we can either find some elements $|\psi\rangle$ such that there exists $|\phi\rangle \in \mathcal{H}_1$, $|\chi\rangle \in \mathcal{H}_2$ such that

$$|\psi\rangle = |\phi\rangle \otimes |\chi\rangle \tag{1.6}$$

We can find some other kind of elements $|\psi\rangle$ such that, $\forall |\phi\rangle, |\chi\rangle$, one has

$$|\psi\rangle \neq |\phi\rangle \otimes |\chi\rangle \tag{1.7}$$

Definition 1.2. (Separated and entangled states) In the case of Eq. (1.6), $|\phi\rangle$ and $|\chi\rangle$ are *separated*, and the pure state $|\psi\rangle$ is *separable*. On the contrary, in the case of Eq. (1.7), $|\psi\rangle$ is an *entangled pure state*.

Generalisation of tensorial product space to mixed states

Everything above has been done considering that the system is in a known state $|\psi\rangle$. If we do not know the exact state, but a probability $p_i \in [0, 1]$ to be in a state $|\psi_i\rangle$, we are facing what is called a mixed state.

Definition 1.3. (Pure state) Let us consider a system which is represented by the determined state $|\psi\rangle$. Such a state is called a *pure state*.

Definition 1.4. (Mixed state) Let us consider a system and a set of n states $\{|\psi_1\rangle, ..., |\psi_n\rangle\}$. Let $\{p_1, ..., p_n\}$ be a set of integers such that $\sum_{i=1}^{n} p_i = 1$. If the system has a probability p_j to be represented by the state $|\psi_j\rangle$, where $j \in \{1, ..., n\}$, then the system is said to be *mixed*, or to be in the *statistical mixture* $|\psi_1\rangle$,...,$|\psi_n\rangle$, or even in the *set of pure states* $\{p_j, |\psi_j\rangle\}_{j\in\{1,...,n\}}$.

Remark 1.2. A pure state is a particular case of mixed states.

We can unify these two notions by defining the density operator, first in the particular case of pure states, and then in the general case of mixed states

Definition 1.5. (Density operator; pure state) Let us consider a system represented by the state $|\psi\rangle$. The operator

$$\widehat{\rho} = |\psi\rangle\langle\psi| \tag{1.8}$$

is the *density operator associated to the pure state* $|\psi\rangle$.
Therefore, we say that the system is in the pure state $|\psi\rangle$, or equivalently represented by the state $\widehat{\rho}$

Definition 1.6. (Density operator; mixed state) Let us consider a set of pure states $\{p_j, |\psi_j\rangle\}_{j\in\{1,...,n\}}$ and a system which has a probability p_j to be represented by the state $|\psi_j\rangle$, with $i \in \{1, ..., n\}$. The operator

$$\widehat{\rho} = \sum_{j=1}^{n} p_j |\psi_j\rangle\langle\psi_j| = \sum_{j=1}^{n} p_j \widehat{\rho}_j \tag{1.9}$$

is the *density operator associated to the mixed state* $\{p_j, |\psi_j\rangle\}_{j\in\{1,...,n\}}$, with $\widehat{\rho}_j$ the density operator associated to the pure state $|\psi_j\rangle$. Therefore, we say that the system is represented by the state $\widehat{\rho}$

Remark 1.3. Since Eq. (1.8) is a *projector*, the sum of Eq. (1.9) is called a *convex sum of projectors*.

Theorem 1.2. *Given any hermitian operator $\widehat{A}$ and assuming the states $|\psi_j\rangle$ composing the density operator verify the Schrödinger's equation, that is $i\hbar\frac{d}{dt}|\psi_j\rangle = \widehat{H}|\psi_j\rangle$, where $\widehat{H}$ is the Hamiltonian operator, the density operator has the following properties :*

Pure state $\widehat{\rho} = \|\psi\rangle$	*Mixed state* $\widehat{\rho} = \sum_{j=1}^{n} p_j\widehat{\rho_j} = \sum_{j=1}^{n} p_j\|\psi_j\rangle\langle\psi_j\|$
$\widehat{\rho} \geq 0$	$\widehat{\rho} \geq 0$
$Tr(\widehat{\rho}) = 1$	$Tr(\widehat{\rho}) = 1$
$Tr(\widehat{\rho}^2) = 1$	$Tr(\widehat{\rho}^2) \leq 1$
$\langle\psi\|\widehat{A}\|\psi\rangle = Tr(\widehat{A}\widehat{\rho})$	$\langle\psi\|\widehat{A}\|\psi\rangle = Tr(\widehat{A}\widehat{\rho}) \geq 0$ *if* $\widehat{A} \geq 0$ (≤ 0 *if* $\widehat{A} \leq 0$)
$i\hbar\frac{d}{dt}\widehat{\rho} = [\widehat{H}, \widehat{\rho}]$	$i\hbar\frac{d}{dt}\widehat{\rho} = [\widehat{H}, \widehat{\rho}]$

Generalisation of separated and entangled states to mixed states

Let two distant physicists create, independently of each other, either a pure or mixed state system $\widehat{\rho}_1$ and $\widehat{\rho}_2$. Then, these two physicists meet and put their system in the same box. Since these two systems have been created independently to each other, one can write the state of the box system by

$$\widehat{\rho} = \widehat{\rho}_1 \otimes \widehat{\rho}_2. \tag{1.10}$$

Such a system is called a *product state*.

Definition 1.7. (Separable state and entangled state) Let $\widehat{\rho}$ be the state of a bipartite system. If there exists $p_1, ...p_n \in \mathbb{C}$ and product states ${\widehat{\rho}_1}^1 \otimes {\widehat{\rho}_2}^1$,...,${\widehat{\rho}_1}^n \otimes {\widehat{\rho}_2}^n$ such that

$$\widehat{\rho} = \sum_{i=1}^{n} p_i \ {\widehat{\rho}_1}^i \otimes {\widehat{\rho}_2}^i \tag{1.11}$$

then the state $\widehat{\rho}$ is *separable*. Otherwise, it is *entangled*.

Remark 1.4. Naturally, this definition can be expanded to multipartite systems. Given a system of N particles whose state is $\widehat{\rho}$, Eq. (1.11) becomes

$$\widehat{\rho} = \sum_{i=1}^{n} \left(p_i \ \bigotimes_{k=1}^{N} {\widehat{\rho}_k}^i \right). \tag{1.12}$$

1.1.3 Entanglement production via Local Operations and Classical Communication

Let the two parties of a bipartite system be each created by Einstein and Schrödinger in their own office, so distant again. What does the global state look like ? It is simply the product state $|\psi_{\text{Einstein}}\rangle \otimes |\psi_{\text{Schrödinger}}\rangle$. Hence from an experimental point of view, it is impossible to get an entangled state without doing anything else than creating distant states. Now Einstein and Schrödinger are allowed to perform some local transformations to their state, that is without implying transformations to the other state. In addition to that, they are allowed to communicate with each other with classical ways, such as phone. This kind of process is called *Local Operations with Classical Communication* abbreviated *LOCC*. A more in-depth discussion about these LOCC will be addressed further. The best they both can do is to prepare mixed states and add some correlations between the mixture they had prepared by using classical communication. These correlations are classical. For instance, the i-th element of Einstein's mixture can share the same probability than the j-th element of Schrödinger's one. So the global state can be written, by reordering indices,

$$\widehat{\rho} = \sum_{i=1}^{n} p_i \ \widehat{\rho}^{\,i}_{\text{Einstein}} \otimes \widehat{\rho}^{\,i}_{\text{Schrödinger}}, \tag{1.13}$$

which is equivalent to Eq. (1.11), hence we are facing a separable state. This result is similar in the case of a multipartite system. It is therefore impossible to create entanglement only from LOCC. One needs multipartite operations to generate entanglement.

1.2 Separability problem

The aim of this section is to determine a way to know if a state is entangled or not. This topic is known as the *separability problem*. An introduction to this topic is proposed in [15], while a very complete overview is presented in [16]. One can also see [12].

1.2.1 Mathematical background

Let us consider a bipartite system. Let $\mathcal{H}$ be a Hilbert space for the global system, and $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces for the two parties. Let also $\widehat{\rho}$ represent the global system, acting in $\mathcal{H}$. One can define from $\widehat{\rho}$ two density operators acting respectively in $\mathcal{H}_1$ and $\mathcal{H}_2$

Definition 1.8. (Reduced density operator) Let $\mathcal{B}_1 = \{|\psi_i^1\rangle\}_{i\in\{1,..,d_1\}}$ and $\mathcal{B}_2 = \{|\psi_j^2\rangle\}_{j\in\{1,..,d_2\}}$ be respectively a basis from Hilbert spaces $\mathcal{H}_1$ of dimension d_1 and $\mathcal{H}_2$ of dimension d_2. One can define $\widehat{\rho}_1$ and $\widehat{\rho}_2$ the *reduced density operators*, or the *partial traces*, such that

$$\langle\psi_i^1|\widehat{\rho}_1|\psi_{i'}^1\rangle = \sum_{k=1}^{d_2}\langle\psi_i^1,\psi_k^2|\widehat{\rho}|\psi_{i'}^1,\psi_k^2\rangle \tag{1.14}$$

and

$$\langle\psi_j^2|\widehat{\rho}_2|\psi_{j'}^2\rangle = \sum_{k=1}^{d_1}\langle\psi_k^1,\psi_j^2|\widehat{\rho}|\psi_k^1,\psi_{j'}^2\rangle \tag{1.15}$$

where $i' \in \{1,...,d_1\}$ and $j' \in \{1,...,d_2\}$.

Remark 1.5. One can show that the operator $\widehat{\rho}_i$ is a density operator acting in $\mathcal{H}_i$. The utility to define these two operators is to extract from $\widehat{\rho}$ all the information about the subsystems 1 and 2.

Partial transposition

The partial transposition will be useful to establish a first criterion of separability.

Theorem 1.3. *Let $\widehat{\rho}$ be the state representing a bipartite system and let $\mathcal{B}_1 = \{|\psi_i^1\rangle\}_{i\in\{1,..,d_1\}}$ and $\mathcal{B}_2 = \{|\psi_j^2\rangle\}_{j\in\{1,..,d_2\}}$ be respectively a basis from the Hilbert spaces $\mathcal{H}_1$ of dimension d_1 and $\mathcal{H}_2$ of dimension d_2 constituting the bipartite system. Given $\{|\psi_i^1\rangle\otimes|\psi_j^2\rangle\}_{i\in\{1,..,d_1\},j\in\{1,..,d_2\}}$ a basis of $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$, $\widehat{\rho}$ can be expanded as*

$$\widehat{\rho} = \sum_{i,j=1}^{d_1}\sum_{k,l=1}^{d_2}\rho_{ijkl}\ |\psi_i^1\rangle\langle\psi_k^2||\psi_j^1\rangle\langle\psi_l^2|. \tag{1.16}$$

Thus one can define the partial transpositions $\widehat{\rho}^{\,T_1}$ and $\widehat{\rho}^{\,T_2}$ of the density operator $\widehat{\rho}$, which is the transposition with respect to one subsystem. That is, one simply switches the indices i and j from ρ_{ijkl}, or the indices k and l.

Definition 1.9. (Partial transposition) The partial transposition with respect to the subsystem 1 is

$$\widehat{\rho}^{\,T_1} = \sum_{i,j=1}^{d_1} \sum_{k,l=1}^{d_2} \rho_{jikl} \; |\psi_i^1\rangle\langle\psi_k^2||\psi_j^1\rangle\langle\psi_l^2| \tag{1.17}$$

while the partial transposition with respect to the subsystem 2 is

$$\widehat{\rho}^{\,T_2} = \sum_{i,j=1}^{d_1} \sum_{k,l=1}^{d_2} \rho_{ijlk} \; |\psi_i^1\rangle\langle\psi_k^2||\psi_j^1\rangle\langle\psi_l^2|. \tag{1.18}$$

Remark 1.6. Naturally, if one uses the notation $.^T$ for the usual operator transposition, one has

$$\widehat{\rho}^{\,T} = (\widehat{\rho}^{\,T_1})^{T_2} = (\widehat{\rho}^{\,T_2})^{T_1}. \tag{1.19}$$

Definition 1.10. (Positive Partial Transpose) A density operator $\widehat{\rho}$ is said to have a *Positive Partial Transpose*, or is said to be *PPT*, if its partial transposition has no negative eigenvalues, that is if its partial transposition is positive semidefinite :

$$\widehat{\rho}^{\,T_1} \geq 0 \iff \widehat{\rho}^{\,T_2} \geq 0 \tag{1.20}$$

Otherwise, $\widehat{\rho}$ is *NTP*, standing for *Negative Partial Transpose*

1.2.2 Separability criteria

There exists a lot of answers to this question, each one introduced to complement the weaknesses of the previous ones. We will highlight this by introducing a first criterion, the PPT criterion, also known as Peres-Horodecki criterion, examine its limits, and introduce another criterion to complete it, the CCNR criterion.

PPT criterion

Theorem 1.4. *(PPT criterion) Let $\widehat{\rho}$ be a separable state. Then $\widehat{\rho}$ is PPT.*

Hence if $\widehat{\rho}$ is NPT, then it is entangled. In other words, computing its spectrum and finding a negative eigenvalue amounts to the conclusion of an entangled state. This strong criterion allows us to partially answer the question of separability. But there is a weakness : this theorem is not always sufficient. This is only true in low dimensional systems.

Theorem 1.5. *(Horodecki) If $\widehat{\rho}$ is PPT, then it is separable only in systems of dimension 2×2 and 2×3.*

This theorem, stated by Peres and proved by Horodecki both in the same paper of 1996, amounts that there exists PPT entangled states. That means the PPT criterion does not detect all entangled states. Despite this weakness, this criterion is the most frequently used separability criterion, since a system of two qubits[6], which is the most common bipartite system whose wave functions are elements from $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, can be said to be entangled or not in every case. A natural question arises : how to detect entangled states when the PPT criterion fails ?

CCNR criterion

We answer this question by the *Computable Cross Norm or Realignment (CCNR)* criterion. To mention it, the name comes from two different ways to derive this criterion. The first is the investigation of cross norms of density operators, and the second is made by realigning the entries of the density operator represented by a matrix.

Let $\widehat{\rho}$ be the state representing a bipartite system and let $\mathcal{B}_1 = \{|\psi_i^1\rangle\}_{i\in\{1,..,d_1\}}$ and $\mathcal{B}_2 = \{|\psi_j^2\rangle\}_{j\in\{1,..,d_2\}}$ be respectively a basis from the Hilbert spaces $\mathcal{H}_1$ of dimension d_1 and $\mathcal{H}_2$ of dimension d_2 constituting the bipartite system. Let us consider without loss of generality that $d_1 \geq d_2$. We can express an element of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ in terms of the basis $\mathcal{B}_1$ and $\mathcal{B}_2$ by the decomposition that follows.

Theorem 1.6. *(Schmidt Decomposition) For all $|\psi\rangle \in \mathcal{H}$,*

$$|\psi\rangle = \sum_{i=1}^{d_2} \alpha_i \ |\psi_i^1\rangle \otimes |\psi_i^2\rangle \tag{1.21}$$

where the coefficients α_i are real, non-negative and unique.

It can be written for the density operator as follows.

Theorem 1.7. *Let $\{G_i^1\}$ and $\{G_j^2\}$ be two sets of hermitian operators both forming an orthonormal basis of the observable spaces[7] of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. For all $\widehat{\rho}$ representing a bipartite system, one has*

$$\widehat{\rho} = \sum_i \alpha_i \ G_i^1 \otimes G_i^2 \tag{1.22}$$

[6] qubits are defined further.

[7] An *observable space* with respect to a Hilbert space is the vector space of all hermitian operators acting in and producing an element of this Hilbert space.

where the coefficients α_i are real, non-negative and unique.

Theorem 1.8. *(CCNR criterion) If $\widehat{\rho}$ is separable, then the coefficients of the Schmidt decomposition meet the following condition :*

$$\sum_i \alpha_i \leq 1. \tag{1.23}$$

Thus if

$$\sum_i \alpha_i \geq 1 \tag{1.24}$$

then the state is entangled.

This criterion is easy to compute and can detect many entangled states where the PPT criterion failed. Nevertheless, the CCNR criterion fails sometimes for two qubits systems, where the PPT criterion never does. It implies a perfect complementarity of these two criteria : one can use both of them to be sure to detect any entangled state.

1.3 Entanglement measures

If a state is entangled, one would want to know how far from a separable state it is, and how to measure what we would call the quantity of entanglement. This is the purpose of the present section. An introduction to this topic is presented in [15], while a more complete overview is presented in [17]. One can also see [12].

1.3.1 General definition of an entanglement measure

Let us define $E : \widehat{\rho} \to E(\widehat{\rho}) \in \mathbb{R}^+$ as a function of a state $\widehat{\rho}$, which returns a positive value depending on the state. The greater the value it returns, the greater the system is entangled. We have to provide it with some natural and intuitive properties it should fit :

- We have to fix E to zero if the state is separable :
$$E(\widehat{\rho}) = 0 \iff \widehat{\rho} \text{ is a separable state} \tag{1.25}$$
- Since it is impossible to generate entanglement via LOCC, $E(\widehat{\rho})$ should not increase under such state operations $\widehat{\rho} \to \text{LOCC}[\widehat{\rho}]$:
$$E(\widehat{\rho}) \geq E(\text{LOCC}[\widehat{\rho}]) \tag{1.26}$$
- There would be no reason for E to be frame dependant, it should be invariant under a local change of basis, that is invariant under local unitary transformations. For a bipartite system,
$$E(\widehat{\rho}) = E(U_1 \otimes U_2 \ \widehat{\rho} \ U_1^\dagger \otimes U_2^\dagger) \tag{1.27}$$
where $U_1, U_2 \in$U(d), the group of d $\times$ d unitary matrices, and where $U_1 \otimes U_2 \equiv \left(\begin{smallmatrix} U_1 & 0 \\ 0 & U_2 \end{smallmatrix}\right)$. One notes that if the second property is fulfilled, then is this one.

- Since $\sum_i p_i E(\widehat{\rho}_i)$ takes into account every pure state entanglement, we lose information about these pure states if we consider now $E(\sum_i p_i \ \widehat{\rho}_i)$. Therefore
$$E(\sum_i p_i \ \widehat{\rho}_i) \leq \sum_i p_i E(\widehat{\rho}_i). \tag{1.28}$$
This is called *convexity*.
- If the entanglement measure of a state is $E(\widehat{\rho})$, and if we copy this state n times, forming the state $\widehat{\rho}^{\otimes n}$, then
$$E(\widehat{\rho}^{\otimes n}) = nE(\widehat{\rho}) \tag{1.29}$$
which is called *additivity*. The enhanced version of this property, *the full additivity*, is the following : given two states $\widehat{\rho}_1$ and $\widehat{\rho}_2$, respectively of entanglement measure $E(\widehat{\rho}_1)$ and $E(\widehat{\rho}_2)$, forming the state $\widehat{\rho}_1 \otimes \widehat{\rho}_2$, one has
$$E(\widehat{\rho}_1 \otimes \widehat{\rho}_2) = E(\widehat{\rho}_1) + E(\widehat{\rho}_2). \tag{1.30}$$

1.3.2 Examples : concurrence and negativity

Every function E which satisfies this definition and at least the three first properties can be used to measure how much a state is entangled. The only thing that matter is how easy it is to compute and use. There are a lot of entanglement measures that has been defined in a lot of different situations. Here, we will give two examples. *Concurrence* is the first one, introduced to emphasize that easily computable measures are popular. The second one, the *negativity*, since it is in continuity with the PPT criterion.

Concurrence

Definition 1.11. (Concurrence, pure bipartite state case) For bipartite pure states, one defines *concurrence* as

$$\mathcal{C}(\psi) = \sqrt{2(1 - \mathrm{Tr}[\widehat{\rho}_1]^2)} \tag{1.31}$$

where $\widehat{\rho}_1$ is the reduced state of $|\psi\rangle$ to the subsystem 1.

Definition 1.12. (Concurrence, mixed bipartite state case) Let $\widehat{\rho}\widehat{S}\widehat{\rho}^{\,*}\widehat{S}$ be an operator and $\lambda_1, \lambda_2, \lambda_3$ and λ_4 be the decreasingly ordered square roots of the eigenvalues of this operator. We denote $S = \sigma_y \otimes \sigma_y$ the *spin-flip* operator. For bipartite mixed states, one defines *concurrence* as

$$\mathcal{C}(\widehat{\rho}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}. \tag{1.32}$$

Negativity

An entangled state from a low dimensional system (such as two qubits) will be detected by the PPT criterion and one could say that this entangled state *violates* the PPT criterion. Negativity is the measure of "how much the PPT criterion is violated".

Definition 1.13. (Negativity) The *negativity* is defined as

$$N(\widehat{\rho}) = \frac{\mathrm{Tr}(\sqrt{\widehat{\rho}^{\,T_2}(\widehat{\rho}^{\,T_2})^\dagger}) - 1}{2} \tag{1.33}$$

which, by construction, is zero if the state is PPT.

1.4 Entanglement classification

Our aim in this section is to mathematically define the *SLOCC classification family* and to show the classification of 2-qubit and 3-qubit system states with SLOCC. Finally, one will conclude this first chapter by introducing the aim of this document, which is the classification of n-qubit system states with algebraic geometry.

1.4.1 Mathematical definition of LOCC and SLOCC families

Definition 1.14. (LOCC equivalence) If a state $\widehat{\rho}$ has a probability of 1 to be transformed into another state $\widehat{\tau}$ under a LOCC, they are said *LOCC-equivalent*, or said to belong to the same *LOCC-class*.

As shown in [18], if $\widehat{\rho}$ and $\widehat{\tau}$ are respectively pure states $|\psi\rangle$ and $|\phi\rangle$, then they are LOCC-equivalent if there exists a local transformation of the basis which transform $|\psi\rangle$ into $|\phi\rangle$, and *vice-versa*. This is the next theorem.

Theorem 1.9. *Let $\mathcal{H}$ be a Hilbert space of dimension n. $|\psi\rangle \in \mathcal{H}$ and $|\phi\rangle \in \mathcal{H}$ are LOCC-equivalent if and only if there exists a set of n unitary matrices $\{U_i,\ i \in \{1, ..., n\} \mid U_i^\dagger U_i = 1\ \forall i\}$ such that*

$$|\psi\rangle = \bigotimes_{i=1}^{n} U_i\ |\phi\rangle. \tag{1.34}$$

Definition 1.15. (SLOCC equivalence) If a state $\widehat{\rho}$ has a probability $0 < p_i < 1$ to be transformed into another state $\widehat{\tau}_i$ under a LOCC, they are said to be *SLOCC-equivalent*, or said to belong to the same *SLOCC-class*.

Remark 1.7. S stands for *Stochastic*.

As shown in [19], two pure states $|\psi\rangle$ and $|\phi\rangle$ are SLOCC-equivalent if and only if they can be converted into each other by an *Invertible Local Operation* (*ILO*).

Theorem 1.10. *Let $\mathcal{H}$ be a Hilbert space of dimension n. $|\psi\rangle \in \mathcal{H}$ and $|\phi\rangle \in \mathcal{H}$ are SLOCC-equivalent if and only if there exists a set of n invertible matrices $\{A_i,\ i \in \{1, ..., n\} \mid det(A_i) \neq 0\ \forall i\}$ such that*

$$|\psi\rangle = \bigotimes_{i=1}^{n} A_i\ |\phi\rangle. \tag{1.35}$$

A difference between these two classes is the following. With a LOCC protocol, one can convert a state $\widehat{\rho}$ into a state $\widehat{\tau}$. With a SLOCC protocol, one can convert a state $\widehat{\rho}$ into a probabilistic distribution of mixed states $\widehat{\tau_i}$ each associated to a probability p_i

Since LOCC classes are also SLOCC classes, but not the converse, there is another difference. The LOCC classification amounts to an infinite number of classes even in the smallest multipartite systems, such as bipartite systems like two qubits, whereas SLOCC classification amounts to an infinite number of classes for multipartite systems composed of more than 3 parties, such as 4-qubit systems.
Note that the 4-qubit system case has been solved (See for instance [20, 21]) and the problem of symmetric states has also been solved (See for instance [22, 23, 24]).

1.4.2 Introduction to qubits

In quantum information, one works with systems called *qubits*, a term first introduced in 1995 in [25], which are systems described by a state from the Hilbert space $\mathcal{H} = \mathbb{C}^2$ with a basis $\mathcal{B} = \{|0\rangle, |1\rangle\}$, where $|0\rangle = \binom{1}{0}$ and $|1\rangle = \binom{0}{1}$. This basis is called the *computational basis*.
They are the quantum counterpart of the classical bit from classical information. While a bit can only be set to 0 or 1, a qubit can be set to a state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{1.36}$$

where α and $\beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. If $\alpha = 0$, then the qubit is similar to a bit set to 1. $\beta = 0$ amounts the qubit to be similar to a bit set to 0. If $\alpha, \beta \neq 0$, then the qubit can not be represented as a classical bit, because such a superposition of states is purely quantum. Measurement of the qubit value leads then the qubit to be either in a state $|0\rangle$ with a probability $|\alpha|^2$ or in a state $|1\rangle$ with a probability $|\beta|^2$.

To implement a qubit, there are a lot of possibilities. Here are some examples :

- Polarization of a photon. A horizontally polarised photon is represented by the state $|0\rangle$, while a vertically polarised photon is represented by $|1\rangle$.
- Photon number. A system with no photon is represented by the state $|0\rangle$, while a system with a photon is represented by $|1\rangle$.
- Spin of an electron. An electron with a spin up ($m_s = \frac{1}{2}$) is represented by the state $|0\rangle$ while the state $|1\rangle$ represents a spin down ($m_s = -\frac{1}{2}$) electron
- Electron number (same as photon number).

Any two-level quantum system can be used as a qubit, or any multi-level quantum system which contains two levels that can be decoupled from the rest (such as photon number).

1.4.3 Classification of 2-qubit and 3-qubit system states with SLOCC

An n-qubit system is represented by an element, a function $|\psi\rangle$, from the Hilbert space

$$\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i \quad \text{where} \quad \mathcal{H}_i = \mathbb{C}^2 \ \forall i \in \{1, ..., n\}. \tag{1.37}$$

Or, in an equivalent way we can use another notation

$$\mathcal{H} = (\mathbb{C}^2)^{\otimes n} \tag{1.38}$$

Let K be the set of n matrices from the general linear group of degree[8] 2 whose entries are complex elements, that is

$$K = \underbrace{\mathrm{GL}_2(\mathbb{C}) \times ... \times \mathrm{GL}_2(\mathbb{C})}_{n \ times} \tag{1.39}$$

From Eq. (1.35), one can see that the elements of K are the matrices defining SLOCC equivalence. Our aim is to identify the action of this group onto $\mathcal{H}$ and to find the different orbits, which are the equivalent classes under the equivalence relation $\sim$ such that

$$\text{Two states are said equivalent } |\psi\rangle \sim |\phi\rangle \iff \text{there exists } g \in K \text{ such that } g|\psi\rangle = |\phi\rangle \tag{1.40}$$

In other words, studying the action of the group K onto $\mathcal{H}$ will allow us to find out the SLOCC equivalence classes for n-qubit.

Classification of 2-qubit system states with SLOCC

The result from [26] is presented here. Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ be the 2-qubit Hilbert space and $K = \mathrm{GL}(2) \times \mathrm{GL}(2)$. Note that for now on, we drop the normalisation constant first for pedagogical purpose and then because we will further work with projective spaces[9].
To determine the orbits, one has to take an element $|\psi\rangle$ from $\mathcal{H}$ and to apply onto it every $g \in K$, and start again for every $|\psi\rangle \in \mathcal{H}$. This process only shows 2 different orbits, *i.e.* two different equivalence classes.

First, the set of all separable states $|\psi\rangle$, such that there exists $|\phi\rangle \in \mathcal{H}_1$, $|\chi\rangle \in \mathcal{H}_2$ such that

$$|\psi\rangle = |\phi\rangle \otimes |\chi\rangle \tag{1.41}$$

This set is called *the set of all separable states*.

[8] This latter being the group of invertible matrices 2×2
[9] A definition is provided further

Second, the set of all non-separable states $|\psi\rangle$, such that there exists $|\phi\rangle, |\phi'\rangle \in \mathcal{H}_1$ and $|\chi\rangle, |\chi'\rangle \in \mathcal{H}_2$ such that

$$|\psi\rangle = |\phi\rangle \otimes |\phi'\rangle + |\chi\rangle \otimes |\chi'\rangle \tag{1.42}$$

This is called *the set of all entangled states.*

This classification means that if a state $|\psi\rangle$ is separable or entangled, then $\forall g \in K$, $g|\psi\rangle$ is respectively separable or entangled. Furthermore, one can show that for every entangled state $|\psi\rangle$, there exists $g \in K$ such that

$$g|\psi\rangle = |\phi\rangle = |00\rangle + |11\rangle \tag{1.43}$$

Which is known as a *Bell state*

Classification of 3-qubit system states with SLOCC

The result of [19] is presented here. Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ be the 3-qubit Hilbert space and $K = \mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(2)$.

There are six different orbits :

- First, the set of all separable states, as in the 2-qubit classification.
- Second, there are two sets of genuinely entangled states.
  A first one compounded with states $|\psi\rangle$ such that there exists $g \in K$ such that

  $$g|\psi\rangle = |\phi\rangle = |000\rangle + |111\rangle \tag{1.44}$$

  where the state $|\phi\rangle$ is called the *Greenberger–Horne–Zeilinger state (or GHZ state)* and denoted $|\mathrm{GHZ}\rangle$.
  A second one compounded with states $|\psi\rangle$ such that there exists $g \in K$ such that

  $$g|\psi\rangle = |\phi\rangle = |001\rangle + |010\rangle + |100\rangle \tag{1.45}$$

  where the state $|\phi\rangle$ is called the *W state* and denoted $|\mathrm{W}\rangle$.
- Third, there are three sets of bi-separable states, that is states of a 3-qubit system where only two qubits are entangled. For such $|\psi\rangle$ states, there exists either g, g' or $g'' \in K$ such that

  $$g|\psi\rangle = |\phi\rangle = |000\rangle + |011\rangle \tag{1.46}$$

  $$g'|\psi\rangle = |\phi\rangle = |000\rangle + |101\rangle \tag{1.47}$$

  $$g''|\psi\rangle = |\phi\rangle = |000\rangle + |110\rangle \tag{1.48}$$

  For instance, we can write $|000\rangle + |011\rangle = |0\rangle \otimes (|00\rangle + |11\rangle)$ to see this bi-separable property.

1.4.4 Classification of n-qubit system states with algebraic geometry

U(1) gauge invariance of quantum theory

U(1) is the group of the complex number of modulus one. Since multiplying a state $|\psi\rangle$ by a *phase factor* $\lambda = \mathrm{e}^{i\phi}$ of modulus one, where ϕ is an angle between 0 and 2π, does not affect the measurements[10], one can say that quantum theory is an invariant theory under U(1), the group of phase factors.

Projective Hilbert space and algebraic geometry

In the following, we will use the U(1) gauge invariance of quantum theory to consider rays in the Hilbert space to be elements of another space, the projective Hilbert space. One notes that rays in the Hilbert space are sets of all states $|\psi\rangle,\ |\phi\rangle$ such that there exists $\lambda \in \mathbb{C}\backslash\{0\}$ such that

$$|\psi\rangle = \lambda|\phi\rangle \tag{1.49}$$

Since algebraic geometry is a mathematical framework to study projective spaces, we will introduce its concepts to fully understand the projective Hilbert space and to use its properties. From there, it will be easy to define and derive some tools from algebraic geometry and multilinear algebra which are SLOCC invariants and which allow one to sort SLOCC equivalent states into families (proper k-secant of Segre variety), and then into subfamilies (l-multilinear ranks).

[10]Since $\langle \mathrm{e}^{i\phi}\psi|\widehat{A}|\mathrm{e}^{i\phi}\psi\rangle = \mathrm{e}^{-i\phi}\mathrm{e}^{i\phi}\langle\psi|\widehat{A}|\psi\rangle = \langle\psi|\widehat{A}|\psi\rangle$, the meaning value of the measurements of an observable A represented by an hermitian operator $\widehat{A}$ is not affected by a phase factor

2 Algebraic geometry

Algebraic geometry is the study of the shape of the zero set of a collection of polynomials, called the *zero locus* of a set of polynomials. For instance, let us consider the following polynomial

$$x^2 + y^2 + z^2 - 1 \quad \text{with } x, y, z \in \mathbb{R}. \tag{2.1}$$

The set of zeros of such a polynomial is the 2-sphere of radius one defined by

$$S^2 = \{(x, y, z) \in \mathbb{R}^2 | x^2 + y^2 + z^2 = 1\}. \tag{2.2}$$

The shape of this set of zeros is represented by Figure 2.1

Figure 2.1: The 2-sphere is the zero locus of the polynomial $x^2 + y^2 + z^2 - 1$

One notes that the study of the zero locus of such polynomials depends on the field whose variables are from. Here we will consider a generic field k, and we will denote k[x] the set of polynomials in a single variable x with coefficient in k. Naturally, we define k[$x_1, x_2, ..., x_n$] to be the set of all polynomials in $x_1, ..., x_n$, with $n \in \mathbb{N}$, with coefficients in k.

For instance, in our previous example, $x_1 = x$, $x_2 = y$, $x_3 = z$ and k=$\mathbb{R}$. Obviously, Eq. (2.1) is an example of element in $\mathbb{R}$[x,y,z]

The following notions of algebraic geometry are developed in a more complete way in [27, 28, 29, 30]. Detailed and pedagogical introduction videos to the topic can be found in [31, 32, 33].

2.1 Affine algebraic varieties

The aim of this present section is to define affine varieties, which play a central role in algebraic geometry. Linear algebraic and topological notions developed in appendices are assumed known in the following.

2.1.1 Zero locus of a set of polynomials

Definition 2.1. (Zero locus of a polynomial) The *zero locus of a polynomial* P in n variables, P∈k$[x_1, ..., x_n]$, is the set Z(P)$\subset k^n$ of points of the n-dimensional affine space such that this polynomial is zero for every element of Z(P). That is

$$\text{Z(P)} = \{a \in k^n \mid P(a) = 0\} \subset k^n. \tag{2.3}$$

Definition 2.2. (Zero locus of a set of polynomials) The *zero locus of a set of polynomials* $S \subset$k$[x_1, ..., x_n]$ is the set Z(S) $\subset k^n$ whose elements are zero locus for every polynomials P∈S. That is,

$$\text{Z(S)} = \{a \in k^n \mid P(a) = 0 \ \forall P \in S\} \subset k^n \tag{2.4}$$

Remark 2.1. The larger is S, the larger is the number of equations that elements from Z(S) has to satisfy. Hence, larger S means smaller Z(S).

Example 2.1. Here are some examples of zero locus of polynomials and set of polynomials

- Eq. (2.2) is an example of Z(P), where P=$x_1^2 + x_2^2 + x_3^2 - 1 \ \in \mathbb{R}[x_1, x_2, x_3]$
- k^n =Z($\emptyset$), where $\emptyset$ is the empty set of polynomials.
- Z(k$[x_1, ..., x_n]$)=$\emptyset$, where $\emptyset$ is the empty set of points from k^n
- Z($y^3 - (x^3 + x^2)$) is the nodal cubic as shown on Figure 2.2

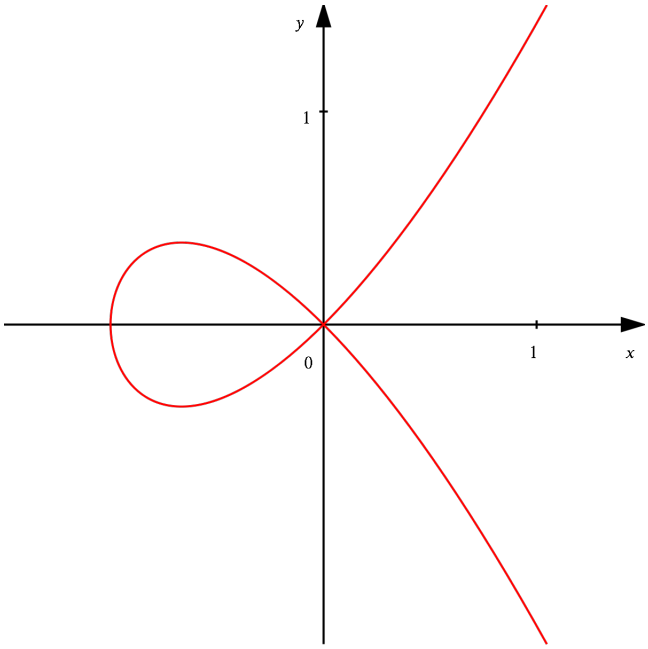

Figure 2.2: Zero locus of P=$y^3 - (x^3 + x^2)$

Definition 2.3. (Affine algebraic set) The zero locus of a set of polynomials defined by Eq. (2.4) is called an *affine algebraic set*

2.1.2 Zariski topology

We can define the Zariski topology to enhance the notion of algebraic affine sets with tools from topology.

Theorem 2.1. *The family of affine algebraic sets is a topology on* k^n.

Proof. Let $\{S_\alpha\}_{\alpha \in J \subset \mathbb{N}}$ be a collection of subsets of $k[x_1, ..., x_n]$. Then[1]

$$\bigcap_{\alpha \in J} \mathrm{Z}(S_\alpha) = \mathrm{Z}(\bigcup_{\alpha \in J} S_\alpha) \tag{2.5}$$

which means that intersections of algebraic affine sets ($\bigcap_{\alpha \in J} \mathrm{Z}(S_\alpha)$) are algebraic affine sets ($\mathrm{Z}(\bigcup_{\alpha \in J} S_\alpha)$)

[1] One can check this following property simply by thinking of it in terms of polynomials and zero locus. $\mathrm{Z}(\bigcup_{\alpha \in J} S_\alpha)$ is the zero locus of all polynomials from every S_α, and $\bigcap_{\alpha \in J} \mathrm{Z}(S_\alpha)$ is the set of common zeros to all S_α. These two sets are the same.

Let S,T∈k$[x_1, ..., x_n]$. Then[2]

$$\mathrm{Z}(S) \cup \mathrm{Z}(T) = \mathrm{Z}(ST) \qquad \text{where} \quad ST = \{fg \mid f \in S,\ g \in T\} \tag{2.6}$$

which means that finite union of algebraic affine sets (Z(S) ∪ Z(T)) are affine algebraic sets (Z(ST)). □

Definition 2.4. (Zariski topology) The *Zariski topology* $\mathcal{T}_Z$ is the topology on k^n whose closed subsets are affine algebraic sets.

As said above, the Zariski topology is a topology on k^n and (k^n, $\mathcal{T}_Z$) is a topological space. If X⊂ k^n, then the induced topology $\mathcal{T}_{\text{induced}} = \{\Omega \cap X \mid \Omega \in \mathcal{T}_Z\}$ is also named the Zariski topology on X ⊂ k^n, but the context will help us to not be confused about this definition.

Theorem 2.2. *($k^n, \mathcal{T}_Z$) is a Noetherian topological space.*

Remark 2.2. It can easily be proved by introducing the *ideal* I(X) of an affine algebraic set X, which is the set of polynomials whose zero locus is X. But since it would not be necessary to introduce ideals for further notions, one won't define it.

Definition 2.5. (Affine algebraic varieties) Let $(X, \mathcal{T}_Z)$, with X⊂ k^n, be a topological space. Irreducible components defined in Eq. (A.10) are named *affine algebraic varieties.*

Recalling the definition of irreducible polynomials, one can make a link between those and affine algebraic varieties.

Theorem 2.3. *If $P \in k[x_1, ..., x_n]$ is irreducible, then Z(P) is an affine algebraic variety, that is Z(P) is irreducible.*

Example 2.2. An example of an affine algebraic variety is given by Z(ST)=Z(S)∪Z(T). Z(S) and Z(T) are the irreducible components of Z(ST). Z(T) and Z(S) are affine algebraic varieties, while Z(ST) is an affine algebraic set. Another example is that $a \in k^n$ is irreducible.

One concludes this section with the definition of a determinantal variety, which will be useful in the chapter 3 to make a link between l-multilinear ranks and algebraic varieties. It is also a good example of algebraic variety.

[2] Z(S) ∪ Z(T) is the zero locus of all polynomials from S plus the zero locus of all polynomials from T. Therefore each point from this union is a zero of all polynomials from S and T. While Z(ST) is the zero locus of all multiplication of polynomials from S and T. Then each zero of polynomials from S or T is a zero of polynomials from ST. Therefore the equality is straightforward.

Definition 2.6. (Determinantal variety) Let $m, n \in \mathbb{N}$ and $r < \min(m, n)$. *The determinantal variety* X_r is the set of all $m \times n$ matrices over a field k with rank less than or equal to r.

Remark 2.3. Since the condition that a matrix have rank less than or equal to r is given by the vanishing of all of its $(r+1) \times (r+1)$ minors, that is the vanishing of polynomials of degree $r+1$, this is indeed an algebraic variety.

2.2 Projective algebraic varieties

2.2.1 Projective space

An important example of a quotient set is the one of projective space. Let us define it before introducing the real projective space. Finally, to make a link between this mathematical framework and our physical motivations, the projective Hilbert space will be introduced.

First, let $k^{n+1}\backslash\{0\}$ be the n+1-dimensional k space without the origin and $(a_0, ..., a_n),\ (b_0, ..., b_n) \in k^{n+1}$. One can define the equivalence relation $\sim$ such that

$$(a_0, ..., a_n) \sim (b_0, ..., b_n) \iff \exists \lambda \in k\backslash\{0\} \text{ such that } (a_0, ..., a_n) = \lambda(b_0, ..., b_n) = (\lambda b_0, ..., \lambda b_n) \quad (2.7)$$

and one denotes $[a_0, ..., a_n]$ the equivalence class of $(a_0, ..., a_n)$ with respect to $\sim$. The definition of the projective space follows.

Definition 2.7. (Projective space) The *n-dimensional projective space* is the quotient set

$$\mathbb{P}^n \equiv (k^{n+1}\backslash\{0\})/\sim \ \equiv^3 \mathbb{P}(k^{n+1}\backslash\{0\}) \quad (2.8)$$

One can identify the projective space of $k^{n+1}\backslash\{0\}$ to the set of all its straight lines. One notes that to go from the projective space to $k^{n+1}\backslash\{0\}$, one can just fix a_0=1. Then the point [1,$a_1, ..., a_n$] corresponds to $(a_1, ..., a_n)$.

Definition 2.8. (Real projective space) Let $\mathbb{R}^{n+1}_\star = \mathbb{R}^{n+1}\backslash\{0\}$ the $n+1$-dimensional real space without the origin and $\sim$ the equivalence relation defined by

$$\forall x, y \in \mathbb{R}^{n+1}_\star,\ x \sim y \iff \exists \lambda \in \mathbb{R}_\star \text{ such that } x = \lambda y. \quad (2.9)$$

The *real projective space*, denoted $\mathbb{R}P^n$, is the quotient set $\mathbb{R}^{n+1}_\star/\sim$. The real projective space together with the quotient topology with respect to this equivalence relation is a topological space.

[3]This notation is used here to highlight the construction of the projective space with the set $k^{n+1}\backslash\{0\}$

Since a quotient set is the set of all equivalence classes with respect to an equivalence relation, one can see the real projective space as the set of all lines of $\mathbb{R}^{n+1}_\star$ passing through the origin (but without it). That is,

$$\mathbb{R}^{n+1}_\star/\sim \; = \; \{y = mx \mid x, y \in \mathbb{R}^{n+1}_\star, \; m \in \mathbb{R}_\star\}. \tag{2.10}$$

Finally, one can see that since lines[4] of $\mathbb{R}^{n+1}_\star$ are associated to points[5] of the projective space $\mathbb{R}P^n$, this latter has one dimension less.

One can now consider $\mathbb{C}^{n+1}$ instead of $\mathbb{R}^{n+1}$. In quantum information, our motivational framework, quantum states of qudits are elements of the Hilbert space $\mathbb{C}^d_\star$. Here we will only consider qubits (d=2), then states from $\mathbb{C}^2_\star$. A system of n qubits is represented by a tensor from the Hilbert space $\mathcal{H} = \mathbb{C}^{2n}_\star \equiv (\mathbb{C}^2_\star)^{\otimes n} = \underbrace{\mathbb{C}^2_\star \otimes ... \otimes \mathbb{C}^2_\star}_{\text{n times}}$.

Denoting $|\psi\rangle \in \mathbb{C}^{2n}_\star$ the state of an n-qubit system, one has the following definition.[6]

Definition 2.9. (Projective Hilbert space of an n-qubit system) Let $\mathbb{C}^{2n}_\star = \mathbb{C}^{2n}\backslash\{0\}$ the Hilbert space of an n-qubit system, without the origin and $\sim$ the equivalence relation defined by

$$\forall |\psi\rangle, |\phi\rangle \in \mathcal{H} = \mathbb{C}^{2n}_\star, \; |\psi\rangle \sim |\phi\rangle \iff \exists \lambda \in \mathbb{C}_\star \text{ such that } |\psi\rangle = \lambda |\phi\rangle \tag{2.11}$$

The *complex projective space*, or the *projective Hilbert space of an n-qubit system*, denoted $\mathbb{C}P^{2n}$ or $\mathbb{P}(\mathcal{H})$, or simply $\mathbb{P}$, is the quotient set $\mathbb{C}^{2n}_\star/\sim$. The complex projective space together with the quotient topology with respect to this equivalence relation is a topological space.

Definition 2.10. (Rays of the Hilbert space) Equivalence classes[7] with respect to $\sim$ defined in Eq. (2.11) are called *rays* of the Hilbert space. Rays of the Hilbert space are denoted $[|\psi\rangle]$, where $|\psi\rangle$ is a representative of this equivalence class.

Remark 2.4. Physically speaking, rays of the Hilbert space represent the same physical state since quantum mechanics are U(1) gauge invariant. Hence it is convenient to make use of the projective Hilbert space rather than the Hilbert space. In the following, one should be careful about the word "states". Elements of the projective Hilbert space are not states, but a set of states which are equivalent in some sense. But since they represent the same *physical* state by the U(1) gauge invariance, one uses again the word "states" to qualify elements of the projective Hilbert space. Therefore, by "the state of a projective Hilbert space", one means "the element of a projective Hilbert space whose representative is a state from an Hilbert space".

[4] Which are subsets of dimension 1

[5] Which are subsets of dimension 0

[6] The same definition holds for a more general projective Hilbert space where $\mathcal{H}$ is not equal to the 2n-dimensional complex set.

[7] That is, elements of the quotient space, here the projective Hilbert space

2.2.2 Zero locus of a set of polynomials

Now we will consider polynomials no more from $k[x_1, ..., x_n]$ but from $k[x_0, ..., x_n]$. The difference lies in the fact that there is one more variable, x_0. Projective spaces have been mathematically introduced to treat compact version of non-compact sets.

Definition 2.11. (Zero of a polynomial) Let $g \in k[x_0, ..., x_n]$ be a homogeneous multivariate polynomial. A point of the projective space p=$[a_0, ..., a_n] \in \mathbb{P}^n$ is a *zero* of g if g(p) = g($\lambda a_0, ..., \lambda a_n$) = 0, $\forall \lambda \in k$. One says that g(p)=0 for every *representative* of p.

One notes that we are dealing with homogeneous polynomials. In the affine case, polynomials were defined as functions. But in the projective case, a polynomial $P \in k[x_0, ..., x_n]$ does not define a function, because of the representative λ[8]. Therefore, homogeneous polynomials define a function in the projective case.[9]

Definition 2.12. (Zero locus of a set of polynomials) Let S $\subset k[x_0, ..., x_n]$ be a set of homogeneous polynomials. The *zero locus* of polynomials of S is

$$\mathrm{Z(S)} \equiv \{p \in \mathbb{P}^n \mid f(p) = 0 \ \forall f \in S\}. \tag{2.12}$$

Definition 2.13. (Projective algebraic set) Zero locus of polynomials defined in Eq. (2.12) are named *projective algebraic sets.*

Remark 2.5. Projective algebraic sets are closed, in the same way affine algebraic sets are closed.

2.2.3 Zariski topology

We will now develop a topology on the projective space $\mathbb{P}(k^{n+1})$. First, let $(f_\alpha)_{\alpha \in J \subset \mathbb{N}}$ be a collection of homogeneous polynomials.

Theorem 2.4.

$$Z(\bigcup_{\alpha \in J} S_\alpha) = \bigcap_{\alpha \in J} Z(S_\alpha) \tag{2.13}$$

i.e. an intersection of closed subsets is closed.

Second, let $S, T \subset k[x_0, ..., x_n]$ be two collections of homogeneous polynomials.

[8] $P(a_0, ..., a_n) \neq P(\lambda a_0, ..., \lambda a_n)$

[9] Since $P(\lambda a_0, ..., \lambda a_n) = \lambda P(a_0, ..., a_n)$ if P is homogeneous, $P(a_0, ..., a_n) \sim P(\lambda a_0, ..., \lambda a_n)$ if P is homogeneous, and then $P(a_0, ..., a_n) = P(\lambda a_0, ..., \lambda a_n)$ in the projective framework. The issue is thus solved.

Theorem 2.5.

$$Z(ST) = Z(S) \cup Z(T) \tag{2.14}$$

i.e. a union of closed subsets is closed.

Definition 2.14. (Zariski topology) The *Zariski topology* on $\mathbb{P}^n$ is the topology denoted $\mathcal{T}_Z$ whose closed subsets are projective algebraic sets. If $X \subset \mathbb{P}^n$, one can define the induced topology on X, which is called the Zariski topology on X.

Definition 2.15. (Quasiprojective algebraic set) An open subset of a projective algebraic set is called a *quasiprojective algebraic set.*

The following theorem leads us to the definition of a projective and to a quasiprojective variety.

Theorem 2.6. *$(\mathbb{P}^n, \mathcal{T}_Z)$ is a Noetherian topological space.*

That means that every subset of $\mathbb{P}^n$ is Noetherian, and thus are any projective algebraic set and any quasiprojective algebraic set. This implies that there exists, for any projective algebraic set or any quasiprojective algebraic set, a unique decomposition into irreducible components. A particular name is given to these irreducible components.

Definition 2.16. (Projective variety and quasiprojective variety) A *projective variety* is an irreducible projective algebraic set. A *quasiprojective variety* is an irreducible quasiprojective algebraic set.

Example 2.3. Since[10] $k^n \subset \mathbb{P}^n$ and since $k^n = \mathbb{P}^n \backslash \mathrm{Z}(x_0)$, k^n is an open subset of $\mathbb{P}^n$. Hence k^n is a quasiprojective variety.

Projective varieties and quasiprojective varieties are of paramount importance for our purpose, which is to use algebraic geometrical tools to classify SLOCC equivalent states. In the following, one will always consider projective varieties. To define these tools, one needs to define a particular morphism, which is the Segre morphism.

[10] $\mathbb{P}^n = k^{n+1} \supset k^n$.

2.3 Morphisms

Here one will define particular maps between algebraic varieties, which are named morphisms. Our motivation is to further define the Segre morphism, a morphism between projective spaces[11].

2.3.1 Definition of local ring and morphism

Let's recall the analytical definition of a regular function.

Definition 2.17. (Regular function) A function is *regular* on a set X if it is analytic[12] and single-valued on X.

To define morphisms, one has to define local rings.

Definition 2.18. (Local ring) Let $V \subset \mathbb{P}^n$ be a projective variety, $P \in V$ a point of V and $U \subset V$ a subset of V and let K(V) be the field of rational functions on V.

- The *local ring of V in P* is the set

$$O_{V,P} = \{\frac{f}{g} \in K(V) \mid g(P) \neq 0\} \tag{2.15}$$

- If U is an open subset of V, the *ring of all regular functions on U* is

$$O_V(U) = \{\frac{f}{g} \in K(V) \mid g(P) \neq 0 \ \forall P \in U\} = \bigcap_{p \in U} O_{V,p} \tag{2.16}$$

Definition 2.19. (Morphism) Let $X, Y \subset \mathbb{P}^n$ be two projective varieties and $\phi : X \to Y$ a map between X and Y. ϕ is a *morphism* if

- ϕ is continuous.
- $\forall U \subset Y$ an open subset of Y, and $\forall f \in O_Y(U)$,

$$f \circ \phi \in O_X(\phi^{-1}(U)) \tag{2.17}$$

Remark 2.6. $\phi^{-1}(U)$ is the subset of X whose points have an image in U by ϕ, and $O_X(\phi^{-1}(U))$ is the set of all rational functions of X regular on $\phi^{-1}(U)$. Hence a continuous map ϕ is a morphism if, for all rational function of Y regular on U$\subset Y$, its composition with ϕ is a rational function of X regular on the subset of X whose points have an image in U by ϕ.

[11] and therefore in our case, between projective Hilbert space.
[12] A function is analytic on X if it is complex differentiable on X

2.3.2 Segre embedding

Definition 2.20. (Segre embedding) Let $U_1, ..., U_n$ be n vector spaces without the origin and $P(U_1, ..., U_n)$ their associated projective vector space. The *Segre embedding* is an injective morphism defined by

$$\sum_{(\dim(U_1),...,\dim(U_j))}^{j} : \mathrm{P}(U_1) \times ... \times \mathrm{P}(U_j) \to \mathrm{P}(U_1 \otimes ... \otimes U_j) \tag{2.18}$$

$$([u_1], ..., [u_j]) \to [u_1 \otimes ... \otimes u_j].$$

Remark 2.7. In the following, we will consider the following definition of the Segre embedding in the case of an n-qubit system, that is by considering n projective spaces $P(\mathbb{C}^2_\star)$.

$$\sum_{(\dim(\mathrm{P}(\mathbb{C}^2_\star)),...,\dim(\mathrm{P}(\mathbb{C}^2_\star)))}^{n} = \sum_{(1,...,1)}^{n} : \mathrm{P}(\mathbb{C}^2_\star) \times ... \times \mathrm{P}(\mathbb{C}^2_\star) \to \mathrm{P}((\mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2)_\star). \tag{2.19}$$

For clarity purpose, we will write $\mathrm{P}(\mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2)$ instead of $\mathrm{P}((\mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2)_\star)$ in the following.

Definition 2.21. (Segre variety) The image of this morphism is called the *Segre variety* and denoted

$$X = \sum_{(\dim(U_1),...,\dim(U_j))}^{j} (\mathrm{P}(U_1) \times ... \times \mathrm{P}(U_j)).$$

Example 2.4. Consider the case of a bipartite qubit system, the Segre variety is the set of all elements $[\psi] \in \mathbb{C}P^3 = P(\mathbb{C}^2 \otimes \mathbb{C}^2)$ whose representative $|\psi\rangle$ can be written as a tensorial product of $|\phi\rangle, |\chi\rangle \in \mathbb{C}^2$, that is $|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$. The Segre morphism is defined here by

$$\sum_{(1,1)}^{2} : \mathrm{P}(\mathbb{C}^2_\star) \times \mathrm{P}(\mathbb{C}^2_\star) \to \mathrm{P}(\mathbb{C}^2 \otimes \mathbb{C}^2) \tag{2.20}$$

or with an equivalent notation

$$\sum_{(1,1)}^{2} : \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^3. \tag{2.21}$$

Remark 2.8. An interesting interpretation of this morphism is the following : let us consider an n-qubit system, the Segre variety is

$$X = \sum_{(1,...,1)}^{n} (\mathrm{P}(\mathbb{C}^2_\star) \times ... \times \mathrm{P}(\mathbb{C}^2_\star)),$$

that is, the set of all elements of the projective Hilbert space whose representative is a separable state that represents such a system. In other words, the Segre variety is the set of all rays of the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2$ whose representative is separable.

2.4 Tangent and secant varieties

Here one will define the secant and tangent varieties. Their interest is to separate the projective Hilbert space into different subspaces. These secant and tangent varieties being SLOCC invariant, this will allow defining a finite number of families and subfamilies of SLOCC classes. This latter remark will be developed in chapter 3.
For a more detailed overview of this topic with a physical point of view, the reader is advised to look at [34, 35, 36, 37]. Let $\mathcal{H}$ be a Hilbert space of dimension N, and $\mathbb{P} = P(\mathcal{H})$ its associated projective space of dimension N-1, the projective Hilbert space. If $|\psi\rangle \in \mathcal{H}$ is nonzero, then its projectivization in $\mathbb{P}$ is denoted $[\psi]$. $|\psi\rangle$ is the representative of $[|\psi\rangle]$.

2.4.1 Secant varieties

Let X$\subset \mathbb{P}$ be a projective variety.

Definition 2.22. (Affine cone) The *affine cone* over X is the set $\widehat{X}$ defined by

$$\widehat{X} = \{(a_0, ..., a_n) \in k^{n+1} \mid [a_0, ..., a_n] \in X\} \cup \{0\}. \tag{2.22}$$

Remark 2.9. The affine cone over X is the set of all elements of k^{n+1} whose line in which they are contained is in X. Since there are only nonzero elements, one adds the origin. For instance, the affine cone of the Segre variety, that is $\widehat{X} \subset \mathcal{H}$ where X the Segre variety, is the set of all separable states of the Hilbert space.

In our quantum language, the affine cone over a set X of elements from the projective Hilbert space is the set of all states from the Hilbert space whose rays in which they are contained is in X. This leads to the following fact : if[13] a basis of $\mathcal{H}$ is in $\widehat{X}$, then any state $|\psi\rangle$ of $\mathcal{H}$ can be written as a linear combination of elements $\{|\chi_i\rangle\}_{i\in J\subset\mathbb{N}}$ of $\widehat{X}$. This allows us to define the notion of rank of a nonzero state from $\mathcal{H}$ with respect to X.

Definition 2.23. (Rank) The *rank* of $|\psi\rangle \in \mathcal{H}$, denoted $\mathrm{rk}_X[\psi]$ or simply $\mathrm{rk}[\psi]$, is defined by

$$\mathrm{rk}_X[\psi] = \mathrm{rk}[\psi] = \min\{k \in \mathbb{R} \mid |\psi\rangle = |\chi_1\rangle + ... + |\chi_k\rangle, \text{ with } [\chi_i] \in X \ \forall i \in \{1, ..., k\}\}. \tag{2.23}$$

Definition 2.24. (Rank subset) The sets

$$X_k = \{[\psi] \in \mathbb{P} \mid \mathrm{rk}[\psi] = k\} \tag{2.24}$$

are the *rank subsets* of $\mathbb{P}$ with respect to X. Naturally, $X_k = \emptyset \ \forall k > N$.

[13] Assumed in what follows.

Remark 2.10. The rank subset X_k is the set of elements of $\mathbb{P}$ whose representative can be written as a linear combination of at least k elements of $\widehat{X}$.

Example 2.5. Let $\widehat{X}$ be the set of all separable states of $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The state $[|000\rangle + |111\rangle] \in P(\mathcal{H})$ is a state of rank 2, hence $[|000\rangle + |111\rangle] \in X_2$.

Theorem 2.7.

- $X_1 = X$.
- $\exists k_m \in \{1, ..., N\}$ *such that* $X_{k_m} \neq \emptyset$ *but* $X_{k_m+1} = \emptyset$.
- $\mathbb{P} = X_1 \cup ... \cup X_{k_m} \backslash X_1 \cap ... \cap X_{k_m} = X_1 \sqcup ... \sqcup X_{k_m}$.

Definition 2.25. (Maximum rank) k_m is the *maximum rank*.

Now we may define the notion of secant variety thanks to the definition of the rank subsets. One notes that the rank subsets are open subsets[14] of $\mathbb{P}$.

Definition 2.26. (k-secant variety) The *k-secant variety* of X is the set $\sigma_k(X)$ defined by

$$\sigma_k(X) = \overline{\bigsqcup_{s \leq k} X_s} \subset \mathbb{P}. \tag{2.25}$$

Remark 2.11. The secant varieties are closed subsets of $\mathbb{P}$. The k-secant variety is the subset of all states of $\mathbb{P}$ whose representative in $\mathcal{H}$ can be written as a linear combination of a maximum of k elements of $\widehat{X}$.

Theorem 2.8.

- $\sigma_1(X) = X_1 = X$.
- $\sigma_k(X) \subset \sigma_{k+1}(X)$.
- $\exists k_g \in \{1, ..., k_m\}$ *such that* $\sigma_{k_g}(X) = \mathbb{P}$ *and* $\sigma_{k_g-1}(X) \neq \mathbb{P}$.

Definition 2.27. (Typical rank) k_g is the *typical rank*.

Naturally one defines sets of elements of $\mathbb{P}$ whose representative can be written as a linear combination of exactly k terms of $\widehat{X}$, that is elements of $\mathbb{P}$ that belong to k-secant of X but not to (k-1)-secant of X. This is called the proper k-secant.

Definition 2.28. (Proper k-secant) The *proper k-secant* of X is the set $\Omega_k(X) = \sigma_k(X) \backslash \sigma_{k-1}(X)$.

[14] $X \subset \overline{X_k}$ and $X \nsubseteq X_k$

2.4.2 Expected dimension of secant varieties

To fully determine an element of $\sigma_k(X)$ one needs to parametrize the elements of the linear combination in $\mathcal{H}$ and its coefficient in $\mathbb{C}_\star$. The number of parameters needed will be the expected dimension of the secant variety.

Let $|\psi\rangle = \lambda_1|\psi_1\rangle + ... + \lambda_k|\psi_k\rangle$, with $\lambda_i \in \mathbb{C}_\star$ and $|\psi_i\rangle \in \widehat{X}$. The projectivization $[|\psi\rangle]$ of such a linear combination is parametrized by k×dim($\widehat{X}$)+k-1 parameters. Indeed, each $|\psi_i\rangle$ can be decomposed in a basis of $\widehat{X}$, which leads to k×dim($\widehat{X}$) parameters. Second, there are k coefficients in $\mathbb{C}_\star$ which also parametrize such a linear combination. One observes that a coefficient can be eliminated by factorization : $[|\psi\rangle] = [\lambda_1|\psi_1\rangle + ... + \lambda_k|\psi_k\rangle] = [|\psi_1\rangle + ... + \frac{\lambda_k}{\lambda_1}|\psi_k\rangle]$. Hence there are k-1 parameters needed to parametrize the coefficients of such a linear combination. Therefore, a total of k×dim($\widehat{X}$)+k-1 parameters is needed to parametrize $[|\psi\rangle]$. Since there is no *a priori* relation between these k×dim(X)+k-1 parameters, one can define the expected dimension of the k-secant variety of X.

Definition 2.29. (Expected dimension of the k-secant variety of X) The *expected dimension* of $\sigma_k(X)$ is

$$\text{edim}(\sigma_k(X)) = \min\{k \times \dim(X) + k - 1, N - 1\}^{15}. \tag{2.26}$$

If edim($\sigma_k(X)$) - dim($\sigma_k(X)$) = R>0, then $\sigma_k(X)$ is called *defective* and R is the *defect*.

Remark 2.12. The expected value for the typical rank, denoted k_{eg} can be calculated. This is the smaller k such that

$$k \times \dim(X) + k - 1 \geq N - 1 \tag{2.27}$$

that is[16],

$$k_{eg} = \lceil \frac{N}{\dim(X) + 1} \rceil. \tag{2.28}$$

Indeed, the number of parameters needed to parametrize an element of $\sigma_k(X)$, that is $k\times\dim(X)+k-1$ should be equal to the number of parameters needed to parametrize an element of $\mathbb{P}$, that is N-1.

2.4.3 K-secant variety of the Segre variety

On one hand, the Segre variety is the set of all elements of the projective Hilbert space whose representative is a separable state, as seen before. On the other hand, the k-secant of a projective variety $\sigma_k(X)$ is the set of all elements $[\psi] \in \mathbb{P}$ whose representative $|\psi\rangle$ can be written as a linear combination of a maximum of k elements of X.
Hence the k-secant of the Segre variety $\sigma_k(\sum^j_{d_1,...,d_j})$ is the set of all elements of $\mathbb{P}$ whose representative $|\psi\rangle$ can be written as a linear combination of a maximum of k separable states. That is, if k>1, the k-secants of the Segre variety are the set of all elements of the projective Hilbert space whose representative is an entangled state (genuinely or not). If k=1, the k-secant of the Segre variety is

[15] If k is such that edim($\sigma_k(X)$) =N-1, then $\sigma_k(X) = \sigma_{k_g}(X)$.
[16] $\lceil x \rceil$ is x rounded up to the nearest natural. For instance, $\lceil 3, 4 \rceil = 4$.

the Segre variety itself as seen in Theorem 2.8.
The proper k-secant of the Segre variety is the set of elements of $\mathbb{P}$ whose representative can be written as a linear combination of exactly k separable states.

Let's take a look at k_{eg} from Eq. (2.28). In our case of an n-qubit system,

- N $=^{17} 2^n$,
- X $= \sum^n_{d_1,\dots,d_n} =^{18} \sum^n_{1,\dots,1}$,
- $\dim(\sum^n_{1,\dots,1}) =^{19}$ n,

therefore

$$k_{eg} = \lceil \frac{2^n}{n+1} \rceil. \tag{2.29}$$

This is an important result, because it gives the smallest k for which the k-secant is $\mathbb{P}$. The plot of k_{eg} with respect to the number of qubits is presented in Fig. 2.3.

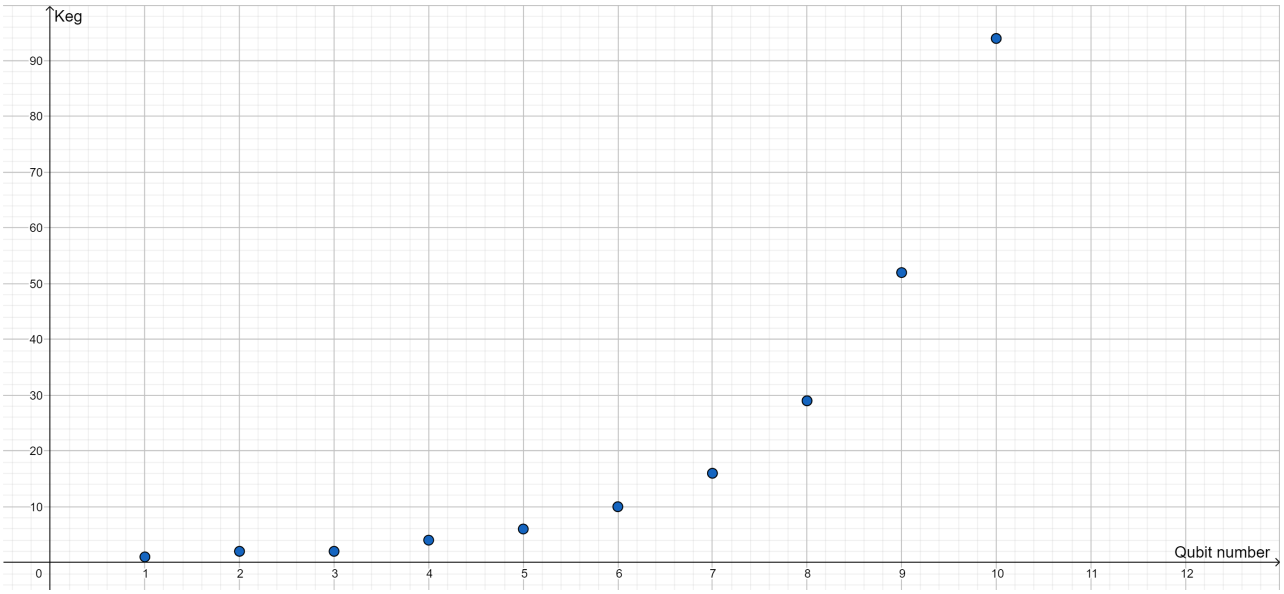

Figure 2.3: Plot of k_{eg} with respect to the number of qubits.

Remark 2.13. In the following, one will make use of some abuses of notation about k-secants and elements of the projective Hilbert space. We will say that a state of a projective Hilbert space, which is not a state but a set of states as mentioned before, is entangled if its representative is an entangled

[17] Since the states $|0\rangle$ and $|1\rangle$ are basis function of the Hilbert space of a 1-qubit system, its dimension is 2. Hence an n-qubit system is of dimension 2^n

[18] Since the dimension of the projective space $\mathbb{C}P^1$ is equal to one.

[19] Since the dimension of the Segre variety, given by Eq. (2.19), is equal to (n+1)-1=n

state. Conversely, if the representative of an element of the projective Hilbert space is separable, then this element is said to be a separable state.

2.4.4 Tangent varieties

First, we will define the notion of tangent space in the framework of algebraic geometry, to define then the notion of tangent variety.

Definition 2.30. (Tangent space) Let X be either an affine or a projective variety, $V \subset X$ a subvariety, and $p \in V$ and $m_p = \{h \in O_{V,p} \mid h(p) = 0\}$. The *tangent space at p* is

$$T_pX = \left(m_p \backslash m_p^2\right)^{\sim} \tag{2.30}$$

where $m_p^2 = \{h_1 h_2 | h_1, h_2 \in m_p\}$ and where $\sim$ denotes the dual space.

Remark 2.14.

- In other words, this definition can be rewritten

$$T_pX = \{\phi : m_p \backslash m_p^2 \to k \mid \phi \ \in \text{Lin(k)}\} = \{\phi : m_p \to k \mid \phi|_{m_p^2} = 0, \ \phi \ \in \text{Lin(k)}\}$$

  where Lin(k) denotes the set of all k-linear maps. Modding out by m_p^2 corresponds to dropping the non-linear terms from the equations defining V, therefore giving a system of linear equations that define the tangent space.

- The tangent space at $p \in X$ has a vector space structure since a dual space is a vector space.

- This definition being not practical, one can find another way to constitute the tangent space at a given point. Let $X \subset k^n$ an affine variety, $f \in k[x_1, ..., x_n]$ and $p \in X$. One defines the differential of f at p by

$$d_p f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) x_i.$$

  One can define the tangent space at $p \in X$ by

$$t_pX = Z(\{d_p f \mid f(t) = 0 \ \forall t \in X\}) \subset k^n \tag{2.31}$$

  Then, one can prove that the map

$$\delta : t_pX \to T_pX : a \to \partial_a, \tag{2.32}$$

  where

$$\partial_a : m_p \backslash m_p^2 \to k : h \to d_p h(a), \tag{2.33}$$

is an isomorphism. Thus, from a practical point of view, we will use the definition from equation (2.31) while, from a theoretical point of view, we will use definition 2.30.

- Both these two definitions have the same interpretation : the tangent space at $p \in X$ contains first-order information. That is, it is the set of all linear curves tangent to p.

Definition 2.31.

- (Smooth point) Let $X \subset k^n$ such that X=Z(f), where $f \in k[x_1, ..., x_n]$ and $p \in X$. p is called *singular* if $\frac{\partial f}{\partial x_i} = 0 \ \forall i = 1, ..., n$. Otherwise, p is *non-singular*, or *smooth*.
- (Smooth variety) A *smooth variety* is a variety whose points are all smooth.

Definition 2.32. (Tangent variety) Let X be a smooth projective variety. The tangent variety of X is

$$\tau(X) = \bigcup_{x \in X} T_x X \tag{2.34}$$

Remark 2.15. To give an illustration of this last definition, let us denote by x(t) a point of X constituting a smooth curve $Y \subset X$, that is $Y = \bigcup_{t:x(t)\in X} x(t) \subset X$. Assume that $x(0) = p \in X$. Denote x'(t) the point which cancels the differential $d_{x(t)}f$ at x(t) of any polynomial f cancelling on X. It follows that x'(t) lies in the tangent space at x(t). Then x'(0)=v denotes a point of the tangent space at x(0)=p That is, $x'(0) \in \tau(X)$. Conversely, any element $v \in \tau(X)$ can be associated with a smooth curve x(t) such that x(0)=p, x'(0)=v. Hence the tangent variety to X can be seen as the set

$$\tau(X) = \{x'(0) \mid \bigcup_{t:x(t)\in X} x(t) \subset X\}.$$

It can be computed by calculating the tangent space at each point of the variety[20].

Example 2.6. Since a set of qubit states multiple of each other is a point of the projective Hilbert space $P(\mathbb{C}^2_\star)$ it is a point x(t) from a curve $\bigcup_{t:x(t)\in X} x(t) \subset P(\mathbb{C}^2_\star)$.
Consider the case of four qubits, that is the projective Hilbert space $P(\mathbb{C}^2_\star \otimes \mathbb{C}^2_\star \otimes \mathbb{C}^2_\star \otimes \mathbb{C}^2_\star)$. One can denote $x(t) = [e_1(t) \otimes e_2(t) \otimes e_3(t) \otimes e_4(t)]$ where $e_i(t) \in \mathbb{C}^2_\star \ \forall i$. Up to a change of basis, one can assume that $e_i(0) = |0\rangle$ and since $\langle 0|1\rangle = 0$, $e_i'(0) = |1\rangle, \ \forall i$. Furthermore, x(t) can be seen as the image of a function $x : k \to X : t \to x(t)$, and therefore the tangent vector x'(t) to the point x(t) is the derivative x'(t) of the function x(t), which can be calculated by the Leibniz rule

$$\begin{aligned} x'(t) = [&e_1'(t) \otimes e_2(t) \otimes e_3(t) \otimes e_4(t) + e_1(t) \otimes e_2'(t) \otimes e_3(t) \otimes e_4(t) \\ &+ e_1(t) \otimes e_2(t) \otimes e_3'(t) \otimes e_4(t) + e_1(t) \otimes e_2(t) \otimes e_3(t) \otimes e_4'(t)] \end{aligned}$$

[20] That is why the definition requires X to be smooth.

Therefore, the tangent vector to $x(0) = [|0000\rangle] \in P(\mathbb{C}^2_\star \otimes \mathbb{C}^2_\star \otimes \mathbb{C}^2_\star \otimes \mathbb{C}^2_\star)$ is

$$x'(0) = [|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle] \in P(\mathbb{C}^2_\star \otimes \mathbb{C}^2_\star \otimes \mathbb{C}^2_\star \otimes \mathbb{C}^2_\star). \tag{2.35}$$

2.4.5 Example for a 2-qubit system

In the case of a two-qubit system, the Segre variety is $X = \sum^2_{(1,\dots,1)}(P(\mathbb{C}^2_\star) \times P(\mathbb{C}^2_\star)) = \{[|00\rangle], [|10\rangle], [|01\rangle], [|11\rangle]\}$. Furthermore, since Eq. (2.29), $k_{eg} = 2$. That means we can recover $P(\mathbb{C}^2_\star \otimes \mathbb{C}^2_\star)$ first by determining $\Omega_1(X)$ and $\Omega_2(X)$, then by determining the tangent variety of these two varieties.
On one hand, $\Omega_1(X) =$ X and $\Omega_2(X) = \{[|00\rangle + |10\rangle], [|00\rangle + |01\rangle], [|00\rangle + |11\rangle], [|10\rangle + |00\rangle], [|10\rangle + |01\rangle], [|10\rangle + |11\rangle], [|01\rangle + |00\rangle], [|01\rangle + |10\rangle], [|01\rangle + |11\rangle], [|11\rangle + |00\rangle], [|11\rangle + |10\rangle], [|11\rangle + |01\rangle]\}$ is the set of all states whose representative is a linear combination of two elements from $\widehat{X}$. On the other hand, $\tau(\Omega_1)$ can be computed easily following the same way as Example 2.6 to get $\tau_1(X)$[21] $= \{[|10\rangle + |01\rangle], [|00\rangle + |11\rangle]\}$. Details are given in the following list :

- $[|00\rangle]' = [|10\rangle + |01\rangle]$
- $[|10\rangle]' = [|00\rangle + |11\rangle]$
- $[|01\rangle]' = [|11\rangle + |00\rangle] = [|10\rangle]'$
- $[|11\rangle]' = [|01\rangle + |10\rangle] = [|00\rangle]'$

We can now compute $\tau_2(X) = \tau(\Omega_2(X))$ in the same way, which will not be detailed here. For instance, an element of $\tau_2(X)$ is $[|00\rangle + |10\rangle]' = [|00\rangle' + |10\rangle']$. It is easy to prove that any element of $\tau_2(X)$ lies in $\sigma_2(X) = \Omega_1 \cup \Omega_2$. In the end, one can check that

$$P(\mathbb{C}^2_\star \otimes \mathbb{C}^2_\star) = \Omega_1(X) \cup \Omega_2(X) \cup \tau_1(X) \cup \tau_2(X) = {}^{22}\Omega_1(X) \cup \Omega_2(X).$$

2.4.6 Example for a 3-qubit system

In the case of a three-qubit system, the Segre variety is $X = \sum^3_{(1,\dots,1)}(P(\mathbb{C}^2_\star) \times P(\mathbb{C}^2_\star) \times P(\mathbb{C}^2_\star)) = \{[|000\rangle], [|100\rangle], [|010\rangle], [|001\rangle], [|110\rangle], [|101\rangle], [|011\rangle], [|111\rangle]\}$. Furthermore, since Eq. (2.29), $k_{eg} = 2$. That means we can recover $P(\mathbb{C}^2_\star \otimes \mathbb{C}^2_\star \otimes \mathbb{C}^2_\star)$ first by determining $\Omega_1(X)$ and $\Omega_2(X)$, then by determining the tangent variety of these two varieties.
On one hand, $\Omega_1(X) =$ X and $\Omega_2(X)$ is the set of all states whose representative is a linear combination of two elements from $\widehat{X}$. For instance, $[|000\rangle + |111\rangle] \in \Omega_2(X)$.

On the other hand, $\tau(\Omega_1)$ can be computed easily following the same way as Example 2.6 to get $\tau_1(X) = \tau(\Omega_1) = \{[|100\rangle + |010\rangle + |001\rangle], [|000\rangle + |110\rangle + |101\rangle], [|110\rangle + |000\rangle + |011\rangle], [|101\rangle + |011\rangle + |000\rangle], [|010\rangle + |100\rangle + |111\rangle], [|001\rangle + |111\rangle + |100\rangle], [|111\rangle + |001\rangle + |010\rangle], [|011\rangle + |101\rangle + |110\rangle]\}$. Details are given in the following list :

[21] This notation will be used later on and allows one to see that this is the tangent variety of the proper 1-secant variety of X

[22] Since $\tau_1(X) \subset \Omega_2(X)$ and since any element of $\tau_2(X)$ lies in $\sigma_2(X)$

- $[|000\rangle]' = [|100\rangle + |010\rangle + |001\rangle]$
- $[|100\rangle]' = [|000\rangle + |110\rangle + |101\rangle]$
- $[|010\rangle]' = [|110\rangle + |000\rangle + |011\rangle]$
- $[|001\rangle]' = [|101\rangle + |011\rangle + |000\rangle]$
- $[|110\rangle]' = [|010\rangle + |100\rangle + |111\rangle]$
- $[|101\rangle]' = [|001\rangle + |111\rangle + |100\rangle]$
- $[|011\rangle]' = [|111\rangle + |001\rangle + |010\rangle]$
- $[|111\rangle]' = [|011\rangle + |101\rangle + |110\rangle]$

We can now compute $\tau_2(X) = \tau(\Omega_2(X))$ in the same way, which will not be detailed here. For instance, an element of $\tau_2(X)$ is $[|000\rangle + |100\rangle]' = [|000\rangle' + |100\rangle']$.
In the end, one can check that

$$P(\mathbb{C}^2_\star \otimes \mathbb{C}^2_\star \otimes \mathbb{C}^2_\star) = \Omega_1(X) \cup \Omega_2(X) \cup \tau_1(X) \cup \tau_2(X).$$

Remark 2.16. As one can see, $\Omega_1(X) \cup \Omega_2(X)$ only contains states whose representative is a linear combination of exactly 2 separable states. Since 2 is the greater value of k_{eg}, to recover some states such as $|100\rangle + |010\rangle + |001\rangle$, one has to consider the first-order information given by the tangent varieties of $\Omega_1(X)$ and $\Omega_2(X)$.

2.5 Mathematical background in a nutshell

As the mathematical background could seem quite dense, the reader who is not comfortable with notions of topology and linear algebra may find here a summary of what has been exposed in Chapter 2 in other words. This section can be skipped otherwise.

First, let us define briefly the projective Hilbert space, motivated by the fact that two states contain the same physical information if they are multiple of each other, and thus are identical in the point of view of quantum mechanics. The *projective Hilbert space*, denoted $\mathrm{P}(\mathcal{H})$, is the projectivization of the Hilbert space $\mathcal{H}$. For instance, in the case of an n-qubit system, $\mathcal{H} = \underbrace{\mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2}_{\text{n times}}$. The *projectivization* of $\mathcal{H}$ is a quite simple notion; take two states of $\mathcal{H}$, say $|\psi\rangle$ and $|\chi\rangle$. If they are multiple of each other, that is there exists $\lambda \in \mathbb{C}_\star = \mathbb{C}\backslash\{0\}$ such that $|\chi\rangle = \lambda|\psi\rangle$, then these two elements of $\mathcal{H}$ are represented by the same element in $\mathrm{P}(\mathcal{H})$. Proceeding so for all elements of $\mathcal{H}$, we build

P($\mathcal{H}$), whose elements are sets of states which are multiple of each other. We denote these elements of the projective Hilbert space by $[|\psi\rangle] = [\psi] = \{|\chi\rangle \in \mathcal{H} \mid \exists \lambda \in \mathbb{C}_\star \ : \ |\chi\rangle = \lambda|\psi\rangle\}$, and $|\psi\rangle$ is said to be the *representative* of $[\psi]$. Therefore, elements of the projective Hilbert space are not states, but sets of states. We will make an abuse of notation in the following; a "state" of the projective Hilbert space will denote an element of this projective space. An *(algebraic) projective variety* is a subset of a projective space which could not be written as the union of two other subsets. A projective *subvariety* is a subset of a projective variety.

Second, we will define some projective varieties. Note that any state can be written as a linear combination of separable states of the Hilbert space, and we will make use of this fact. The *Segre variety* X (see Remark 2.8 for notation) can be interpreted as the set of all elements of the projective space whose representative is a separable state. In the following we will make an abuse of notation; such elements of the projective Hilbert space will be denoted "separable state" of the projective Hilbert space. The *k-secant of the Segre variety*, denoted $\sigma_k(X)$, is the set of elements of the projective Hilbert space whose representative can be written as a linear combination of at most k separable states. The *proper k-secant*, denoted $\Omega_k(X)$, is the set of elements of the projective Hilbert space whose representative can be written as a linear combination of precisely k separable states. The values of k are bounded; the *maximum value of k* is[23] $k_{eg} = \lceil \frac{2^n}{n+1} \rceil$, where n is the number of qubits. For instance, if n=3, $k_{eg} = \lceil \frac{2^3}{3+1} \rceil = \lceil \frac{8}{4} \rceil = 2$. A plot of k_{eg} is presented in Fig. 2.3 .The *tangent variety* to a proper k-secant, denoted $\tau(\Omega_k(X)) \equiv \tau_k(X)$, is the set of all tangent points to all points of this proper k-secant. For instance, the tangent variety to a k-secant of the Segre variety in the case of a 2-qubit and a 3-qubit system has been calculated in Examples 2.4.5 and 2.4.6.

[23] $\lceil \ \rceil$ means that we take the rounded up value

3 Fine-structure classification of multiqubit entanglement by algebraic geometry

This chapter is devoted to present the main result of [6], which is the classification of n-qubit system states. The organisation of this chapter is the following. First, we will roughly define a tool from multilinear algebra, the l-multilinear ranks. We will work out some examples to make the reader comfortable with this definition and we will finally show it to be SLOCC invariant. In addition to that, since proper k-secant are also SLOCC invariant (see below), we will establish a SLOCC invariant filtering of states. This will lead to a splitting of the projective Hilbert space and to a congregation of equivalent SLOCC classes. We will then work out some examples, which is the classification of two, three and four qubit system states.

3.1 l-multinear rank

Let $\mathcal{H}_n = \bigotimes_{i=1}^{n} \mathbb{C}^2$ be the Hilbert space whose elements are states representing n-qubit systems. Such states can be written

$$|\psi\rangle = \sum_{\alpha=1}^{n} \sum_{i_\alpha=0}^{1} a_{i_1,...,i_n} |i_1\rangle \otimes ... \otimes |i_n\rangle. \tag{3.1}$$

In the following, we will use the notation $|i_1...i_n\rangle = |i_1\rangle \otimes ... \otimes |i_n\rangle$. Let define an l-tuple $I = \{r_1, ..., r_l\} \subset \mathbb{R}^l$ with $l \in \{1, ..., n-1\}$ and $r_i \in \{1, ..., n\}$ $\forall i \in \{1, ..., l\}$. Let also define an (n-l)-tuple $I' = \{r_{l+1}, ..., r_n\} \subset \mathbb{R}^{n-l}$ with $r_i \in \{1, ..., n\}$ $\forall i \in \{l+1, ..., n\}$. We suppose these two sets to be ordered, that is $r_i < r_j$ $\forall i < j$. It follows from this definition that $I \cup I' = \{1, ..., n\}$. Therefore, one can write

$$\mathcal{H}_n = \mathcal{H}_I \otimes \mathcal{H}_{I'}$$

where we defined $\mathcal{H}_I = \bigotimes_{\alpha=r_1}^{r_l} \mathbb{C}^2 = \bigotimes_{I} \mathbb{C}^2$ and $\mathcal{H}_{I'} = \bigotimes_{I'} \mathbb{C}^2$.

Definition 3.1. (Matricization of a state) Let $I = \{r_1, ..., r_l\}$ be a l-tuple defined as above. Without loss of generality, consider $r_i = i \ \forall i \in \{1, ..., l\}$. *A matricization of a state with respect to I* is an application

$$M_I : \mathcal{H}_n \to \mathbb{R}^{2l \times m}$$

$$: |\psi\rangle \to \begin{pmatrix} a_{0,...,0} & a_{\underbrace{\mathbf{0,...,0}}_{l \text{ times}},1,0,...,0} & a_{\mathbf{0,...,0},0,1,0,...,0} & \cdots & a_{\mathbf{0,...,0},1,1,...,1} \\ a_{\mathbf{1,0,...,0},0,...,0} & a_{\mathbf{1,0,...,0},1,0,...,0} & a_{\mathbf{1,0,...,0},0,1,0,...,0} & \cdots & a_{\mathbf{1,0,...,0},1,1,...,1} \\ a_{\mathbf{0,1,0,...,0},0,...,0} & a_{\mathbf{0,1,0,...,0},1,0,...,0} & a_{\mathbf{0,1,0,...,0},0,1,0,...,0} & \cdots & a_{\mathbf{0,1,0,...,0},1,1,...,1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{\mathbf{1,...,1},0,...,0} & a_{\mathbf{1,...,1},1,0,...,0} & a_{\mathbf{1,...,1},0,1,0,...,0} & \cdots & a_{\mathbf{1,...,1},1,1,...,1} \end{pmatrix} \tag{3.2}$$

where m depends on n and l in the following way[1] :

n	l	m
2	1	2
3	1	4
3	2	2
4	1	4
4	2	8

Remark 3.1. 1. There exists $\binom{l}{n} = \frac{n!}{l!(n-l)!}$ different matricizations with respect to an l-tuple I for a state $|\psi\rangle \in \mathcal{H}_n$. **2.** In the following, we will only deal with l equals to 1 or 2, thus other values of l in the following examples and in the previous definition are not taken into account.

Example 3.1. For the case of 2 qubits (n=2), there is only one possible value for l, which is l=1. Therefore, there are two different matricizations of a state $|\psi\rangle \in \mathcal{H}_2$, first with respect to $I_1 = \{1\}$, second with respect to $I_2 = \{2\}$. Respectively, they write

$$M_{I_1}(\psi) = \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix}$$

and

$$M_{I_2}(\psi) = \begin{pmatrix} a_{0,0} & a_{1,0} \\ a_{0,1} & a_{1,1} \end{pmatrix}.$$

Example 3.2. If n=3, there are two possible values for l, l=1 and l=2.
If l=1, there are 3 different matricizations of a state $|\psi\rangle \in \mathcal{H}_3$ with respect to $I_1 = \{1\}$, $I_2 = \{2\}$ and

[1]Here are presented values of m for the case of 2, 3 and 4 qubit system state matricizations since we will only deal with those. There is no generic formula.

$I_3 = \{3\}$. They write

$$M_{I_1}(\psi) = \begin{pmatrix} a_{0,0,0} & a_{0,1,0} & a_{0,0,1} & a_{0,1,1} \\ a_{1,0,0} & a_{1,1,0} & a_{1,0,1} & a_{1,1,1} \end{pmatrix}$$
$$M_{I_2}(\psi) = \begin{pmatrix} a_{0,0,0} & a_{1,0,0} & a_{0,0,1} & a_{1,0,1} \\ a_{0,1,0} & a_{1,1,0} & a_{0,1,1} & a_{1,1,1} \end{pmatrix}$$
$$M_{I_3}(\psi) = \begin{pmatrix} a_{0,0,0} & a_{1,0,0} & a_{0,1,0} & a_{1,1,0} \\ a_{0,0,1} & a_{1,0,1} & a_{0,1,1} & a_{1,1,1} \end{pmatrix}.$$

If l=2, there are 3 different matricizations of $|\psi\rangle$ with respect to $I_1 = \{1,2\}$, $I_2 = \{1,3\}$ and $I_3 = \{2,3\}$. They write

$$M_{I_1}(\psi) = \begin{pmatrix} a_{0,0,0} & a_{0,0,1} \\ a_{1,0,0} & a_{1,0,1} \\ a_{0,1,0} & a_{0,1,1} \\ a_{1,1,0} & a_{1,1,1} \end{pmatrix}$$
$$M_{I_2}(\psi) = \begin{pmatrix} a_{0,0,0} & a_{0,1,0} \\ a_{1,0,0} & a_{1,1,0} \\ a_{0,0,1} & a_{0,1,1} \\ a_{1,0,1} & a_{1,1,1} \end{pmatrix}$$
$$M_{I_3}(\psi) = \begin{pmatrix} a_{0,0,0} & a_{1,0,0} \\ a_{0,1,0} & a_{1,1,0} \\ a_{0,0,1} & a_{1,0,1} \\ a_{0,1,1} & a_{1,1,1} \end{pmatrix}.$$

Example 3.3. If n=4, there are three possible values for l, l=1, l=2 and l=3. Matricizations of the l=3 case being equivalent, up to a transposition, to matricizations of the l=1 case, it will not be developed here.
If l=1, there are 4 different matricizations of a state $|\psi\rangle \in \mathcal{H}_4$ with respect to $I_1 = \{1\}$, $I_2 = \{2\}$, $I_3 = \{3\}$ and $I_4 = \{4\}$. They write

$$M_{I_1}(\psi) = \begin{pmatrix} a_{0,0,0,0} & a_{0,1,0,0} & a_{0,0,1,0} & a_{0,0,0,1} & a_{0,1,1,0} & a_{0,1,0,1} & a_{0,0,1,1} & a_{0,1,1,1} \\ a_{1,0,0,0} & a_{1,1,0,0} & a_{1,0,1,0} & a_{1,0,0,1} & a_{1,1,1,0} & a_{1,1,0,1} & a_{1,0,1,1} & a_{1,1,1,1} \end{pmatrix}$$
$$M_{I_2}(\psi) = \begin{pmatrix} a_{0,0,0,0} & a_{1,0,0,0} & a_{0,0,1,0} & a_{0,0,0,1} & a_{1,0,1,0} & a_{1,0,0,1} & a_{0,0,1,1} & a_{1,0,1,1} \\ a_{0,1,0,0} & a_{1,1,0,0} & a_{0,1,1,0} & a_{0,1,0,1} & a_{1,1,1,0} & a_{1,1,0,1} & a_{0,1,1,1} & a_{1,1,1,1} \end{pmatrix}$$
$$M_{I_3}(\psi) = \begin{pmatrix} a_{0,0,0,0} & a_{1,0,0,0} & a_{0,1,0,0} & a_{0,0,0,1} & a_{1,1,0,0} & a_{1,0,0,1} & a_{0,1,0,1} & a_{1,1,0,1} \\ a_{0,0,1,0} & a_{1,0,1,0} & a_{0,1,1,0} & a_{0,0,1,1} & a_{1,1,1,0} & a_{1,0,1,1} & a_{0,1,1,1} & a_{1,1,1,1} \end{pmatrix}$$
$$M_{I_4}(\psi) = \begin{pmatrix} a_{0,0,0,0} & a_{1,0,0,0} & a_{0,1,0,0} & a_{0,0,1,0} & a_{1,1,0,0} & a_{1,0,1,0} & a_{0,1,1,0} & a_{1,1,1,0} \\ a_{0,0,0,1} & a_{1,0,0,1} & a_{0,1,0,1} & a_{0,0,1,1} & a_{1,1,0,1} & a_{1,0,1,1} & a_{0,1,1,1} & a_{1,1,1,1} \end{pmatrix}.$$

If l=2, there are 6 different matricizations of $|\psi\rangle$ with respect to $I_1 = \{1,2\}$, $I_2 = \{1,3\}$, $I_3 = \{1,4\}$,

$I_4 = \{2,3\}$, $I_5 = \{2,4\}$ and $I_6 = \{3,4\}$. They write

$$M_{I_1}(\psi) = \begin{pmatrix} a_{0,0,0,0} & a_{0,0,1,0} & a_{0,0,0,1} & a_{0,0,1,1} \\ a_{1,0,0,0} & a_{1,0,1,0} & a_{1,0,0,1} & a_{1,0,1,1} \\ a_{0,1,0,0} & a_{0,1,1,0} & a_{0,1,0,1} & a_{0,1,1,1} \\ a_{1,1,0,0} & a_{1,1,1,0} & a_{1,1,0,1} & a_{1,1,1,1} \end{pmatrix}$$

$$M_{I_2}(\psi) = \begin{pmatrix} a_{0,0,0,0} & a_{0,1,0,0} & a_{0,0,0,1} & a_{0,1,0,1} \\ a_{1,0,0,0} & a_{1,1,0,0} & a_{1,0,0,1} & a_{1,1,0,1} \\ a_{0,0,1,0} & a_{0,1,1,0} & a_{0,0,1,1} & a_{0,1,1,1} \\ a_{1,0,1,0} & a_{1,1,1,0} & a_{1,0,1,1} & a_{1,1,1,1} \end{pmatrix}$$

$$M_{I_3}(\psi) = \begin{pmatrix} a_{0,0,0,0} & a_{0,1,0,0} & a_{0,0,1,0} & a_{0,1,1,0} \\ a_{1,0,0,0} & a_{1,1,0,0} & a_{1,0,1,0} & a_{1,1,1,0} \\ a_{0,0,0,1} & a_{0,1,0,1} & a_{0,0,1,1} & a_{0,1,1,1} \\ a_{1,0,0,1} & a_{1,1,0,1} & a_{1,0,1,1} & a_{1,1,1,1} \end{pmatrix}$$

$$M_{I_4}(\psi) = \begin{pmatrix} a_{0,0,0,0} & a_{1,0,0,0} & a_{0,0,0,1} & a_{1,0,0,1} \\ a_{0,1,0,0} & a_{1,1,0,0} & a_{0,1,0,1} & a_{1,1,0,1} \\ a_{0,0,1,0} & a_{1,0,1,0} & a_{0,0,1,1} & a_{1,0,1,1} \\ a_{0,1,1,0} & a_{1,1,1,0} & a_{0,1,1,1} & a_{1,1,1,1} \end{pmatrix}$$

$$M_{I_5}(\psi) = \begin{pmatrix} a_{0,0,0,0} & a_{1,0,0,0} & a_{0,0,1,0} & a_{1,0,1,0} \\ a_{0,1,0,0} & a_{1,1,0,0} & a_{0,1,1,0} & a_{1,1,1,0} \\ a_{0,0,0,1} & a_{1,0,0,1} & a_{0,0,1,1} & a_{1,0,1,1} \\ a_{0,1,0,1} & a_{1,1,0,1} & a_{0,1,1,1} & a_{1,1,1,1} \end{pmatrix}$$

$$M_{I_6}(\psi) = \begin{pmatrix} a_{0,0,0,0} & a_{1,0,0,0} & a_{0,1,0,0} & a_{1,1,0,0} \\ a_{0,0,1,0} & a_{1,0,1,0} & a_{0,1,1,0} & a_{1,1,1,0} \\ a_{0,0,0,1} & a_{1,0,0,1} & a_{0,1,0,1} & a_{1,1,0,1} \\ a_{0,0,1,1} & a_{1,0,1,1} & a_{0,1,1,1} & a_{1,1,1,1} \end{pmatrix}.$$

Definition 3.2. (Matrix rank) The *rank of a matrix*, denoted rk, is defined as the maximum number of linearly independent column vectors or the maximum number of linearly independent row vectors.

Definition 3.3. (l-multilinear rank) The *l-multilinear rank* of a state $|\psi\rangle \in \mathcal{H}_n$ is the $\binom{l}{n}$-tuple whose elements are ranks of the $\binom{l}{n}$ different matricizations of this state. It reads

$$\left(rk(M_{I_1}), ..., rk(M_{I_{\binom{l}{n}}}) \right) \tag{3.3}$$

Example 3.4. Let n=2. The 1-multilinear rank of a state $|\psi\rangle \in \mathcal{H}_2$ is $(rk(M_{I_1}), rk(M_{I_2}))$, where those matrices have been defined in Example 3.1 following Eq.3.2. One can check that there exists no state whose 1-multilinear rank is (1,2) or (2,1).
Let n=3. The 1-multilinear rank of a state $|\psi\rangle \in \mathcal{H}_3$ is $(rk(M_{I_1}), rk(M_{I_2}), rk(M_{I_3}))$. The 2-multilinear rank of $|\psi\rangle$ is $(rk(M_{I_1}), rk(M_{I_2}), rk(M_{I_3}))$. Those matrices have been defined in Example 3.2 following Eq.3.2. One can check that the 1-multilinear rank is equal to the 2-multilinear rank in this case.
Let n=4. The 1-multilinear rank of a state $|\psi\rangle \in \mathcal{H}_4$ is $(rk(M_{I_1}), rk(M_{I_2}), rk(M_{I_3}), rk(M_{I_4}))$. The 2-multilinear rank of $|\psi\rangle$ is $(rk(M_{I_1}), rk(M_{I_2}), rk(M_{I_3}), rk(M_{I_4}), rk(M_{I_5}), rk(M_{I_6}))$. Those matrices have

been defined in Example 3.3 following Eq.3.2. One can check that 2-multilinear rank is symmetric in this case.

Example 3.5. Let $|\psi\rangle = |000\rangle + |111\rangle \in \mathcal{H}_3$. The matricizations in the case of l=1 are

$$M_{I_1}(\psi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$M_{I_2}(\psi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$M_{I_3}(\psi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since rk(M_{I_1})=2, rk(M_{I_2})=2 and rk(M_{I_3})=2, the 1-multinear rank of $|\psi\rangle$ is (2,2,2).
The matricizations in the case of l=2 are

$$M_{I_1}(\psi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$M_{I_2}(\psi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$M_{I_3}(\psi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since rk(M_{I_1})=2, rk(M_{I_2})=2 and rk(M_{I_3})=2, the 2-multinear rank of $|\psi\rangle$ is (2,2,2).

Example 3.6. Let $|\psi\rangle = |100\rangle + |111\rangle \in \mathcal{H}_3$. The matricizations in the case of l=1 are

$$M_{I_1}(\psi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$
$$M_{I_2}(\psi) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$M_{I_3}(\psi) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since rk(M_{I_1})=1, rk(M_{I_2})=2 and rk(M_{I_3})=2, the 1-multinear rank of $|\psi\rangle$ is (1,2,2).

The matricizations in the case of l=2 are

$$M_{I_1}(\psi) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$M_{I_2}(\psi) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$M_{I_3}(\psi) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since rk(M_{I_1})=2, rk(M_{I_2})=2 and rk(M_{I_3})=1, the 2-multinear rank of $|\psi\rangle$ is (2,2,1).

Remark 3.2. The fact that these two states have different l-multilinear ranks is not meaningless, and this meaning will be highlighted in the following.

Remark 3.3. Following these definitions and examples, we are able to make some observations about l-multilinear ranks;

- The elements of the l-multilinear ranks of a state $|\psi\rangle$ are all equal to one for all l if and only if $|\psi\rangle$ is a separable state. Indeed, if it is a separable state, only one $a_{i_1,...,i_n}$ is different from zero, and therefore the matrix rank is equal to one for all matricizations of $|\psi\rangle$.
- A state is genuinely entangled if and only if elements of all l-multilinear ranks of this state are greater than one. Indeed, by making some calculations in the following, we will see that any genuinely entangled state follows this remark.
- The greater elements of the l-multilinear ranks of a state are, the greater entanglement of this state is. Indeed, Examples 3.5 and 3.6 show that a genuinely entangled state (Example 3.5) has elements of its 1-multilinear rank and 2-multilinear rank greater than those of a biseparable entangled state (Example 3.6).
- l-multilinear ranks of a state $|\psi\rangle \in \mathcal{H}_n$ are invariant under SLOCC. Indeed, let I be an l-tuple and I' an (n-l)-tuple as defined in the beginning of this chapter, $A_i \in \mathrm{SL}(2,\mathbb{C}^2)\ \forall i \in \{1,...,n\}$ and $|\chi\rangle = (\otimes_{i\in I} A_i)|\psi\rangle$. By definition (Eq. 1.35), these two states are SLOCC equivalent. The matricization of $|\chi\rangle$ with respect to I reads $M_I(\chi) = (\otimes_{i\in I} A_i) M_I(\psi)(\otimes_{i\in I'} A_i)$, which leaves the rank invariant, hence the conclusion.

Remark 3.4. From now on, the l-multilinear ranks of a state will be denoted by the 1-multilinear rank followed by the 2-multilinear rank. For instance, $|\psi\rangle = |000\rangle + |111\rangle$ have l-multilinear ranks (2,2,2)-(2,2,2).

One concludes this section about l-multilinear ranks by making a link with algebraic variety. On first hand, l-multilinear ranks do not depend on any phase factor or any constant multiplying the state vector. Therefore, l-multilinear ranks can be defined in the projective Hilbert space as the l-multilinear ranks of the representative of an element of the projective Hilbert space. On the other hand, statements about ranks can be rephrased as a statement about minors, that is determinants[2]. Therefore, l-multilinear ranks determine a determinantal variety (Def. 2.6) in the projective Hilbert space. Indeed, multipartite pure states which have l-multilinear ranks bounded by a given integer sequence make a subvariety of $\mathrm{P}(\mathcal{H}_n)$. Furthermore, these determinantal varieties are subvarieties of secant varieties of the Segre variety.

[2]The rank r of a matrix is the largest number such that some $r \times r$ minor does not vanish.

3.2 Main results

3.2.1 Splitting of the projective Hilbert space

First, as one can see in Section 2.4.5 and 2.4.6, any state from the n-qubit projective Hilbert space $P(\mathbb{C}^2_\star \otimes ... \otimes \mathbb{C}^2_\star)$ belongs either to a proper k-secant variety, or to a tangent variety of these latter if the number of elements in the linear combination is greater than k_{eg}. This is the first classification of states into entanglement families, and therefore the first splitting of the projective Hilbert space into a finite number of parts, that is into a finite number of projective algebraic varieties which are the proper k-secants. This finite number is k_{eg}, and therefore the number of families for an n-qubit system is plotted in Fig.2.3. For instance, $[|000\rangle + |111\rangle]$ and $[|001\rangle]$, since their representative writes as a linear combination of a different number of separable states, belong to two different proper secant families, that is two different parts of the projective Hilbert space, respectively to $\Omega_2(X)$ and $\Omega_1(X) = X$.

Second, given two states from the same proper k-secant or the same tangent variety, one can try to distinguish them by their l-multilinear ranks and then to create subfamilies whose elements belong to the same proper k-secant and have the same l-multilinear ranks. This number of subfamilies is finite if n is finite since elements of l-multilinear ranks are finite. It splits again each proper k-secant part of the projective Hilbert space into a finite number of pieces, that is into a finite number of projective determinantal varieties determined by the different l-multilinear ranks. For instance, $[|000\rangle + |111\rangle]$ and $[|100\rangle + |111\rangle]$ belongs to the same family $\Omega_2(X)$, but their l-multilinear ranks are different (See Examples 3.5 and 3.6). They respectively belongs to subfamilies (2,2,2)-(2,2,2) and (1,2,2)-(2,2,1).

3.2.2 SLOCC classification

Remember that our initial issue is the SLOCC classification, which fails to be finite for any n-qubit system. The solution to this issue is demonstrated in the appendix of [6] (sixth page). Here are presented the main results to use the splitting of the projective Hilbert space in the SLOCC classification.

- The k-secants of the Segre variety are SLOCC invariant and thus are proper k-secants. That is, if two states are SLOCC equivalent (See Eq. (1.35)), then they are in the same family. The same holds for a tangent variety of a proper k-secant variety. Indeed, it follows from the fact that if points of a given projective variety X are invariant under the action of a group, then so is any subvariety built from points of X. The converse is not true, because since SLOCC can not create entanglement, two states of the same family with a different kind of entanglement can not be in the same SLOCC class. SLOCC classes congregate naturally into proper k-secant families.
- According to Remark 3.3, the l-multilinear ranks are SLOCC invariant. That is, two states are SLOCC invariant if and only if they are in the same l-multilinear ranks subfamily. That means

that SLOCC classes in each proper k-secant family gather into l-multilinear ranks subfamilies.

This main result means that some SLOCC equivalence classes are equivalent in the point of view of proper k-secant (*i.e.* elements of some SLOCC equivalence classes belong to the same proper k-secant family.) and in the point of view of l-multilinear ranks (*i.e.* elements of some SLOCC equivalence classes belong to the same l-multilinear ranks subfamily). Since the number of families and subfamilies are both finite, the number of distinct SLOCC classes is finite, for all n. From an infinite number of SLOCC classes, some of which being equivalent in the point of view of families and subfamilies, one gets a finite number of distinct SLOCC classes. Finally, since a proper k-secant is a projective subvariety of the projective Hilbert space and l-multilinear ranks determine a subvariety of a proper k-secant, the set of all SLOCC classes equivalent to another is precisely an algebraic projective subvariety. This subvariety is the determinantal subvariety of a proper k-secant variety of the Segre variety determined by the l-multilinear rank of a state belonging to one of these classes. This makes a natural link between SLOCC classes and algebraic varieties.

Remark 3.5. As stated in [6], elements of l-multilinear ranks of a given state which belongs to a given proper k-secant are at most k.

The following examples are developed in order to give the reader an illustration of this classification. First, Example 3.7 illustrates that two SLOCC equivalent states belong to the same subfamily (that is, they belong to the same proper k-secant and they have the same l-multilinear ranks), and that two states which have different l-multilinear ranks are SLOCC distinct. Second, Example 3.8 illustrates that two SLOCC distinct states belong to different subfamilies. Finally, Example 3.9 illustrates that two states that belong to different families are SLOCC distinct.

Example 3.7. Let $[|\psi\rangle] = [|000\rangle + |011\rangle]$ and $[|\chi\rangle] = [|100\rangle + |111\rangle]$ two biseparable states (which are SLOCC equivalent states as stated in [19]). First, they both belong to the proper 2-secant of the Segre variety $\Omega_2(\sum^3_{(1,1,1)})$ thus they belong to the same family. Second, they have the same l-multilinear ranks. Indeed, the matricizations of $|\psi\rangle$ are the following. First, if l=1,

$$M_{I_1}(\psi) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$M_{I_2}(\psi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$M_{I_3}(\psi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which leads to a 1-multilinear rank (1,2,2).Then, if l=2,

$$M_{I_1}(\psi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$M_{I_2}(\psi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$M_{I_3}(\psi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix},$$

this leads to a 2-multilinear rank (2,2,1). Therefore $[|\psi\rangle]$ belong to the subfamily (1,2,2)-(2,2,1), which is the same as $[|\chi\rangle]$ (See Example 3.6).
Let us now take two states with different l-multilinear ranks (but which belong to the same proper k-secant family). For instance, consider l-multilinear ranks (1,2,2)-(2,2,1) and (2,1,2)-(2,1,2). States with those l-multilinear ranks are respectively, for instance, $[|100\rangle + |111\rangle]$ and $[|010\rangle + |111\rangle]$, which are two biseparable SLOCC distinct states, according to [19].

Example 3.8. Let $[|\psi\rangle] = [|100\rangle + |111\rangle]$ and $[|\chi\rangle] = [|010\rangle + |111\rangle]$ be two states which are SLOCC distinct according to [19]. Following the second part of Example 3.7, these two states have respectively l-multilinear ranks (1,2,2)-(2,2,1) and (2,1,2)-(2,1,2), therefore they have distinct l-multilinear ranks.

Example 3.9. Let $[|\psi\rangle] = [|100\rangle + |010\rangle + |001\rangle]$ and $[|\chi\rangle] = [|000\rangle + |111\rangle]$. $[|\psi\rangle]$ belongs to the tangent variety of the proper 2-secant variety, while $[|\chi\rangle]$ belongs to the proper 2-secant variety. Therefore they do not belong to the same family. Again according to [19], these two states belong to distinct SLOCC classes.

3.2.3 Algorithm

As one will see in examples of two, three and four qubits, there is a repetitive algorithm which allows us to sum up this finite SLOCC classification.

- First, one has to find the different distinct SLOCC classes of every proper k-secant family.
- Second, one has to determine elements of the tangent varieties that are not contained in the proper k-secant they are built on.
- Finally, one has to calculate the different possible l-multilinear ranks for each family.

3.3 Applications

Here we apply this classification to the case of two, three and four qubits. Since details about calculations have been given in the last sections, we present here the results one can find by easy, but sometimes tedious, calculations.

3.3.1 Two-qubits

Proper k-secant families

In the case of two qubits (n=2), $k_{eg} = \lceil \frac{2^2}{2+1} \rceil = 2$. Hence the highest k for which the proper k-secant of the Segre variety is non-empty is k=2. There are then two families :

- The family of the proper 1-secant of the Segre variety $\Omega_1(\sum^2_{(1,1)}) = \sum^2_{(1,1)}$ which is the set of all separable states of the projective Hilbert space.
- The family of the proper 2-secant of the Segre variety $\Omega_2(\sum^2_{(1,1)})$.

The second family can thus be interpreted as the set of all entangled states of the projective Hilbert space.

Tangent variety families

As developed in Example 2.4.5, the tangent variety families are contained in the proper k-secant families.

l-multilinear ranks subfamilies

According to Remark 3.5, the first family exclusively contains states whose 1-multilinear rank is (1,1). The second family exclusively contains states whose 1-multilinear rank is (2,2). Indeed, according to Remark 3.4, there exists no state which has a 1-multilinear rank (1,2) or (2,1). Furthermore, according to Remark 3.3, only separable states have 1-multilinear rank (1,1). Therefore, there is only one subfamily for a family, which is the family itself.

Conclusion about the fine structure

There are two distinct SLOCC classes. The first one is the set of all separable states, the second is the set of all entangled states. An example of an element from the first subfamily is $[|00\rangle]$, while an example of an element from the second family is $[|00\rangle + |11\rangle]$. This fine-structure is summarized in Fig.3.1.

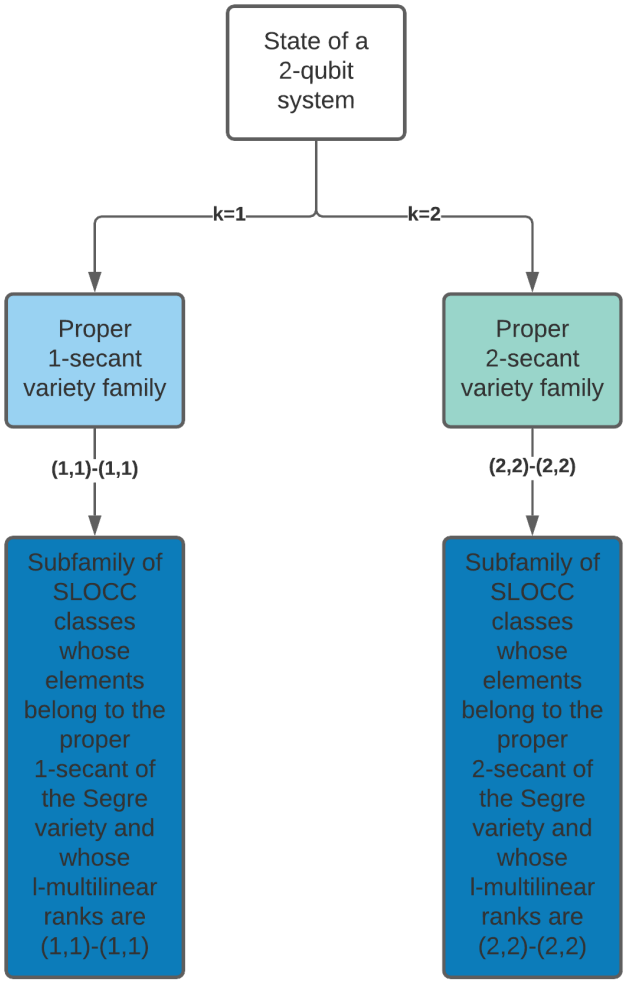

Figure 3.1: Classification diagram for 2-qubit system states. The darker the colour is, the finer the classification step is.

3.3.2 Three-qubits

Proper k-secant families

In the case of three qubits (n=3), $k_{eg} = \lceil \frac{2^3}{3+1} \rceil = 2$. Hence the highest k for which the proper k-secant of the Segre variety is non-empty is k=2. There are thus two families *a priori*:

- The family of the proper 1-secant of the Segre variety $\Omega_1(\sum^3_{(1,1,1)}) = \sum^3_{(1,1,1)}$ which is the set of all separable states of the projective Hilbert space.
- The family of the proper 2-secant of the Segre variety $\Omega_2(\sum^3_{(1,1,1)})$.

The second family can thus be interpreted as the set of all entangled states of the projective Hilbert space.

Tangent variety families

As developed in the Example 2.4.6, the tangent variety to the proper 2-secant is not contained in $\Omega_2(\sum^3_{(1,1,1)})$. This implies that there is a third family $\tau_2(\sum^3_{(1,1,1)})$ rising from the fact that $\tau_2(\sum^3_{(1,1,1)}) \backslash \Omega_2(\sum^3_{(1,1,1)})$ is non empty. The state which belong to this third family is[3] $[|W_3\rangle]$.

l-multilinear ranks subfamilies

According to Remark 3.5, the first family $\sum^3_{(1,1,1)}$ exclusively contains states whose 1-multilinear rank is (1,1,1). According to Remark 3.4, the second family $\sigma_2(\sum^3_{(1,1,1)})$ contains states whose l-multilinear ranks are (2,2,2)-(2,2,2), (1,2,2)-(1,2,2), (2,1,2)-(2,1,2) or (2,2,1)-(2,2,1). According to Remark 3.3, the third family only contains states whose l-multilinear ranks are (2,2,2)-(2,2,2) since they are genuinely entangled. Therefore, the first and the third family each constitute a subfamily. The subfamilies for the second family are the following :

- First, the set of all states whose l-multilinear ranks are (2,2,2)-(2,2,2). This is the set of all genuinely entangled states, the $[|GHZ_3\rangle]$ states.
- Second, the set of all states whose l-multilinear ranks are (1,2,2)-(1,2,2). This is the set of all biseparable states with the first party separated from the other two, the $[|B_1\rangle]$ states. An example is given by $[|100\rangle + |111\rangle]$ in Example 3.6.

[3] The lower index $_3$ means that this is the $|W\rangle$ state in the case of a 3-qubit system.

- Third, the set of all states whose l-multilinear ranks are (2,1,2)-(2,1,2). This is the set of all biseparable states with the second party separated from the other two, the $|B_2\rangle$ states. An example is given by $[|010\rangle + |111\rangle]$ in Example 3.8.
- Finally, the set of all states whose l-multilinear ranks are (2,2,1)-(2,2,1). This is the set of all biseparable states with the third party separated from the other two, the $|B_3\rangle$ states. An example is given by $[|001\rangle + |111\rangle]$ whose calculations are similar to those of Example 3.6.

Conclusion about the fine structure

There are 6 distinct SLOCC classes. The first family is one of those SLOCC classes, the second family contains 4 of those SLOCC classes, each representing a subfamily and the third is one of those SLOCC classes. This is summarized in the following table, with $|\text{Sep}\rangle$ a notation for a representing separable state. This is also summarized in Fig.3.2.

First family $\sum^3_{(1,1,1)}$	Second family $\Omega_2(\sum^3_{(1,1,1)})$	Third family $\tau_2(\sum^3_{(1,1,1)})$
$\|\text{Sep}\rangle$	$\|GHZ_3\rangle$ $\|B_i\rangle^3_{i=1}$	$\|W_3\rangle$

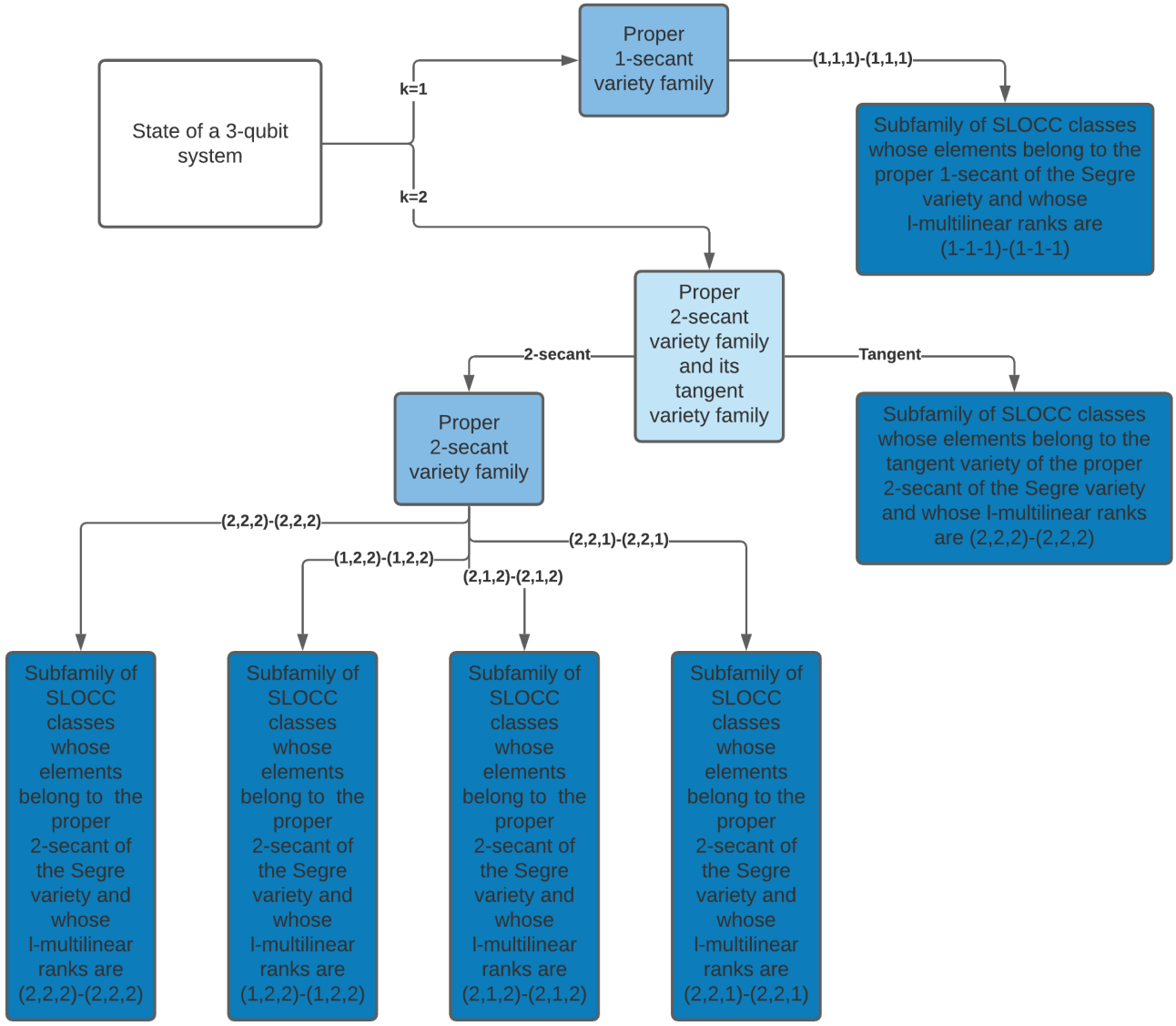

Figure 3.2: Classification diagram for 3-qubit system states. The darker the colour is, the finer the classification step is.

3.3.3 Four-qubits

Proper k-secant families

In the case of four qubits (n=4), $k_{eg} = \lceil \frac{2^4}{4+1} \rceil = 4$. Hence the highest k for which the proper k-secant of the Segre variety is non-empty is k=4. There are then four families *a priori*:

- The family of the proper 1-secant of the Segre variety $\Omega_1(\sum^4_{(1,1,1,1)}) = \sum^4_{(1,1,1,1)}$ which is the set of all separable states of the projective Hilbert space.
- The family of the proper 2-secant of the Segre variety $\Omega_2(\sum^4_{(1,1,1,1)})$.
- The family of the proper 3-secant of the Segre variety $\Omega_3(\sum^4_{(1,1,1,1)})$.
- The family of the proper 4-secant of the Segre variety $\Omega_4(\sum^4_{(1,1,1,1)})$.

The union $\Omega_2(\sum^4_{(1,1,1,1)}) \cup \Omega_3(\sum^4_{(1,1,1,1)}) \cup \Omega_4(\sum^4_{(1,1,1,1)})$ can thus be interpreted as the set of all entangled states of the projective Hilbert space.

Tangent variety families

As the calculations are a bit long and tedious, though simple, we will not prove what is exposed here. Here are only presented the results. The tangent variety of the proper 2-secant is not contained in $\Omega_2(\sum^4_{(1,1,1,1)})$. This implies that there is a fifth family $\tau_2(\sum^4_{(1,1,1,1)})$ rising from the fact that $\tau_2(\sum^4_{(1,1,1,1)}) \backslash \Omega_2(\sum^4_{(1,1,1,1)})$ is non empty.
Furthermore, the tangent variety to the proper 3-secant is not contained in $\Omega_3(\sum^4_{(1,1,1,1)})$. This implies that there is a sixth family $\tau_3(\sum^4_{(1,1,1,1)})$ rising from the fact that $\tau_3(\sum^4_{(1,1,1,1)}) \backslash \Omega_3(\sum^4_{(1,1,1,1)})$ is non empty.

l-multilinear ranks subfamilies

- According to Remark 3.3, the first family $\sum^4_{(1,1,1,1)}$ exclusively contains states whose 1-multilinear rank is (1,1,1,1). For instance, it contains the state $[|0000\rangle]$.
- The second family $\Omega_2(\sum^4_{(1,1,1,1)})$ can be split into 11 subfamilies, which are
  - The set of genuinely entangled states $[|\mathrm{GHZ}_4\rangle]$:
    - (2,2,2,2)-(2,2,2,2,2,2), which contains for instance $[|0000\rangle + |1111\rangle]$.
  - The set of biseparable GHZ states $[|\mathrm{B}_i^{\mathrm{GHZ}_3}\rangle^4_{i=1}]$:
    - (1,2,2,2)-(2,2,2,2,2,2), which contains for instance $[|1000\rangle + |1111\rangle]$.
    - (2,1,2,2)-(2,2,2,2,2,2), which contains for instance $[|0100\rangle + |1111\rangle]$.
    - (2,2,1,2)-(2,2,2,2,2,2), which contains for instance $[|0010\rangle + |1111\rangle]$.

    * (2,2,2,1)-(2,2,2,2,2,2), which contains for instance $[|0001\rangle + |1111\rangle]$.
  - The set of triseparable states $[|\mathrm{T}_i\rangle_{i=1}^{6}]$:
    * (1,1,2,2)-(1,2,2,2,2,1), which contains for instance $[|1100\rangle + |1111\rangle]$.
    * (1,2,1,2)-(2,1,2,2,1,2), which contains for instance $[|1010\rangle + |1111\rangle]$.
    * (2,1,1,2)-(2,2,1,1,2,2), which contains for instance $[|0110\rangle + |1111\rangle]$.
    * (1,2,2,1)-(2,2,1,1,2,2), which contains for instance $[|1001\rangle + |1111\rangle]$.
    * (2,1,2,1)-(2,1,2,2,1,2), which contains for instance $[|0101\rangle + |1111\rangle]$.
    * (2,2,1,1)-(1,2,2,2,2,1), which contains for instance $[|0011\rangle + |1111\rangle]$.

- The third family $\Omega_3(\sum_{(1,1,1,1)}^{4})$ can be split into 4 subfamilies.
    * (2,2,2,2)-(3,3,3,3,3,3), which contains for instance $[|1000\rangle + |0110\rangle + |1101\rangle]$.
    * (2,2,2,2)-(2,3,3,3,3,2), which contains for instance $[|0000\rangle + |0011\rangle + |1111\rangle]$.
    * (2,2,2,2)-(3,2,3,3,2,3), which contains for instance $[|0000\rangle + |1010\rangle + |1111\rangle]$.
    * (2,2,2,2)-(3,3,2,2,3,3), which contains for instance $[|0000\rangle + |1001\rangle + |1111\rangle]$.

- The fourth family $\Omega_4(\sum_{(1,1,1,1)}^{4})$ can be split into 10 subfamilies.
    * (2,2,2,2)-(2,4,4,4,4,2), which contains for instance $[|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle]$.
    * (2,2,2,2)-(4,2,4,4,2,4), which contains for instance $[|0000\rangle + |0101\rangle + |1010\rangle - |1111\rangle]$.
    * (2,2,2,2)-(4,4,2,2,4,4), which contains for instance $[|0000\rangle + |1001\rangle + |0110\rangle - |1111\rangle]$.
    * (2,2,2,2)-(3,4,4,4,4,3).
    * (2,2,2,2)-(4,3,4,4,3,4).
    * (2,2,2,2)-(4,4,3,3,4,4).
    * (2,2,2,2)-(4,4,4,4,4,4).

  - The set of biseparable states $[|\mathrm{Bell}\rangle \otimes |\mathrm{Bell}\rangle] \equiv [|\mathrm{BB}_i\rangle_{i=1}^{3}]$:
    * (2,2,2,2)-(1,3,3,3,3,1), which contains for instance $[|0000\rangle + |1100\rangle + |0011\rangle + |1111\rangle]$.
    * (2,2,2,2)-(3,1,3,3,1,3), which contains for instance $[|0000\rangle + |1010\rangle + |0101\rangle + |1111\rangle]$.
    * (2,2,2,2)-(3,3,1,1,3,3), which contains for instance $[|0000\rangle + |1001\rangle + |0110\rangle + |1111\rangle]$.

- The fifth family $\tau_2(\sum_{(1,1,1,1)}^{4})$ can be split into 5 subfamilies.
  - The set of genuinely entangled states $[|\mathrm{W}_4\rangle]$:
    * (2,2,2,2)-(2,2,2,2,2,2), which contains for instance $[|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle]$.
  - The set of biseparable states $[|\mathrm{B}_i^{\mathrm{W}_3}\rangle_{i=1}^{4}]$:
    * (1,2,2,2)-(2,2,2,2,2,2), which contains for instance $[|0100\rangle + |0010\rangle + |0001\rangle]$.
    * (2,1,2,2)-(2,2,2,2,2,2), which contains for instance $[|1000\rangle + |0010\rangle + |0001\rangle]$.
    * (2,2,1,2)-(2,2,2,2,2,2), which contains for instance $[|1000\rangle + |0100\rangle + |0001\rangle]$.
    * (2,2,2,1)-(2,2,2,2,2,2), which contains for instance $[|1000\rangle + |0100\rangle + |0010\rangle]$.

- The sixth family $\tau_3(\sum^4_{(1,1,1,1)})$ can be split into 4 subfamilies.
    * Tangent elements to states whose l-multilinear ranks are (2,2,2,2)-(3,3,3,3,3,3).
    * Tangent elements to states whose l-multilinear ranks are (2,2,2,2)-(2,3,3,3,3,2).
    * Tangent elements to states whose l-multilinear ranks are (2,2,2,2)-(3,2,3,3,2,3).
    * Tangent elements to states whose l-multilinear ranks are (2,2,2,2)-(3,3,2,2,3,3).

Conclusion about the fine structure

There are 35 distinct SLOCC classes distributed into 6 families. The first family $\sum^4_{(1,1,1,1)}$ is a subfamily itself, the second family $\Omega_2(\sum^4_{(1,1,1,1)})$ can be split into 11 subfamilies, The third family $\Omega_3(\sum^4_{(1,1,1,1)})$ can be split into 4 subfamilies, The fourth family $\Omega_4(\sum^4_{(1,1,1,1)})$ can be split into 10 subfamilies, The fifth family $\tau_2(\sum^4_{(1,1,1,1)})$ can be split into 5 subfamilies and the sixth family $\tau_3(\sum^4_{(1,1,1,1)})$ can be split into 4 subfamilies. This is summarized in Fig.3.3. The reader is advised to look at the caption of this figure.

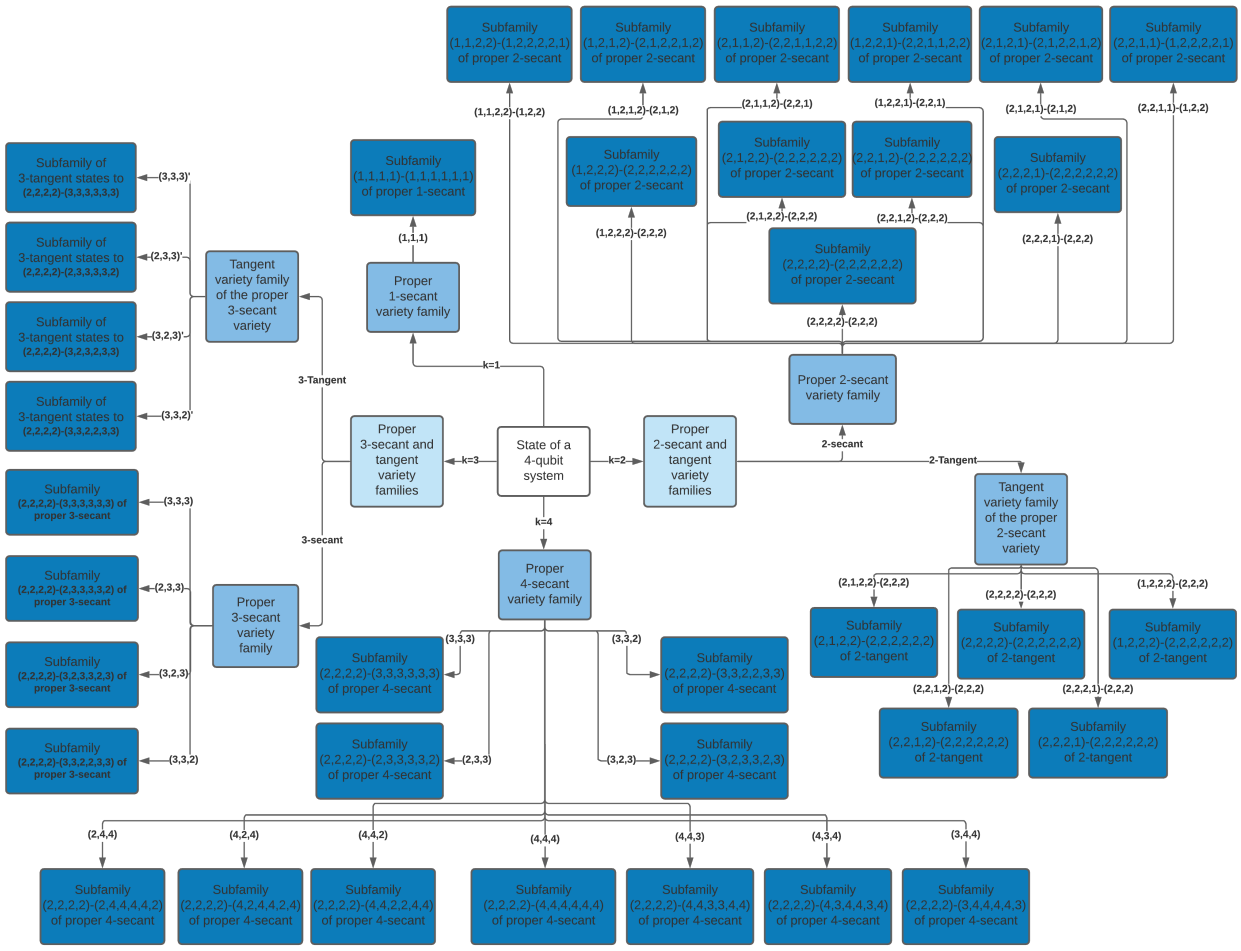

Figure 3.3: Classification diagram for 4-qubit system states. The darker the colour is, the finer the classification step is. The name of the subfamilies is different from the other diagrams because of the lack of space. We denote "3-tangent" the tangent states to some elements of the proper 3-secant variety. Sometimes, we use a simplified notation for the l-multilinear ranks; if there is no source of confusion, we use the fact that the 2-multilinear rank is symmetric in the n=4 case. For instance, (2,2,2,2)-(2,3,3,3,3,2) is simply denoted (2,3,3) if there is no confusion about the 1-multilinear rank. Finally, (2,3,3)' denotes the symmetric notation of 2-multilinear rank associated to the tangent to states whose symmetric notation of l-multilinear rank is (2,3,3).

Conclusion

The main purpose of this thesis was to shed light on the mathematical and physical frame in which the article [6] has been developed. In order to understand the context of this article and to grasp its role, the notions of entanglement, separability, entanglement measure and classification have been addressed. Then, a study of the mathematics of entanglement from the point of view of algebraic geometry has been performed. Finally, a great result emerged from these mathematical notions in this physical context, allowing splitting of the projective Hilbert space finite parts, leading to a finite entanglement classification by associating them with SLOCC classes.

In chapter one, we explored the notion of entanglement in two different ways. First, we looked for a physical insight behind this mysterious notion from quantum mechanics. This has been achieved by asking ourselves how particular is the correlation between two entangled photons, and then, by analysing a paradox (EPR paradox) and its solution (both Alain Aspect's experiment and Bell's inequality). Second, we defined quantum entanglement formally and mathematically. We defined entanglement in the case of pure states, then we generalized this notion with the density operator, again first in the pure state case, and then in the mixed state case. We completed this notion with the introduction of LOCC and the fact that these LOCC can not create entanglement. Then, we wanted to know if a state is entangled or not, and if so, how much is it entangled. These two questions are respectively the separability problem and the entanglement measure. For the separability problem, we developed a mathematical background before introducing two complementary criteria, the PPT criterion and the CCNR criterion. After that, we defined the general notion of entanglement measure before introducing two examples, which are concurrence and negativity. Finally, we set the foundations of entanglement classification, first by defining mathematically LOCC and SLOCC families, and then by summarizing known results about the classification of these families in the case of two and three qubits. Since the classification of SLOCC families in the general case of n qubits fails to be finite, we highlighted the role of algebraic geometry in this classification issue.

In chapter two, we detailed and defined notions from algebraic geometry that are used in the main result of [6]. First, we defined a key concept, affine algebraic varieties, which are irreducible zero sets of a set of polynomials. Then, we defined the projective space and projective algebraic varieties in the same way we did in the affine case. We also introduced examples of projective space, such as projective Hilbert space, which is a cornerstone in our study of entanglement classification. After that, we defined morphisms to define the Segre embedding, whose image, the Segre variety, is the set of all separable states. Then, proper k-secant varieties of the Segre variety and their tangent variety were defined in such a way that every element of the projective Hilbert space is contained in one of those proper k-secant or tangent varieties, hence the projective Hilbert space can be split into parts.

In chapter three, we developed a SLOCC fine-structure classification of n-qubit states thanks to these tools from algebraic geometry and multilinear algebra. We started this chapter with the notion of l-multilinear ranks defined in the general framework of the matricization of a given n-qubit state

as an order-n tensor in multilinear algebra. Given the finite expected dimension of proper k-secant varieties, and given the finite number of distinct possibilities of the l-multilinear ranks, these tools leads to a finite classification of elements of projective Hilbert space. Since the proper k-secant and the l-multilinear ranks are SLOCC invariant, it leads to the classification of distinct SLOCC classes by identifying a SLOCC class to a determinantal subvariety of a k-secant variety of the Segre variety. This last step solves our initial issue, which was the infinite number of SLOCC classes for n-qubit systems, if n>4. Some applications were given to see how this classification works practically, first in the trivial case of two-qubit systems, then in the case of three-qubit systems, and finally in the case of four-qubit systems.

An example of perspective that could be achieved from this result is the construction of a new entanglement witness for detecting entanglement in a multipartite mixed state system.

A

Appendix

Algebraic reminder

Here are presented algebraic notions used in this thesis. For a more detailed overview, the reader is invited to look at [38].

Algebraic structures

Definition A.1. (Group) A *group* G is a monoïde with the additional axiom

- There exists an additive inverse b$\in G$ of a$\in G$, that is a+b=0 $\forall a \in G$. b is denoted -a.

G is said to be *abelian* if + is commutative.

Definition A.2. (Ring) A *ring* R is a set of elements with binary internal operations addition + and multiplication . satisfying the ring axioms, which are

- + is commutative and associative.
- There exists an additive identity in R denoted 0, that is a+0=a $\forall a \in R$.
- There exists an additive inverse b$\in R$ of a$\in R$, that is a+b=0 $\forall a \in R$. b is denoted -a.
- . is associative.
- There exists a multiplicative identity in R denoted 1, that is a.1=a $\forall a \in R$.
- multiplication is distributive with respect to addition.

Furthermore, if . is commutative, then R is said to be *an abelian or commutative ring.*

Definition A.3. (Field) A *field* k is an abelian ring whose nonzero elements are invertible. That is, + and . satisfy

- + is commutative and associative.
- There exists an additive identity in k denoted 0, that is a+0=a $\forall a \in k$.
- There exists an additive inverse b$\in k$ of a$\in k$, that is a+b=0 $\forall a \in k$. b is denoted -a.
- . is associative and commutative.
- There exists a multiplicative identity in k denoted 1, that is a.1=a $\forall a \in k$.
- There exists a multiplicative inverse in k denoted a^{-1}, that is a.a^{-1}=1 $\forall a \in k$.
- Multiplication is distributive with respect to addition.

Definition A.4. (Vector space) Let k be a field. A *vector space* V over k is a set of elements called vectors together with two internal operations, addition + : V×V → V : (v,w) → v+w and multiplication by a scalar .λ : V → V : v → $\lambda.v$, which satisfy the following axioms.

- Associativity and commutativity of addition.
- There exists an additive identity of v in V denoted 0, that is v+0=v $\forall v \in V$.
- There exists an additive inverse of v in V denoted -v, that is v+(-v)=0 $\forall v \in V$.
- Compatibility of scalar multiplication with field multiplication : $\forall a, b \in$F, $\forall v \in$V, a(b**v**) = (ab)**v**.
- There exists a multiplicative identity in k denoted 1, that is v.1=v $\forall v \in V$.
- Distributivity of scalar multiplication with respect to vector addition.
- Distributivity of vector addition with respect to scalar multiplication.

Definition A.5. (Basis of a vector space) A basis $\mathcal{B}$ of a vector space V over k is a linearly independent subset of V spanning V. That is, a subset $\mathcal{B} \subset V$ is a basis if

- $\forall v \in V$ $\exists a_1, ..., a_n \in k$ and $\exists v_1, ..., v_n \in \mathcal{B}$ such that $v = a_1v_1 + ... + a_nv_n$. (spanning property)
- For all subset $v_1, ..., v_m$ of $\mathcal{B}$, $\forall c_1, ..., c_n \in k$, $c_1v_1 + ... + c_mv_m = 0 \implies c_1 = ... = c_n = 0$. (linear Independence property)

If the vector space contains at least one basis with a finite number of elements, then the vector space is called *finite dimensional.* Otherwise, it is called *infinite dimensional.*

Definition A.6. (Dimension of a vector space) The *dimension of a vector space* is the number of vectors of a basis of V over its base field.

Definition A.7. (Affine space) Let V be a vector space over a field k. An *affine space* $\mathbb{A}$ is a non empty set of elements called *points* together with a vector space V and a map $\mathbb{A} \times \mathbb{A} \to$V : $(A, B) \to \overrightarrow{AB}$ such that

- $\forall A, B, C \in \mathbb{A},\ \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$
- $\forall A \in \mathbb{A}$, and $\forall v \in$V, there exists a unique $B \in \mathbb{A}$ such that v=$\overrightarrow{AB}$

Remark A.1. $\mathbb{A}^{\times n}$ is the notation for the cartesian product of n affine spaces $\underbrace{\mathbb{A} \times ... \times \mathbb{A}}_{\text{n times}}$

Definition A.8. (Dimension of an affine space) The *dimension* of an affine space is the dimension of the associated vector space. A n-dimensional affine space is denoted $\mathbb{A}^n$

Definition A.9. k^n is the affine space

$$k^n = \{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mid a_i \in k\ \forall i \in \{1, ..., n\}\}. \tag{A.1}$$

Polynomials

The set of all polynomials in variables $x_1, ..., x_n$, denoted k$[x_1, ..., x_n]$ above, is an abelian ring. Each polynomial P∈k$[x_1, ..., x_n]$ can be written as a sum of monomials as follows

$$\sum_{(i_1, i_2, ...i_n) \in I \subset \mathbb{N}^n} a_{i_1} a_{i_2} ... a_{i_n} x_1^{i_1} x_2^{i_2} ... x_n^{i_n} \quad \text{with } a_{i_j} \in k\ \ \forall j \in \{1, ..., n\}. \tag{A.2}$$

While the degree of a polynomial in a single variable P∈k$[x]$ is easily defined as the largest natural $d \in \mathbb{N}$ such that a_d is non zero in the expression

$$P = \sum_{i \in \mathbb{N}} a_i x^i, \tag{A.3}$$

one has to define degree in a more general way for multivariate polynomials.

Definition A.10. (Degree) The *degree of a monomial* $a_{i_1} a_{i_2} ... a_{i_n} x_1^{i_1} x_2^{i_2} ... x_n^{i_n}$ is the sum

$$\sum_{j=0}^{n} i_j \tag{A.4}$$

The degree of a polynomial is the highest monomial degree.

Definition A.11. (Homogeneous polynomial) A polynomial is called *homogeneous* if all of its monomials are of the same degree.

Definition A.12. (Reducible and irreducible polynomials) Let $P \in k[x_1, ..., x_n]$ be a non-constant[1] multivariate polynomial. P is *reducible* if there exists $f_1, ..., f_N \in k[x_1, ..., x_n]$ such that

$$P = F_1 ... F_N \tag{A.5}$$

Otherwise, P is *irreducible*.

Definition A.13. (Homogeneous component of a polynomial) Let $P \in k[x_1, ..., x_n]$ be a multivariate polynomial in degree d. P can be written as a sum of terms of the same degree, that is

$$P = P^{(1)} + ... + P^{(d)} \tag{A.6}$$

where $P^{(i)}$ is in degree i. Those $P^{(i)}$ are the *homogeneous components* of P.

[1] P is constant if $P = \lambda$, $\lambda \in k$

Topological reminder

Here are presented used notions about topology. For a more detailed overview, the reader is invited to look at [39].

Topological spaces

Definition A.14. (Topology and open subsets) Let X be a set.

- A *topology* $\mathcal{T}$ is a family of subsets of X, that is $\mathcal{T} \subset 2^X$, such that
  - $\emptyset$ and X are in $\mathcal{T}$
  - Any union of elements of $\mathcal{T}$ is in $\mathcal{T}$: $\forall A_i \in \mathcal{T},$ with $i \in J \subset \mathbb{N}$, $\bigcup_{i \in J} A_i \in \mathcal{T}$
  - Any finite intersection of elements of $\mathcal{T}$ is in $\mathcal{T}$: $\forall A_i \in \mathcal{T},$ with $i \in \{1, ..., n\}$, $\bigcap_{i=0}^{n} A_i \in \mathcal{T}$
- A *topological space* (X, $\mathcal{T}$) is a set X together with a topology $\mathcal{T} \subset 2^X$.
- The elements of a topology $\mathcal{T}$ on X are called the *open subsets* of $\mathcal{T}$.

Definition A.15. (Closed subsets) Let (X, $\mathcal{T}$) be a topological space. A subset F $\subset X$ is *closed* if X\F is an open subset, that is, an element of $\mathcal{T}$.

Theorem A.1. *Let (X,$\mathcal{T}$) be a topological space. Any intersection of closed subsets are closed subsets. Any finite union of closed subsets are closed subsets.*

Proof. Let A,B$\in \mathcal{T}$.
First A $\cup$ B $\in \mathcal{T}$ (and is then an open subset). Since X$\backslash(A \cup B) = X\backslash A \cap X\backslash B$, and since X$\backslash(A \cup B)$, $X\backslash A$ and $X\backslash B$ are closed subset, then intersection of closed subsets are closed subsets.
Second A$\cap$B $\in \mathcal{T}$ (and is then an open subset). Since X$\backslash(A \cap B) = X\backslash A \cup X\backslash B$, and since X$\backslash(A \cap B)$, $X\backslash A$ and $X\backslash B$ are closed subset, then a finite union of closed subsets are closed subsets.
□

Thus, we can define a topology with closed subsets instead of open subsets.

Definition A.16. (Topology and closed subsets) Let X be a set. A *topology* $\mathcal{T}$ is a family of subsets of X, that is $\mathcal{T} \subset 2^X$, such that

- $\emptyset$ and X are in $\mathcal{T}$
- Any finite union of closed subsets of $\mathcal{T}$ is in $\mathcal{T}$: $\forall A_i \in \mathcal{T},$ with $i \in J \subset \mathbb{N}$, $\bigcup_{i \in J} A_i \in \mathcal{T}$

- Any intersection of closed subsets of $\mathcal{T}$ is in $\mathcal{T}$: $\forall A_i \in \mathcal{T},$ with $i \in \{1, ..., n\}$, $\bigcap_{i=0}^{n} A_i \in \mathcal{T}$

The elements of a topology $\mathcal{T}$ on X are called the *closed subsets* of $\mathcal{T}$.

Definition A.17. (Order between topologies) Let X be a set. Let $\mathbb{T}_X$ or simply $\mathbb{T}$ be the set of all topologies on X. Inclusion between elements of this latter set can define a way to order them :

$$\text{Let } \mathcal{T}, \mathcal{T}' \in \mathbb{T}. \text{ If } \mathcal{T} \subset \mathcal{T}', \text{ then } \mathcal{T}' \text{ is finer than } \mathcal{T} \text{ and } \mathcal{T} \text{ is coarser than } \mathcal{T}' \tag{A.7}$$

and we note $\mathcal{T} \leq \mathcal{T}'$.

Remark A.2. Naturally, if one takes $\mathcal{T}, \mathcal{T}' \in \mathbb{T}$ such that $\mathcal{T} \not\subset \mathcal{T}'$ and $\mathcal{T}' \not\subset \mathcal{T}$, then this is not possible to order them together.

Definition A.18. (Continuous map) A map f : $X \subset k^n \to Y \subset k^n$ is *continuous* if U=f$^{-1}(V) \subset X$[2] is an open subset in X, $\forall V \subset Y$ an open subset of Y.

Remark A.3. The same definition lies if one considers closed subsets.

Theorem A.2. *Let (Y,$\mathcal{T}'$) be a topological space, X a set and f : X$\to$Y a map from X to Y. There exists a unique topology on X, denoted $f^{-1}(\mathcal{T}')$, such that f is a continuous map and which is coarser than any other topology for which f is also continuous.*
Furthermore, one has $f^{-1}(\mathcal{T}') = \{f^{-1}(A)$ |Ã $\in \mathcal{T}'\}$

Theorem A.3. *Let (X,$\mathcal{T}$) be a topological space, Y a set and f : X$\to$Y a map from X to Y. There exists a unique topology on Y, $f(\mathcal{T})$ such that f is a continuous map and which is finer than any other topology for which f is also continuous.*
Furthermore, one has $f(\mathcal{T}) = \{A \subset Y \mid f^{-1}(A) \in \mathcal{T}\}$

Definition A.19. (Initial and final topology) $f^{-1}(\mathcal{T}')$ is the initial topology with respect to f. It is the coarser topology on the domain[3] of the map f for which f is continuous. $f(\mathcal{T})$ is the final topology with respect to f. It is the finer topology on the codomain[4] of the map f for which f is continuous.

Definition A.20. (Closure) Let $(X, \mathcal{T})$ be a topological space. The *closure* $\overline{A}$ of $A \subset$X is the intersection of every closed subsets of X which contains A.

Definition A.21. (Dense) A is said to be *dense* if $\overline{A} = X$.

[2] $f^{-1}(V)$ is the set of elements of X whose image via f is in V$\subset Y$
[3] The domain of f : X $\to Y$ is X
[4] The codomain of f : X$\to Y$ is Y

Irreducible subsets and Noetherian spaces

Definition A.22. (Reducible set) Let (X, $\mathcal{T}$) be a topological space. X is *reducible* if there exists closed subsets of X, $X_i \subsetneq X,\ i \in J \in \mathbb{N}$, such that

$$X = \bigcup_{i \in J} X_i. \tag{A.8}$$

If not, then X is *irreducible*. If X is reducible, then these subsets X_i are called *irreducible components of X*, and Eq. (A.8) is a *decomposition of X into its irreducible components.*

Definition A.23. (Noetherian topological space) A topological space $(X, \mathcal{T})$ is *Noetherian* if every descending chain of closed subsets $X \supsetneq X_1 \supsetneq X_2 \supsetneq ...$ becomes stationary, that is

$$\forall \text{ descending chain } X \supsetneq X_1 \supsetneq X_2 \supsetneq ...,\ \exists N \in \mathbb{N}\ :\ X_N = X_{N+1} = ... \tag{A.9}$$

Remark A.4. If (X,$\mathcal{T}$) is Noetherian, then $\forall Y \subset X$, (Y,$\mathcal{T}_{\text{induced}}$) is Noetherian. Then, from Example 2.2, one can see that any affine algebraic set is Noetherian.

Theorem A.4. *Let (X,$\mathcal{T}$) be a Noetherian topological space. Then X is a finite union of irreducible closed subsets, that is*

$$\exists X_1, ..., X_n \quad \textit{such that} \quad X = X_1 \cup X_2 \cup ... \cup X_n. \tag{A.10}$$

If $X_i \not\subset X_j,\ \forall i \neq j$ then this decomposition into irreducible components is unique.

Now one can define the dimension of a topological space. This definition has to satisfy a trivial need : if X is irreducible and Y⊂X is irreducible, then dimension of Y has to be lower or equal to dimension of X. But if Y⊄X, than we have to define dimension in another way.

Definition A.24. (Dimension of irreducible topological spaces) Let (X,$\mathcal{T}$) be an irreducible topological space with $X \neq \emptyset$. Its *dimension*, denoted dim(X), is the largest $n \in \mathbb{Z}$ such that

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq ... \subsetneq X_n = X \tag{A.11}$$

where X_i is an irreducible closed subset of X, $\forall i \in \{1, ..., n\}$.

Definition A.25. (Dimension of Noetherian topological spaces) Let (X,$\mathcal{T}$) be a Noetherian topological space. Its *dimension*, denoted dim(X), is the maximal dimension of its irreducible components. That is,

$$\dim(\mathrm{X}) = \max\{d_i\} \tag{A.12}$$

where d_i is the dimension of the i-th irreducible component.

Equivalence relation and quotient set

Definition A.26. (Equivalence relation) Let X be a set. An *equivalence relation* is a binary relation $\sim$ on X such that, $\forall a, b, c \in X$

- $\sim$ is reflexive : $a \sim a$
- $\sim$ is symmetric : $a \sim b \iff b \sim a$
- $\sim$ is transitive : if $a \sim b$ and $b \sim c$, then $a \sim c$

Definition A.27. (Equivalence classes) Every element in equivalence relation with $a \in X$ belongs to the same *equivalence class* and are said *equivalent with respect to* $\sim$. The equivalence class defined by the subset of X whose elements are equivalent to $a \in X$ with respect to $\sim$ is denoted [a], that is

$$\forall a \in X,\ [a] = \{b \in X \mid b \sim a\}. \tag{A.13}$$

In this case, a is called a *representative* of the equivalence class [a], such as every other equivalent element which can represents this equivalence class.

Definition A.28. (Quotient set) Let X be a set. The set of all equivalence classes of X with respect to the equivalence relation $\sim$ is the *quotient set* denoted $X/\sim$, that is

$$X/\sim = \{[a] \mid a \in X\}. \tag{A.14}$$

Definition A.29. (Quotient map) Let (X,$\mathcal{T}$) be a topological space and $\sim$ an equivalence relation on X. The *quotient map*, denoted π, is the map that send a point of X to its equivalence class with respect to $\sim$. That is,

$$\pi\ :\ X \to X/\sim :\ a \to [a]. \tag{A.15}$$

Definition A.30. (Quotient topology) The *quotient topology* on X/$\sim$ is the final topology with respect to π and therefore denoted $\pi(\mathcal{T})$.

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