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Mémoire

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FACULTY OF SCIENCES
DEPARTMENT OF MATHEMATICS

A Study of Persistent Homology through Persistence Modules

A thesis submitted in fulfillment of the requirements for the
Master's degree in mathematical sciences, research focus

Academic year 2024-2025

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Under the supervision of:
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Acknowledgments

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Contents

1 | Introduction

The humorous trait of the following meme[1] about Schrödinger's cat is that when asked to interpret what the second message states, the common response is to say "it depends". Indeed, depending on the scale, we could say that the message contains 30 times "yes" or one time "no". It is a toy example of the importance of the scale when analysing a data set. However, finding the "right" scale is a difficult task in data analysis and is usually left at the user's discretion, often through trial and error. Multiscale allows for the study of data at all scales at once and gives an understanding of the role of the scale on output. The goal of persistent homology is to provide such a multiscale analysis of data sets through homology.

Messages to this chat and calls are now secured with end-to-end encryption. Tap for more info.

Is the cat alive? 01:37

```
yesyes      yes     yesyes
yes yes     yes  yes     yes
yes   yes   yes  yes      yes
yes    yes  yes  yes      yes
yes     yes yes   yes    yes
yes      yesyes     yesyes   01:37
```

Figure 1.1: A meme on Schrödinger's cat as an insightful introduction to topological data analysis and multiscale analysis.

Persistent homology is a crucial tool in the field of topological data analysis (TDA). As its name suggests, TDA is at the intersection of topology and data sciences, using topological tools, like homology, to infer some information about the "shape" of some data sets. This field has applications in several branches of Science and is still an extremely fruitful field of research. It is, for example, used in Medicine as a method for the detection of cancer cells in [45], it is used in Finance

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[1] `https://www.reddit.com/r/sciencememes/comments/16qi5ai/ill_take_it_as_both_yes_and_no/?utm_source=share&utm_medium=web3x&utm_name=web3xcss&utm_term=1&utm_content=share_button`

to analyse crash of the stock exchange [32], in Political Science as a way to analyse gerrymandering (political redisctricting) [26], in Image Analysis [25], in Machine Learning [35], and much more. The DONUT website[2] (Database of Original and Non-Theoretical Uses of Topology) provides numerous other examples.

1.1 Outline

The goal of this master thesis is to provide an in-depth and (mostly) self-contained introduction to the main results of persistent homology. Of course, much more could have been presented, as discussed in Section 5. The present text mainly focuses on the algebraic aspects of persistent homology as a way to justify the use of persistent homology in topological data analysis. Therefore, the topological aspects, as well as the implementation of the algorithm and how to make it more efficient, are surveyed, and we will not dive into them. Some interesting surveys of the topics are [49, 33, 47].

This document is divided into four main chapters. The first one is the introduction, in which I give some context around the study of persistent homology. In particular, I present Morse theory as well as the data inference problem, which is the main motivation for the study of persistent homology. I then give the first definitions of persistent homology and persistence module. These persistence modules will provide a way to study persistent homology from a purely algebraic setting, which we will do during the rest of this master thesis.

As the definition of persistence module is equivalent in certain cases to the one of a representation of a certain type of quiver, the second chapter is focused on the study of quivers and their representations. The ultimate goal of this chapter is to prove Gabriel's theorem, which gives a characterisation of the quivers of finite representation type.

In the third chapter, we study the decomposition of persistence modules into indecomposable ones. The main modules of interest will be interval modules that provide both an interpretation of the module and a description of it. It also allows simpler computation on it. We first show that if a decomposition exists, then it must be unique up to reordering and isomorphism. Unfortunately, there is not always such a decomposition. We provide some conditions for it to exist, which cover the cases we encounter in the usual applications and several other theoretical cases. Using this type of decomposition, we can define (a simple version of) the persistence diagram and barcode of a persistence module.

In the final chapter, the goal is to provide the correct framework to say that "two persistence modules are close if their persistence diagram is close", which is crucial in the data inference problem. While changing the viewpoint of persistence modules from a representation-theoretic point of view to a category-theoretic perspective, we define a notion of (extended pseudo-) metric on them and compute the value of that "metric" between persistence modules. We then define another metric on the persistence diagram. To conclude, we show that there is an isometry between some particular subspaces of the persistence diagram space and the space of persistence modules.

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[2] `https://donut.topology.rocks/`.

There are also two appendices providing the basic results of algebraic geometry, which we use in Chapter 2, and of category theory, as well as homological algebra, which we will use throughout the whole document. In those annexes, I will only focus on giving the main and relevant results, mostly without proofs. The goal is to provide the context of the results that will be used to provide the relevant background in algebra.

1.2 Context

Persistent homology is based on the concept of *persistence*, which was independently put forward at the end of the 1990s in the work of P. Frosini and M. Ferri [30] using size functions, in the work of V. Robins [51] and in the work of Edelsbrunner using alpha shapes [27]. Intuitively, persistence measures how certain characteristics are a defining part of the object we are studying. The motivation of persistent homology is to provide a tool to allow a multi-scale analysis of data points.

Persistent homology can also be seen as a generalisation of Morse[3] theory. We will first focus on that theory to introduce the key ideas of persistent homology.

1.2.1 Morse Theory

In this section, M will denote a smooth manifold of dimension d. We mainly follow [28, 43, 47].

For a smooth real function on M, a *critical point* is a point p for which the differential of the function is zero. Such a point is *non-degenerate* if the Hessian matrix of the function at p in a local chart is of full rank. We call the *index* of f at p the number of negative eigenvalues of the Hessian matrix at p.

Definition 1.2.1. A *Morse function* is a smooth function from M to $\mathbb{R}$ that has only a finite number of critical points, all of which are non-degenerate and of different critical values.

Note that to be complete, we should check that the notion of critical points and non-degeneracy does not depend on the charts used. The condition of the manifold being smooth guarantees that fact.

Example 1.2.2. The function $f_1\colon \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ is a Morse function while the function $f_2\colon \mathbb{R} \to \mathbb{R}, x \to x^3$ is not, as the point 0 is degenerate. Similarly, the function $f_3\colon \mathbb{R} \to \mathbb{R}, x \mapsto x\cos(5x)$ is not a Morse function as it has an infinite number of critical points. The function $f_4\colon \mathbb{R} \to \mathbb{R}, x \mapsto (x+1)^2(x-1)^2$ is not Morse either as it has 2 critical points of critical value 0 (see the Figure 1.2).

Another typical example in several dimensions is the function h giving the height (i.e. the z coordinate) of the points on the torus[4] $\begin{pmatrix} (2+\cos(u))\cos(v) \\ \sin(u) \\ (2+\cos(u))\sin(v)+3 \end{pmatrix}$

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[3] Marston Morse (1892-1977) was an American mathematician specialised in calculus. He gave his name to the Thue-Morse sequence; note, however, that he is unrelated to Morse code (which was created by Samuel Morse (1791-1872)).

for $0 \leq u, v < 2\pi$ (see Figure 1.3). Intuitively, a direct computation shows that the critical points of h are obtained when $\sin(u) = 0 = \cos(v)$, i.e when ($u = 0$ or $u = \pi$) and ($v = \pi/2$ or $v = 3\pi/2$). Therefore, the critical points are $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$, each with a different critical value.

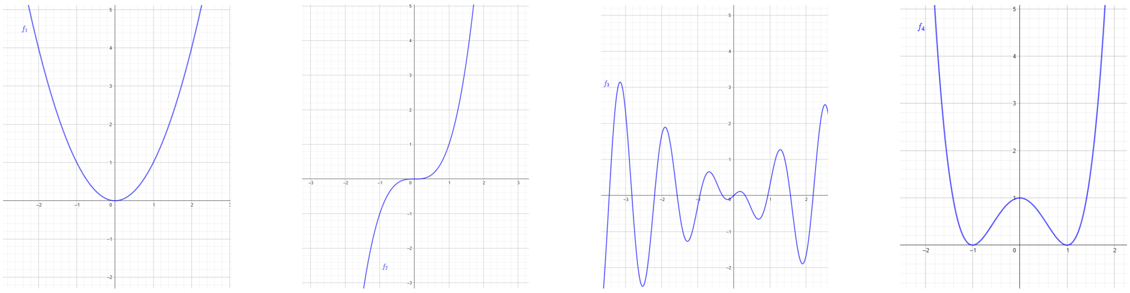


Figure 1.2: The functions f_1, f_2, f_3 and f_4. Only f_1 is a Morse function.

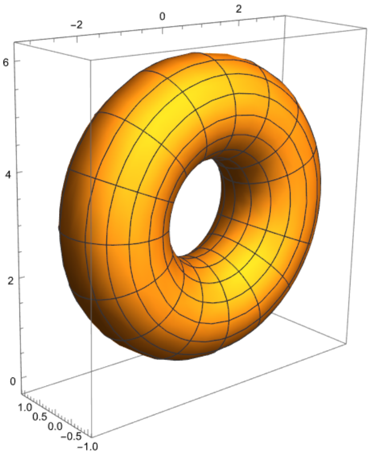


Figure 1.3: A torus.

The last example is an occurrence of a common use of Morse function to describe a manifold, and it gives some intuition on what we will do later. We compute the "height" of the point on the manifold and then try to retrieve some topological information of the manifold by analysing the topology of the set "below" a certain value.

More formally, given a function $f\colon M \to \mathbb{R}$, we can define its *sublevel set* at $t \in \mathbb{R}$ as

$$M^t = f^{-1}\big((-\infty, t]\big).$$

---

[4] Note that to be totally rigorous, we should only take open subsets of $\mathbb{R}^2$ as the domain for the charts and therefore have $0 < u, v < 2\pi$ and introduce another chart to add the missing points. As this is not the focus of this master thesis and the context is still clear, we will allow this abuse of notation.

Thanks to the conditions on Morse functions, we can deduce some important properties of the sublevel sets and their topology. Both theorems are stated and proved in [43].

Theorem 1.2.3. Suppose that f is a Morse function, let $a < b$ be two real numbers such that $f^{-1}([a,b])$ is compact and contains no critical points of f, then M^a is diffeomorphic to M^b. To be more precise, M^a is a deformation retract, therefore, the inclusion map $M^a \to M^b$ is a homotopy equivalence.

Recall that a *deformation retract* of a topological space X is a subspace A for which there exists a continuous function $F\colon X \times [0,1] \to X$ such that for all $x \in X$ and $a \in A$, $F(x,0) = x, F(x,1) \in A$ and $F(a,1) = a$.

Therefore, with a Morse function f on a compact manifold, we can choose regular values

$$t_0 < t_1 < \cdots < t_m$$

bracketing the critical values and if we denote by

$$M^j = f^{-1}\big((-\infty, t_j]\big)$$

the sublevel set containing the j first critical values, then the different sets are homotopic independent of the choice of the t_i.

Theorem 1.2.4. Let f be a Morse function, p a critical point with critical value c and index λ. Let $\epsilon > 0$ be such that $f^{-1}[c-\epsilon, c+\epsilon]$ is a compact subset of M and contains no critical points others than p. Then $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ-cell (i.e. a closed ball of dimension λ) attached.

By "attaching" a cell, I mean that we have a continuous morphism φ from the boundary of the closed ball of dimension λ to $M^{c-\epsilon}$ and $M^{c+\epsilon}$ will be

$$(M^{c-\epsilon} \sqcup D^\lambda)/\sim$$

where we have $x \sim \varphi(x)$.

Therefore, if λ is the index of the jth critical point, then to go from M^j to M^{j+1}, we have either that the rank of $H_\lambda(M^{j+1})$ is equal to the rank of $H_\lambda(M^j)+1$ or that the rank of $H_{\lambda-1}(M^{j+1})$ is equal to the rank of $H_{\lambda-1}(M^j) - 1$ and the homology groups of other degrees are isomorphic. In the former case, we will call the critical point *positive* and, in the latter, we will call it *negative*.

Indeed, we can deduce it using the *cellular chain complex* associated with the 2 sublevel sets and the rank-nullity theorem. The cellular chain complex over a field $\mathbf{k}$ associated with a cell complex has dimension at position n, the number of n-cells in the cell complex. We will not define the boundary maps. In our case, if $(C_\bullet, \partial)$ is the cellular chain complex of M^j and $(C'_\bullet, \partial')$ is the cellular chain complex of M^{j+1}, we have $\dim(C_n) = \dim(C'_n)$ for all $n \neq \lambda$ and $\dim(C'_\lambda) = \dim(C_\lambda)+1$. Moreover, as we just added a λ-cell to M^j to get M^{j+1}, the boundary maps are the same, except for $\partial_\lambda\colon C_\lambda \to C_{\lambda-1}$. We have that $\partial'_\lambda|_{C_\lambda} = \partial_\lambda$. Therefore, we have

$$H_n(C_\bullet) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1}) = H_n(C'_\bullet)$$

for all $n \neq \lambda$ and $n \neq \lambda - 1$. Let us investigate the dimension of $H_\lambda(C'_\bullet)$ and $H_{\lambda-1}(C'_\bullet)$. As $\partial'_\lambda|_{C_\lambda} = \partial_\lambda$, we have $\dim(\operatorname{im}(\partial'_\lambda)) \geq \dim(\operatorname{im}(\partial_\lambda))$ and $\dim(\ker(\partial'_\lambda)) \geq \dim(\ker(\partial_\lambda))$. By the rank-nullity theorem, we have

$$\begin{aligned}\dim(\operatorname{im}(\partial'_\lambda)) + \dim(\ker(\partial'_\lambda) &= \dim(C'_\lambda)\\ &= \dim(C_\lambda) + 1\\ &= \dim(\operatorname{im}(\partial_\lambda)) + \dim(\ker(\partial_\lambda) + 1.\end{aligned}$$

We therefore either have

$$\dim(\ker(\partial'_\lambda)) = \dim(\ker(\partial_\lambda)) + 1 \text{ and } \dim(\operatorname{im}(\partial'_\lambda)) = \dim(\operatorname{im}(\partial_\lambda))$$

or

$$\dim(\ker(\partial'_\lambda)) = \dim(\ker(\partial_\lambda)) \text{ and } \dim(\operatorname{im}(\partial'_\lambda)) = \dim(\operatorname{im}(\partial_\lambda)) + 1.$$

In the first case, we have

$$\begin{aligned}\dim(H_\lambda(M^{j+1})) &= \dim\left(\ker(\partial'_\lambda)/\operatorname{im}(\partial'_{\lambda+1})\right)\\ &= \dim\left(\ker(\partial'_\lambda)\right) - \dim\left(\operatorname{im}(\partial'_{\lambda+1})\right)\\ &= \dim\left(\ker(\partial_\lambda)\right) + 1 - \dim\left(\operatorname{im}(\partial_{\lambda+1})\right)\\ &= \dim\left(\ker(\partial_\lambda)/\operatorname{im}(\partial_{\lambda+1})\right) + 1\\ &= \dim(H_\lambda(M^j)) + 1\end{aligned}$$

and $H_{\lambda-1}(M^{j+1}) = \ker(\partial'_{\lambda-1})/\operatorname{im}(\partial'_\lambda) = \ker(\partial_{\lambda-1})/\operatorname{im}(\partial_\lambda) = H_{\lambda-1}(M^j)$.

In the second case, we have $H_\lambda(M^{j+1}) = H_\lambda(M^j)$ and

$$\begin{aligned}\dim(H_{\lambda-1}(M^{j+1})) &= \dim\left(\ker(\partial'_{\lambda-1})/\operatorname{im}(\partial'_\lambda)\right)\\ &= \dim\left(\ker(\partial'_{\lambda-1})\right) - \dim\left(\operatorname{im}(\partial'_\lambda)\right)\\ &= \dim\left(\ker(\partial_{\lambda-1})\right) - \left(\dim\left(\operatorname{im}(\partial_\lambda)\right) + 1\right)\\ &= \dim\left(\ker(\partial_{\lambda-1})/\operatorname{im}(\partial_\lambda)\right) - 1\\ &= \dim(H_{\lambda-1}(M^j)) - 1.\end{aligned}$$

Example 1.2.5. The function f of Figure 1.4 is a Morse function with 6 critical values; the minima with index 0 and the maxima with index 1. By Theorem 1.2.3, we can then construct 7 sublevel sets, homotopy independent of the choice of the values bracketing the critical values:

$$\begin{aligned}M^0 &= (-\infty, \alpha_0]\\ M^1 &= (-\infty, \alpha_1] \cup [c_1, d_1]\\ M^2 &= (-\infty, \alpha_2] \cup [a_2, b_2] \cup [c_2, d_2]\\ M^3 &= (-\infty, \alpha_3] \cup [a_3, d_3]\\ M^4 &= (-\infty, \alpha_4] \cup [a_4, d_4] \cup [e_4, f_4]\\ M^5 &= (-\infty, d_5] \cup [e_5, f_5]\\ M^6 &= (-\infty, f_6].\end{aligned}$$

Let us recall that the rank of $H_0(M)$ gives the number of connected components of M. We then see that critical values 1,2, and 4, i.e. the minima, are positive as the sublevel set M^j has one more connected component than M^{j-1} for $j \in \{1, 2, 4\}$

while the critical values 3,5, and 6, i.e. the maxima, are negative as the sublevel set M^j has one less connected component than M^{j-1} for $j \in \{3,5,6\}$.

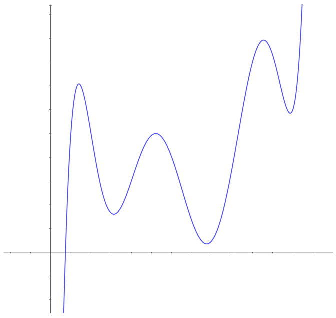


Figure 1.4: Generic example of a Morse function $f\colon \mathbb{R} \to \mathbb{R}$.

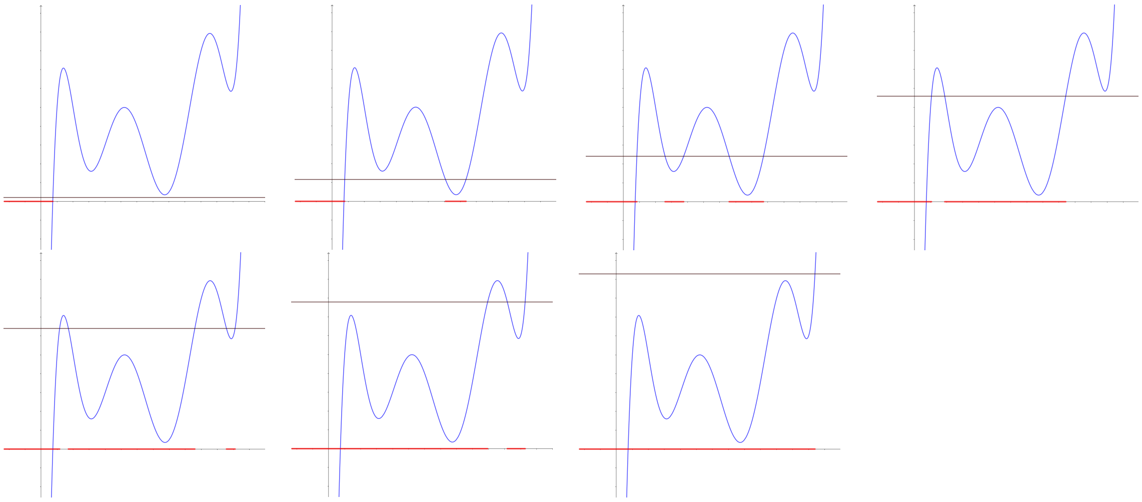


Figure 1.5: The different sublevel sets (in red) of the Morse function f. In brown is the line $y = a$ used to define M^a.

Example 1.2.6. For the torus of Figure 1.3, let us compute the index of each critical point (H represent the hessian matrix of h):

$$\begin{aligned}
H_{\binom{0}{\pi/2}} &= \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} & \rightarrow \lambda = 2 \\
H_{\binom{0}{3\pi/2}} &= \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} & \rightarrow \lambda = 0 \\
H_{\binom{\pi}{\pi/2}} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \rightarrow \lambda = 1 \\
H_{\binom{\pi}{3\pi/2}} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \rightarrow \lambda = 1
\end{aligned}$$

Moreover, the point of parameter $\binom{0}{\pi/2}$ represents the point $\begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$ on the torus, the point of parameter $\binom{0}{3\pi/2}$ represents the point $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ on the torus, the point of parameter $\left(\begin{smallmatrix}\pi\\ \pi/2\end{smallmatrix}\right)$ represents the point $\left(\begin{smallmatrix}0\\ 0\\ 4\end{smallmatrix}\right)$ on the torus and the point of parameter $\left(\begin{smallmatrix}\pi\\ 3\pi/2\end{smallmatrix}\right)$ represents the point $\left(\begin{smallmatrix}0\\ 0\\ 2\end{smallmatrix}\right)$ on the torus.

This allows us, thanks to Theorem 1.2.4, to compute the different sublevel sets, up to homotopy (here, $\sim$ denotes that it is homotopy equivalent).

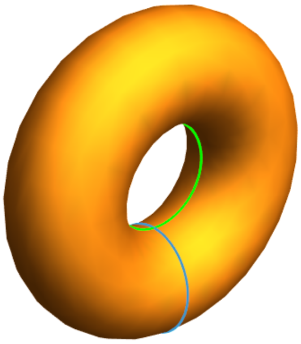
$$\begin{aligned} &M^0 = \emptyset, \\ &M^1 \sim \{*\}, \\ &M^2 \sim S^1, \\ &M^3 \sim S^1 \vee S^1, \\ &M^4 \sim T. \end{aligned}$$

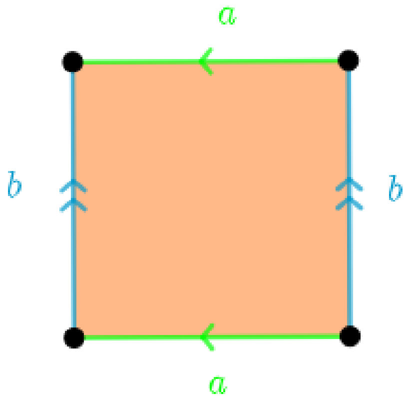
To go from M^0 to M^1, we attach a 0-cell (i.e. a point) to the empty set: we get a point. To go from M^1 to M^2, we attach a 1-cell (i.e. a compact interval) to the point we had in M^1; this gives a circle S^1. To go from M^2 to M^3, we add another 1 cell to the circle we had in M^2; this gives an object homotopic to 2 tangent circles (i.e. the shape "8"), more formally $S^1 \vee S^1$.

Finally, to obtain M^4, we glue the boundary of a closed disk D^2 to $S^1 \vee S^1$ to form a torus. The way to glue it is rather technical. First, the boundary of D^2 is the circle S^1. We identify it with $I = [0, 1)$. We can see $S^1 \vee S^1$ as the set $A = \{(u, v) : 0 \leq u, v < 1, uv = 0\}$. To glue the 2 objects, we will identify each point of the boundary of D^2 with a point of $S^1 \vee S^1$ through the following continuous morphism:

$$\varphi \colon I \to A, t \mapsto \begin{cases} (4t, 0) & 0 \leq t < 1/4, \\ (0, 4(t - 1/4)) & 1/4 \leq t < 1/2, \\ (1 - 4(t - 1/2), 0) & 1/2 \leq t < 3/4, \\ (0, 1 - 4(t - 3/4)) & 3/4 \leq t < 1. \end{cases}$$

A maybe more intuitive way to see the glueing of the boundary of the 2-cell to $S^1 \vee S^1$ to form a torus is by seeing S^1 as a square and identifying the opposite edges with the same orientation as the one in the figure.





Let us now compute the different homologies of the different sublevel sets. I will not give the details of the computations as they are standard.

$$\begin{array}{rcccl}
H_n(M^0) & = & H_n(\emptyset) & = & 0 \quad n \in \mathbb{Z}. \\
H_n(M^1) & = & H_n(\{*\}) & = & \begin{cases} \mathbf{k} & n = 0, \\ 0 & \text{else.} \end{cases} \\
H_n((M^2) & = & H_n(S^1) & = & \begin{cases} \mathbf{k} & n = 0, 1, \\ 0 & \text{else.} \end{cases} \\
H_n(M^3) & = & H_n(S^1 \vee S^1) & = & \begin{cases} \mathbf{k} & n = 0, \\ \mathbf{k} \oplus \mathbf{k} & n = 1, \\ 0 & \text{else.} \end{cases} \\
H_n(M^4) & = & H_n(T) & = & \begin{cases} \mathbf{k} & n = 0, 2, \\ \mathbf{k} \oplus \mathbf{k} & n = 1, \\ 0 & \text{else.} \end{cases}
\end{array}$$

In particular, it means that all the critical values are positive.

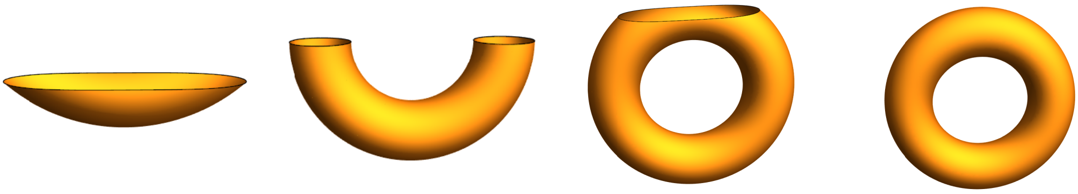


Figure 1.6: The different sublevel sets of the torus T.

We can now introduce the notion of persistence. Let us denote by

$$\varphi_p^{i,j} : H_p(M^i) \to H_p(M^j)$$

the map induced by the inclusion $M^i \subset M^j$ whenever $i \leq j$. More explicitly, in our case, we have

$$\varphi_p^{i,j} : H_p(M^i) \to H_p(M^j), x + \mathrm{im}(\partial_{p+1}^i) \mapsto x + \mathrm{im}(\partial_{p+1}^j)$$

where ∂_{p+1}^i denotes the boundary $\partial_{p+1} : C_{p+1}^i \to C_p^i$ in the cellular chain complex associated to M^i. It is well defined because we have $\ker(\partial_p^i) \subset \ker(\partial_p^j)$ and $\mathrm{im}(\partial_{p+1}^i) \subset \mathrm{im}(\partial_{p+1}^j)$ as we have $C_p^i \subset C_p^j$ for all p and $\partial_p^j|_{C_p^i} = \partial_p^i$.

Let α be a homology class $\alpha \in \sqcup_{0 \leq j \leq m} H_p(M^j)$. Suppose that $\alpha \in H_p(M^t)$. We say that α is *born* at M^j if

$$\alpha \in \mathrm{im}(\varphi_p^{j,t}) \backslash \mathrm{im}(\varphi_p^{j-1,t}).$$

If such a class α is born at M^j, we say that it *dies* entering M^k if

$$\varphi_p^{t,k-1}(\alpha) \notin \mathrm{im}(\varphi_p^{j-1,k-1}) \text{ and } \varphi_p^{t,k}(\alpha) \in \mathrm{im}(\varphi_p^{j-1,k}).$$

We further say that the homology class α is *alive* from j to k.

Let $\alpha \in H(M^j)$ be an element that is born at j and dies at k. As we work with Morse functions, we have the following decomposition:

$$H_p(M^j) = \mathrm{im}(\varphi_p^{j-1,j}) \oplus \rangle \alpha \langle.$$

The sum is direct as, if the intersection $\mathrm{im}(\varphi_p^{j-1,j}) \cap \rangle \alpha \langle$ is not null, then there is a $a\alpha \in \mathrm{im}(\varphi_p^{j-1,j})$ with $a \in \mathbf{k} \backslash \{0\}$ and therefore $\alpha \in \mathrm{im}(\varphi_p^{j-1,j})$. We have the equality of the vector spaces using the dimension. Using the reasoning made after Theorem 1.2.4, we get that

$$\dim(\ker(\partial_p^j)) = \dim(\ker(\partial_p^{j-1})) + 1 \text{ and } \dim(\mathrm{im}(\partial_p^j)) = \dim(\mathrm{im}(\partial_p^{j-1}))$$

or

$$\dim(\ker(\partial_p^j)) = \dim(\ker(\partial_p^{j-1})) \text{ and } \dim(\mathrm{im}(\partial_p^j)) = \dim(\mathrm{im}(\partial_p^{j-1})) + 1.$$

Recall that $\varphi_p^{j-1,j}$ is defined by $\varphi_p^{j-1,j} \colon \ker(\partial_p^{j-1}) / \mathrm{im}(\partial_{p-1}^{j-1}) \to \ker(\partial_p^j) / \mathrm{im}(\partial_{p-1}^j), x + \mathrm{im}(\partial_{p+1}^{j-1}) \mapsto x + \mathrm{im}(\partial_{p+1}^j)$. As $\alpha \notin \mathrm{im}(\varphi_p^{j-1,j})$, it implies that we are in the first case. It further states that $\varphi_p^{j-1,j}$ is injective. Therefore, we have

$$\begin{aligned}
\dim(H_p(M^j)) &= \dim(H_p(M^{j-1})) + 1 \\
&= \dim(\mathrm{im}(\varphi_p^{j-1,j})) + \dim(\ker(\varphi_p^{j-1,j})) + 1 \\
&= \dim(\mathrm{im}(\varphi_p^{j-1,j})) + \dim(\rangle \alpha \langle).
\end{aligned}$$

Therefore, for all $t < k$, we have

$$\mathrm{im}(\varphi_p^{j,t}) = \mathrm{im}(\varphi_p^{j-1,t}) \oplus \rangle \varphi_p^{j,t}(\alpha) \langle.$$

Indeed, the sum is direct. Let $\beta \in \mathrm{im}(\varphi_p^{j-1,t}) \cap \rangle \varphi_p^{j,t}(\alpha) \langle$ and suppose β non-zero. In particular, $\beta = a\varphi_p^{j,t}(\alpha)$ for $a \in \mathbf{k} \backslash \{0\}$ and $\beta = \varphi_p^{j-1,t}(\gamma)$ for a $\gamma \in H_p(M^{j-1})$. It gives that $\varphi_p^{j,t}(\alpha) = \varphi_p^{j-1,t}(\frac{\gamma}{a}) \in \mathrm{im}(\varphi_p^{j-1,t})$, which is a contradiction with the condition that α dies at k. We have the inclusion $\mathrm{im}(\varphi_p^{j,t}) \supset \mathrm{im}(\varphi_p^{j-1,t}) \oplus \rangle \varphi_p^{j,t}(\alpha) \langle$ directly, let us check the other inclusion. Let $\beta \in \mathrm{im}(\varphi_p^{j,t})$. We have $\beta = \varphi_p^{j,t}(\gamma)$ for a $\gamma \in H_p(M^j)$. Using the equality $H_p(M^j) = \mathrm{im}(\varphi_p^{j-1,j}) \oplus \rangle \alpha \langle$, we get $\gamma = \gamma' + a\alpha$ with $\gamma' \in \mathrm{im}(\varphi_p^{j-1,j})$. therefore, $\beta = \varphi_p^{j,t}(\gamma) = \varphi_p^{j,t}(\gamma') + \varphi_p^{j,t}(a\alpha) \in \mathrm{im}(\varphi_p^{j-1,t}) \oplus \rangle \varphi_p^{j,t}(\alpha) \langle$.

Finally, let $\beta \in H_p(M^{j-1})$ be such that $\varphi_p^{j-1,k}(\beta) = \varphi_p^{j,k}(\alpha)$. Then, the element $\alpha' = \alpha - \varphi_p^{j-1,j}(\beta)$ will be also an element born at j and dying at k, therefore the above reasoning still works and we have further that $\varphi_p^{j,k}(\alpha') = 0$.

Definition 1.2.7. If the homology class α is born at M^j and dies entering M^k, then the *persistence* of α is $f(c_k) - f(c_j)$, where c_i is the ith critical point.

We can then construct a *persistence diagram*, $\mathrm{dgm}_p(f)$ which includes all the points of the form $(f(x), f(y))$ where x is a positive critical point of index p that is paired with the negative critical point y of index $p+1$. If the positive critical point is not paired with another negative critical point, we pair it with $+\infty$. We also include the first diagonal in the diagram.

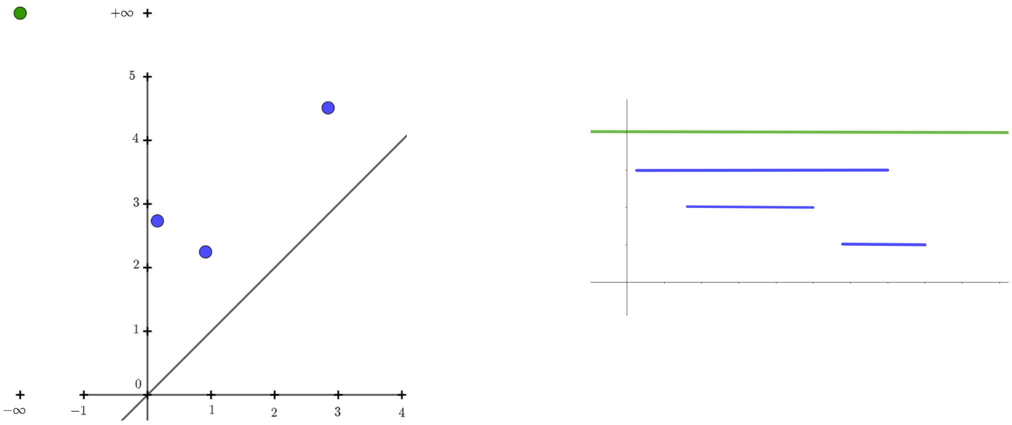


Figure 1.7: Persistence diagram (left) and barcode (right) of the function f for H_0. In green, the component present at any sublevel set.

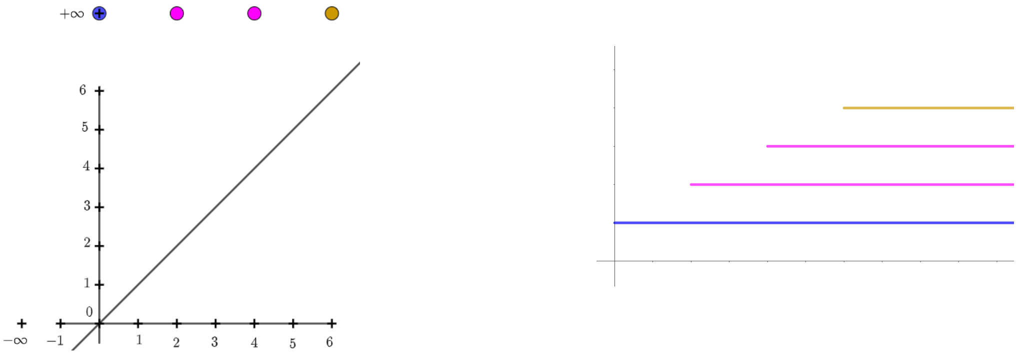


Figure 1.8: Persistence diagram (left) and barcode (right) of the function torus T. In blue, the components related to H_0, in pink the ones related to H_1 and in brown the one related to H_2.

Another way to represent the homology is to have segments $[f(x), f(y)]$ or $[f(x), +\infty)$ for all homology classes. We call this diagram the *barcode* of M.

One important result that will justify the use of persistence diagrams and barcodes in topological data analysis is that two "close" functions will have diagrams that are "close" to each other. To be more precise, we first need to introduce some metric on the functions and, most importantly, on the persistence diagrams.

Definition 1.2.8. The *Bottleneck distance* between the persistence diagrams of f and g is

$$d_B(\mathrm{dgm}(f), \mathrm{dgm}(g)) = \inf_{\nu} \sup_{x} ||x - \nu(x)||_\infty .$$

Where ν is a bijection between the 2 diagrams.

Theorem 1.2.9. We have the inequality

$$d_B(\mathrm{dgm}(f), \mathrm{dgm}(g)) \leq ||f - g||_\infty$$

for all Morse functions on M.

As stated above, it implies that if we have two close functions, their persistence diagrams will also be close. In other words, the persistence diagram of a function is stable under small changes in the function.

1.2.2 Tame Functions

The results found on Morse functions are sometimes a bit too restrictive due to the conditions they require. Some more general results were found. To start with, we will define a class of functions containing the Morse functions and then generalise the previous results.

Let $\mathbb{X}$ be a topological space and $f\colon \mathbb{X} \to \mathbb{R}$. As before, we denote by $\mathbb{X}^t = f^{-1}\big((-\infty, t]\big)$ the sublevel set defined by t.

Definition 1.2.10. The function f is *tame* if there are only a finite number of t across which the homology groups of $\mathbb{X}^t$ are not isomorphic and such that all of these groups have finite ranks.

As for the Morse functions, we can denote by $c_1 < c_2 < \cdots < c_m$ the values across which the homology of $\mathbb{X}^t$ is different,then we can choose $t_0 < t_1 < \cdots < t_{m+1}$ bracketing the c_i; i.e. with $t_0 = -\infty, t_{m+1} = +\infty$ and $t_{i-1} < c_i < t_i$ for all $1 \leq i \leq m$. We also denote by $\mathbb{X}^j$ the sublevel set $\mathbb{X}^{t_j}$. Similarly to the Morse theory case, the homology of $\mathbb{X}^{t_j}$ is independent of the choice of t_j.

Let us also denote by $\varphi_p^{i,j}\colon H_p(\mathbb{X}^i) \to H_p(\mathbb{X}^j)$ the map induced by $\mathbb{X}^i \subset \mathbb{X}^j$ on the pth homology, whenever $i \leq j$. The images of $\varphi_p^{i,j}$ are called a *persistent homology group*. We denote by $\beta_p^{i,j} = \operatorname{rank}(\varphi_p^{i,j})$ a *persistent Betti number* of f. Note that the collection of $\beta_0^{i,j}$ for $i \leq j$ is also called the *size function* of f.

Similarly to the case with Morse functions, we can define the persistence diagram of the function. All the non-diagonal points will be of the form (t_j, t_k); the main difference with the previous case is that we can have points of multiplicity greater than 1. We want as the multiplicity of (t_j, t_k) the number of independent homology classes that are born at j and die at k. To make sense of that, let us show the following result.

Lemma 1.2.11. Let V^j be a vector space such that $\operatorname{im}(\varphi_p^{j',j}) = \operatorname{im}(\varphi_p^{j'-1,j}) \oplus V^j$ (for a $j' \leq j$). In particular, elements of V^j are elements born at j'. Then, there is a basis $\mathcal{A} = \{a_1, \cdots, a_n\}$ of V^j such that if $i' \leq i$, then $a_{i'}$ dies before or at the same time than a_i and such that $\varphi_p^{j,t}$ is injective on $\rangle a_i, \cdots, a_n \langle$ if a_i dies after t.

Proof. We will prove it by induction on the dimension of V^j. The case $n = 1$ is similar to the Morse case. Suppose that we have the result for all V'^j of dimension strictly less than n and take V^j of dimension n.

Let t be the time of death of a non-zero element of V^j with the shortest lifespan. Let us show that $\varphi_p^{j,t'}$ is injective on V^j for all $t' < t$. Let $a \in \ker(\varphi_p^{j,t'}|_{V^j})\backslash\{0\}$. We then have $\varphi_p^{j,t'}(a) = 0 \in \operatorname{im}(\varphi_p^{j-1,t'})$, a contradiction with the minimality of t. Therefore, $\varphi_p^{j,t'}$ is injective on V^j and in particular, $\varphi_p^{j,t-1}$ is injective on V^j. Let $W = \varphi_p^{j,t-1}(V^j)$ we have that $\varphi_p^{j,t-1}|_{V^j}\colon V^j \xrightarrow{\sim} W$ is an isomorphism and W has the same dimension as V^j.

Let us consider V^t such that

$$\operatorname{im}(\varphi_p^{j',t}) = \operatorname{im}(\varphi_p^{j'-1,t}) \oplus V^t \text{ and } \varphi_p^{j,t}(V^j) \supset V^t.$$

Such a vector space exists as we have $\operatorname{im}(\varphi_p^{j',t}) = \operatorname{im}(\varphi_p^{j'-1,t}) + \varphi_p^{j,t}(V^j)$. Moreover, we have $\dim(V^t) < \dim(V^j)$ as by definition of t, there is a non-zero element $\alpha \in V^j$ that dies entering t, therefore such that $\varphi_p^{j,t}(\alpha) \in \operatorname{im}(\varphi_p^{j'-1,t})$. We then either have $\varphi_p^{j,t}(\alpha) = 0$ and $\dim(V^j) > \dim(\varphi_p^{j,t}(V^j))$ or $\varphi_p^{j,t}(\alpha) \neq 0$ and $\varphi_p^{j,t}(\alpha) \notin V^t$, otherwise the sum would not be direct.

By induction, there is a basis $\mathcal{B} = \{b_{m+1}, \cdots, b_n\}$ of V^t respecting the conditions. As $b_{m+1}, \cdots, b_n$ are elements of V_t, they are in particular elements of $\varphi_p^{j,t}(V^j)$ and therefore we can take $a_{m+1}, \cdots, a_n$ elements of V^j such that $\varphi_p^{j,t}(a_i) = b_i$ for all $m+1 \leq i \leq n$. They trivially form a set of linearly independent elements.

Consider W' such that

$$\rangle \varphi_p^{j,t-1}(a_{m+1}), \cdots, \varphi_p^{j,t-1}(a_n) \langle \, \oplus W' = W = \varphi_p^{j,t-1}(V^j).$$

As $\dim(W) = \dim(V^j)$, we have $\dim(W') = m$. Let $\{c_1, \cdots c_m\}$ be a basis of W. Moreover, $\varphi_p^{t-1,t}(c_i) = u_i + \sum_{s=m+1}^{n} \mu_s b_s$ with $u \in \operatorname{im}(\varphi_p^{j'-1,t})$. Then consider $c'_i = c_i - \sum_{s=m+1}^{n} \mu_s \varphi_p^{j,t-1}(a_s)$. Then the c'_i will still form a linearly independent subset of W such that $W = \rangle c'_1, \cdots, c'_m \langle \, \oplus \, \rangle \varphi_p^{j,t-1}(a_{m+1}), \cdots, \varphi_p^{j,t-1}(a_n) \langle$. Defining $a_i \in V^j$ such that $\varphi_p^{j,t-1}(a_i) = c'_i$ for all $1 \leq i \leq m$, we have formed a basis of V^j such that $a_1, \cdots, a_m$ are elements dying at t as $\varphi_p^{j,t}(a_i) = u_i \in \operatorname{im}(\varphi_p^{j'-1,t})$ for all $1 \leq i \leq m$.

The basis $\mathcal{A} = \{a_1, \cdots, a_n\}$ of V^j that we constructed respects the conditions. It is such that if $i' \leq i$, then $a_{i'}$ dies before or at the same time as a_i. Moreover, $\varphi_p^{j,t'}$ is injective on V^j for $t' < t$ and then, as $\varphi_p^{j,t}$ is a bijection on $\rangle a_{m+1}, \cdots, a_n \langle$, we have the rest of the condition for $t' \geq t$. □

REMARK 1.2.12. Using the previous lemma, by tweaking a bit the V^j, we can find a V'^j with nicer properties.

If $\mathcal{A} = \{a_1, \cdots, a_n\}$ is a basis of V^j respecting the conditions of the lemma, let $\varphi_p^{j,k}(a_i) = \varphi_p^{j'-1,k}(b_i)$ for some $b_i \in H_p(\mathbb{X}^{j'-1})$ if a_i dies at k. Then we can construct the basis $\mathcal{A}' = \{a'_1, \cdots, a'_n\}$ of V'^j by defining $a'_i = a_i - \varphi_p^{j'-1,j}(b_i)$. It will still be a linearly independent set such that $\operatorname{im}(\varphi_p^{j',j}) = \operatorname{im}(\varphi_p^{j'-1,j}) \oplus V'^j$. Moreover, the basis $\mathcal{A}'$ respects the same condition as above but, in addition, for all $0 \leq i \leq n$, if a'_i dies at k, then $\varphi_p^{j,t}(a'_i) = 0$ for all $k \leq t$.

To compute the multiplicity $\mu_p^{j,k}$ of the point (t_j, t_k), we will use the persistent Betti numbers. Let V_j be such that $H_p(\mathbb{X}^j) = \operatorname{im}(\varphi_p^{j-1,j}) \oplus V_j$. As we work with tame functions, we have that $\dim V_i < \infty$. Using the previous lemma, we will take as a basis of V_j the basis $\{a_1, \cdots a_p, b_1, \cdots b_q, c_1 \cdots c_r\}$ such that the a_i are elements dying before j, the b_i are elements dying at k and the c_i are elements dying after k. We therefore want a formula to isolate the b_i. We have

$$\operatorname{im}(\varphi_p^{j,k-1}) = \rangle \varphi_p^{j,k-1}(b_1), \cdots, \varphi_p^{j,k-1}(b_q), \varphi_p^{j,k-1}(c_1), \cdots, \varphi_p^{j,k-1}(c_r) \langle \, \oplus \, \operatorname{im}(\varphi_p^{j-1,k-1})$$

as the $\varphi_p^{j,k-1}(a_i)$ are in $\operatorname{im}(\varphi_p^{j-1,k-1})$. Similarly, we have

$$\operatorname{im}(\varphi_p^{j,k}) = \rangle \varphi_p^{j,k}(c_1), \cdots, \varphi_p^{j,k}(c_r) \langle \, \oplus \, \operatorname{im}(\varphi_p^{i-1,j}).$$

Moreover, still using the previous lemma, we know that the elements of the set $(\varphi_p^{j,k-1}(b_1), \cdots, \varphi_p^{j,k-1}(b_q), \varphi_p^{j,k-1}(c_1), \cdots \varphi_p^{j,k-1}(c_r))$ are linearly independent. Similarly, the $(\varphi_p^{j,k}(c_1), \cdots \varphi_p^{j,k}(c_r))$ are linearly independent.

Therefore, we have

$$\begin{aligned}\beta_p^{i,j-1} - \beta_p^{i-1,j-1} - \beta_p^{i,j} + \beta_p^{i-1,j} &= q + r + \dim(\mathrm{im}(\varphi_p^{i,j-1})) - \dim(\mathrm{im}(\varphi_p^{i,j-1})) \\ &\quad -(r + \dim(\mathrm{im}(\varphi_p^{i,j}))) + \dim(\mathrm{im}(\varphi_p^{i,j-1})) \\ &= q.\end{aligned}$$

It means that we can compute $\mu_p^{j,k}$ thanks to the formula

$$\mu_p^{j,k} = \beta_p^{i,j-1} - \beta_p^{i-1,j-1} - \beta_p^{i,j} + \beta_p^{i-1,j}.$$

1.2.3 The Data Inference Problem

Suppose we have a finite set of points X sampled (in a uniform way) on an unknown compact manifold M in a bounded metric space (T, ρ). We would like to infer the topology of that manifold. The following method, explained in [49], gives an efficient way to infer the homology of the manifold. For every positive number $\epsilon > 0$, we can consider

$$X^{(\epsilon)} = \{t \in T : \rho(t, X) \leq \epsilon\}$$

and analyse its topology. Of course, the topology of $X^{(\epsilon)}$ depends on ϵ: taking $\epsilon = 0$, we just have the set X of points, it has homology $H_0(X^{(\epsilon)}) = \mathbf{k}^{|X|}$ and $H_n(X^{(\epsilon)}) = 0$ for $n \neq 0$. Similarly, for ϵ big enough, $X^{(\epsilon)}$ will just be the whole space T. These two cases are not really insightful. However, probably that a better suited value of ϵ will give more relevant information. We then fall back to the problem raised in the introduction: the choice of parameter is a difficult one. We will therefore use a multiscale analysis.

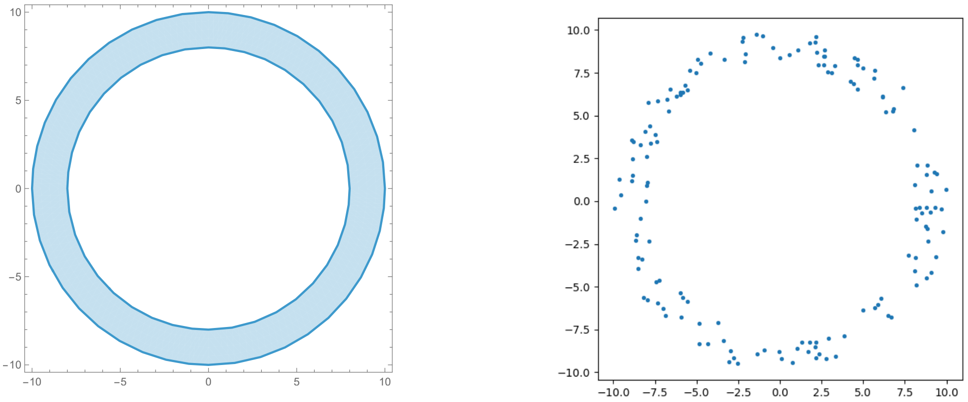


Figure 1.9: The annulus A (left) and 150 points randomly selected points of A (right).

Example 1.2.13. As a guiding example, we will try to compute the homology of the annulus A centered in $(0,0)$ with inner radius 8 and outer radius 10, thanks to 150 (pseudo-)randomly selected points[5] of A (see Figure 1.9). Let X be the set of randomly selected points.

We can see what the $X^{(\epsilon)}$ look like for different ϵ.

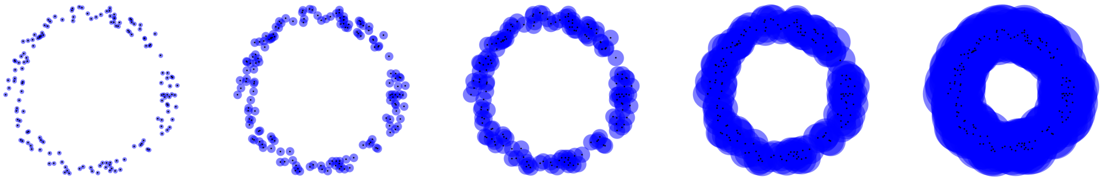


Figure 1.10: Representation of the sets $X^{(\epsilon)}$ for $\epsilon = 0.25, 0.5, 1, 2, 4$.

We will therefore compute the homology of $X^{(\epsilon)}$ for all $\epsilon \geq 0$. To compute the homology of $X^{(\epsilon)}$, the first step is to transform this space into a finite combinatorial model of its topology. A great theoretical tool to do that is through *simplices*. We will construct the *Vietoris*[6]*-Rips*[7]*sequence* from the set X by

$$R_\epsilon(X) = \{\{x_0, \cdots, x_k\} \subset X : k \in \mathbb{N}, \rho(x_i, x_j) \leq \epsilon \text{ for all } 0 \leq i, j \leq k\}.$$

More explicitly, the n-simplices $R_\epsilon(X)_n$ are the $\{\{x_0, \cdots, x_n\} \subset X : \rho(x_i, x_j) \leq \epsilon \text{ for all } 0 \leq i, j \leq k\}$ and the face maps are

$$d_i \colon R_\epsilon(X)_n \to R_\epsilon(X)_{n-1}, \{x_0, \cdots, x_i, \cdots, x_n\} \mapsto \{x_0, \cdots, \widehat{x_i}, \cdots, x_n\},$$

where $0 \leq i \leq n$ and $\widehat{x_i}$ means that we omit that term. Note that to be totally complete, we should also define degeneracy maps.

Another way to see $R_\epsilon(X)$ is to take all the subsets Y of X such that the intersection of all the circles centred at points of Y with radius $\epsilon/2$ is non-empty (and keep the same face maps).

Example 1.2.14. Let us look at Figure 1.11 at some Vietoris-Rips complexes obtained from the points of X.

Once we have a simplicial complex, we can compute its homology algorithmically. Indeed, for the simplicial set $R_\epsilon(X)$, we construct the chain complex $A_\bullet$ (see Definition B.5.4) by $A_n = 0$ for $n < 0$ and A_n is the free abelian group with coefficients in $\mathbf{k}$ on the set $R_\epsilon(X)_n$ (i.e. elements of A_n are formal linear combination of elements of $R_\epsilon(X)_n$ with coefficients in $\mathbf{k}$). For $0 \leq i \leq n$, we can extend the definition of d_i to A_n by

$$d_i\left(\sum_{x \in R_\epsilon(X)_n} k_x x\right) = \sum_{x \in R_\epsilon(X)_n} k_x d_i(x).$$

---

[5] The 150 points were selected using the function “RandomPoint” from Wolfram Mathematica.

[6] Leopold Vietoris (1891-2002) was an Austrian mathematician. He greatly contributed to the field of topology (both general and algebraic). He also lends his name to the Mayer-Vietoris sequence.

[7] Eliyahu Rips (1948-2024) was an Israeli mathematician specialised in geometric group theory. He is also known for co-authoring a paper known as the “Bible Code” that decrypts a supposed code hidden in the Torah.

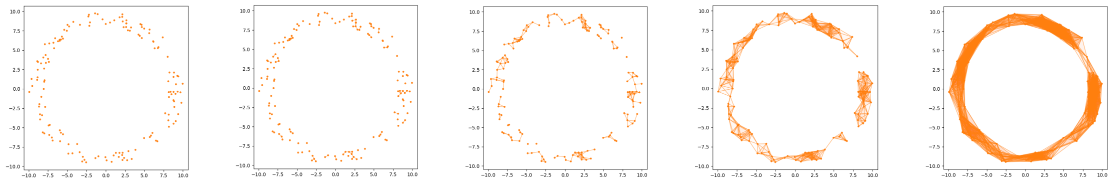


Figure 1.11: Representation of $R_\epsilon(X)$ for $\epsilon = 0.25, 0.5, 1, 2, 4$.

We then define the differential maps by $\partial_n = \sum_{i=0}^{n}(-1)^i d_i$ (it is standard to check the equality $\partial_n \partial_{n+1} = 0$). Once we have that complex A, we can compute its homology.

Moreover, for all $\epsilon < \epsilon'$, we have $R_\epsilon \subset R_{\epsilon'}$. In other words, we have a filtration of complexes. As in Definition 1.2.7, for $i \leq j$, we will also denote by $\varphi_p^{i,j}\colon H_p(R_i) \to H_p(R_j)$ the map induced by the inclusion $R_i \subset R_j$. Similarly to the previous case, we say that a p-homology class $\alpha \in H_p(R_t)$ is born at $i \leq t$ if $\alpha \in \mathrm{im}(\varphi_p^{i,t})$ and $\alpha \notin \mathrm{im}(\varphi_p^{i',t})$ for all $i' < i$. If α is born in i, we say that it dies entering j (with $j > t$) if $\varphi_p^{t,j'}(\alpha) \notin \mathrm{im}(\varphi_p^{i',j'})$ for all $i' < i$ and $j' < j$, and there is some $i' < i$ such that $\varphi_p^{t,j}(\alpha) \in \mathrm{im}(\varphi_p^{i',j})$.

Now that we have this notion of persistence, we can also construct a persistence diagram for X, which we will denote by $\mathrm{dgm}(X)$. Points "far" from the diagonal will represent homological features which are likely to be common with the manifold.

Example 1.2.15. The persistence diagram related to the set X is given in Figure[8] 1.12. There are some points near the diagonal, especially at the beginning. Those represent all the distinct points that each gives a different connect component and thus a different H_0 class. Those are "quickly" dying as the ϵ gets bigger. We also see some H_1 and H_2 classes near the diagonal due to some noise coming from the approximation of A by X. However, the two points that clearly stand out are the two points at the top left corner. They are far enough from the diagonal to justify the interpretation that A has probably as homology $H_n(A) = \mathbf{k}$ for $n = 0, 1$ and zero otherwise.

Remark 1.2.16. As X is finite, we do not need to compute R_ϵ for all $\epsilon \in \mathbb{R}^+$ as the simplicial complex will only change when we add some subset of points to it, which happens only when ϵ gets bigger than the distance between two points of X. As they are maximum $|X|(|X| - 1)$ of those, we only need to compute $O(n^2)$ Vietoris-Rips complex (where n is the number of points in X).

Example 1.2.17. Let us sample points around the figure ∞. The persistence diagram (Figure 1.13) also gives a clear indication of the homology of the shape.

More than just being able to infer the homology of the manifold, persistence diagrams also offer some insight into how the homology feature is "important" in the characterisation of the manifold.

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[8] I used the package scikit-tda [52] in python to have the different persistent diagram.

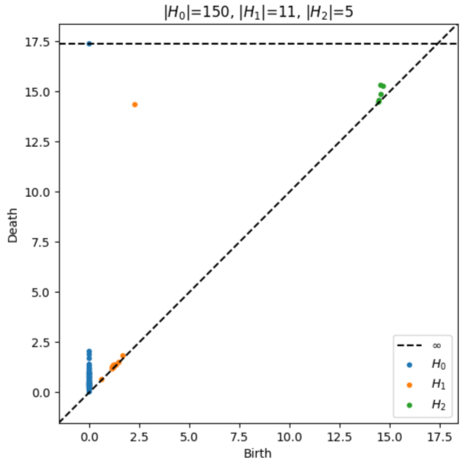


Figure 1.12: Persistence diagram related to the set X of points sampling the annulus A.

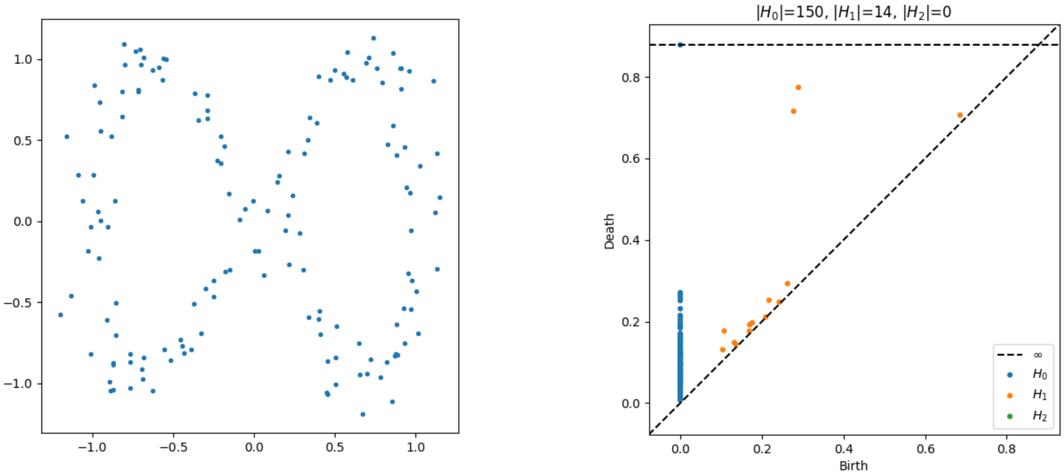


Figure 1.13: Points sampling the figure "∞" and their related persistence diagram.

Example 1.2.18. Let us go back to the example of the computation of the persistence diagram of the annulus. However, this time, we will consider the annulus A' centred in $(0, 0)$ with inner radius 4 and outer radius 10. The hole of the annulus is much less important in A' than in A. This translates to the persistence diagram (Figure 1.14) by having the point related to the H_1 class much closer to the diagonal as the shape is "closer" to being a disk with no H_1 homology.

This method is robust in the sense that if X is dense enough in M, then the persistence diagram related to X is close enough to the one of M. For the simplicity of the argument, we will assume that M is embedded in a compact subset T of R^n for some $n > 0$. We can define the functions

$$f : T \to \mathbb{R}, t \mapsto 2d(t, X)$$

and

$$g : T \to \mathbb{R}, t \mapsto 2d(t, M)$$

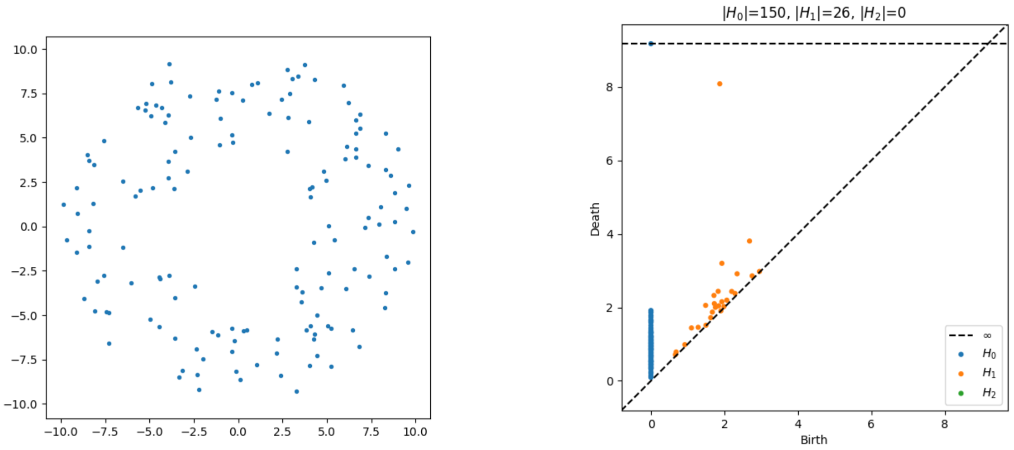


Figure 1.14: Points sampling A' and their related persistence diagram.

where d is the Euclidean distance. We have that $T_g^\epsilon = g^{-1}([0, \epsilon]) = X^{(\frac{1}{2}\epsilon)}$. We also know that the homology of $X^{(\frac{1}{2}\epsilon)}$ can be computed by R_ϵ. Therefore, the persistence diagram of g is the same as the one related to X. If f and g are Morse function, using Theorem 1.2.9, we then have that

$$d(\mathrm{dgm}(f), \mathrm{dgm}(X)) \leq ||f - g||_\infty.$$

In particular, if $X^{(\epsilon)} \supset M$, we have $|f(t) - g(t)| = 2|d(t, M) - d(t, X)|$. Moreover, as M is compact, there is $m \in M$ such that $d(t, M) = d(t, m)$. As $X^{(\epsilon)} \supset M$, we have that there is $x \in X$ such that $d(x, m) \leq \epsilon$. It implies that

$$d(t, M) \leq d(t, X) \leq d(t, x) \leq d(t, m) + \epsilon = d(t, M) + \epsilon.$$

Therefore, $||f - g|| \leq 2\epsilon$.

The only problem is that f and g are not necessarily Morse functions, thus Theorem 1.2.9 does not always apply. One of the goals of the field of topological data analysis is to generalise that theorem. We will provide such a generalisation in Section 4.5.2. Note that finer results concerning the persistence diagram of data point sets are developed in [17].

1.3 First Definitions

Let us now define more generally what persistent homology is. Afterwards, we will link persistent homology to persistence modules, allowing us to have algebraic tools to study persistent homology.

In order to properly define persistent homology, we need the notion of filtration.

> **Definition 1.3.1.** A *filtration* of a set S is an indexed family of sets S_t for $t \in T$ a totally ordered set such that $S_i \subset S_j$ if $i \leq j$.

I defined the notion of filtration of a set, but we can consider the filtration of vector spaces, ideals, etc. Note also that we ask T to be a totally ordered set, but there are some generalisations where the set is a poset.

Let us now define the persistent homology in generality.

Definition 1.3.2. Given a filtration $\mathcal{F}$ of topological spaces X_t (resp. filtration of complexes C_t) indexed by T, the *persistent homology* of degree n is the homology groups

$$\{H_n(X_t) : t \in T\}$$

(resp. $\{H_n(C_t) : t \in T\}$) with, for all $i \leq j \in T$, maps $f_i^j \colon H_n(X_i) \to H_n(X_j)$ (resp. $f_i^j \colon H_n(C_i) \to H_n(C_j)$) that are induced by the respective inclusions.

Note that here we did not specify the type of homology used to compute the groups $H_n(X_t)$. Changing the type of homology will give a different persistent homology. The focus of persistent homology is the filtration that is essential to define it. This notion of filtration is then used to have a notion of "lifespan" of some feature. Using the birth and death of elements, as defined above, we can construct persistence diagrams and barcodes.

A First Step Towards Algebra

In 2005, a shift in the study of persistent homology occurred: rather than studying them from a geometric and data analytic point of view, the focus was put on the algebraic side. Indeed, to a filtration of a space, we can associate a persistence module that will carry that same information as the persistent homology related to that filtration. The upside of the persistence module is that it is easier to manipulate and allows better generalisation. In this master thesis, we will focus on studying persistence modules.

Let us first recall the definition of a (left) R-module as it will be crucial in the rest of the document.

Definition 1.3.3. Let R be a ring (with identity). A left *module* is an abelian group $(M, +)$ endowed with an operation $\cdot \colon R \times M \to M$ that satisfies the following properties for all $r, s \in R$ and $x, y \in M$.

M1) $r \cdot (x + y) = r \cdot x + r \cdot y$,

M2) $(r + s) \cdot x = r \cdot x + s \cdot x$,

M3) $(rs) \cdot x = r \cdot (s \cdot x)$,

M4) $1 \cdot x = x$.

If M and N are two left R-modules, the map $\varphi \colon M \to N$ is a *morphism of left modules* if for all $r, s \in R$ and $x, y \in M$, we have

$$\varphi(r \cdot x + s \cdot y) = r \cdot \varphi(x) + s \cdot \varphi(y).$$

Modules can be seen as a generalisation of vector spaces where instead of working over a field, we work over a (possibly non-commutative) ring.

Persistence modules were first defined in 2005 in [58] as a family of R-modules M^i (for $i \in \mathbb{Z}$), together with homomorphisms $\varphi^i \colon M^i \to M^{i+1}$. The motive behind this definition is that if we have a persistent homology $(H_p(\mathbb{X}^i))_i$ of a tame function (with $H_p(-)$ being the homology over the ring R), then it forms a persistence module.

REMARK 1.3.4. If we define the operations + and $\cdot$ component-wise. It directly gives that the family M of R-modules M^i with homomorphisms $\varphi^i \colon M^i \to M^{i+1}$ is indeed a R-module.

REMARK 1.3.5. The module $M = (H_p(\mathbb{X}^i))_i$ is equivalent to the module

$$M' = H_p(\mathbb{X}^0) \oplus H_p(\mathbb{X}^1) \oplus \cdots \oplus H_p(\mathbb{X}^m) \tag{1.1}$$

on the ring $R[t]$ with an action by t being, for each $\alpha \in H_p(\mathbb{X}^i)$,

$$t^k \alpha = \begin{cases} \varphi_p^{i,i+k}(\alpha) & \text{if } i+k \leq m \\ 0 & \text{else.} \end{cases}$$

Indeed, first M' is a $R[t]$-module. First, $(M', +)$ is an abelian group. Then, let $P(t) = \sum_{i=0}^{n} r_i t^i \in R[t]$ and $x = (x_j)_{j \in \{0, \cdots, m\}} \in M'$. We have

$$P(t) \cdot x = \left(\sum_{k=0}^{\min(j,n)} \varphi_p^{j-k,j}(r_k \cdot x_{j-k}) \right)_j .$$

By the linearity of the definition of the operation and as each $H_p(\mathbb{X}^j)$ is an R-module, the conditions are fulfilled.

The 2 structures carry the same information: given the persistence module M, we can retrieve the module M' using the construction given above. Given the module M', we can use its decomposition to retrieve the $H_p(M^i)$ and

$$\varphi^i = (t \cdot -)|_{H_p(\mathbb{X}^j)} \colon H_p(\mathbb{X}^j) \to H_p(\mathbb{X}^{j+1}), x \mapsto t \cdot x.$$

Using categories, the concept of persistent modules was later generalised in [13] and [18]. The set $\mathbb{R}$ can be seen as a category with objects the real numbers and a unique morphism between p and q if and only if $p \leq q$ (see Example B.1.3 for more information).

DEFINITION 1.3.6. A *persistence module* over the real line is a functor from $\mathbb{R}$ to the category of $\mathbf{k}$-vector spaces.

More explicitly, it is a family of $\mathbf{k}$-vector spaces $(V_t : t \in \mathbb{R})$ as well as a doubly indexed family of morphisms $(v_s^t \colon V_s \to V_t | s \leq t)$ such that $v_s^t \circ v_r^s = v_r^t$ for all $r \leq s \leq t$ and $v_t^t = \mathrm{id}_{V_t}$.

If $R = \mathbf{k}$ is a field, this latter definition is clearly a generalisation of the first definition: let $\mathbb{M} = (M^i)_{i \in \mathbb{N}}$ be a family of R-modules with homomorphisms $\varphi^i \colon M^i \to M^{i+1}$. Then $\mathbb{V} = (V_t)_{t \in \mathbb{R}}$ with $V_t = M_{\lfloor t \rfloor}$ and

$$v_s^t = \begin{cases} \varphi^{\lfloor t \rfloor - 1} \circ \varphi^{\lfloor t \rfloor - 2} \circ \cdots \circ \varphi^{\lfloor s \rfloor} & \lfloor s \rfloor < \lfloor t \rfloor, \\ \mathrm{id}_{V_t} & \lfloor s \rfloor = \lfloor t \rfloor \end{cases}$$

defines a persistence module.

REMARK 1.3.7. Note that even though we work with $\mathbf{k}$-vector spaces at each level, the whole object is still a module and not a vector space as we will consider it as a $k[t]$-module.

Example 1.3.8. Let X be a topological space and $f\colon X \to \mathbb{R}$ a function. Consider the *sublevel sets* $X^t = \{x \in X : f(x) \leq t\}$ with the inclusion maps $i_s^t\colon X^s \to X^t$, call it the *sublevelset filtration* of (X, f) and denote it by $\mathbb{X}_{\text{sub}}$. Let $H_p(-, \mathbf{k})$ be the pth homology with coefficients in $\mathbf{k}$ functor. Then $\mathbb{V} = H_p(\mathbb{X}_{\text{sub}}, \mathbf{k})$ is a persistence module.

Remark 1.3.9. In the rest of this master thesis, H_p will always denote the p-homology functor over a field $\mathbf{k}$. We need this to have the different decomposition results that we will develop in Chapter 3.

If $(T, \leq)$ is a totally ordered set, it can be turned into a category $\mathcal{C}_T$ in the same way as $\mathbb{R}$. Similarly to the case with $\mathbb{R}$, we can define a T-persistence module as being a functor from $\mathcal{C}_T$ to $\mathbf{Vect_k}$. Usually, we work with subsets of $\mathbb{R}$.

Moreover, to an ordered set A, we can associate its *order topology* which has as basis of open subsets given by the subsets of the form $\{x : x < b\}$, $\{x : x > a\}$ and $\{x : a < x < b\}$. We say that A is *discrete* if the order topology on A is the discrete topology. Another way to put it is that for all $x \in A$, there must exists $a, b \in A$ such that $\{y : a < y < b\}$ only contains x, or x is a maximum of A and there is an $a \in A$ such that $\{y : y > a\}$ only contains x, or in the last case, x is a minimum of A and there is an $b \in A$ such that $\{y : y < b\}$ only contains x.

If the totally ordered set T is discrete, we can associate its Hasse quiver. This quiver will be linear in the sense that its underlying undirected multi-graph will be a graph and, more specifically, a tree (so connected and acyclic) with every vertex of degree at most two. Moreover, all the arrows are going in the same direction. Note that the quiver can be either finite or infinite. In that case, a T-persistence module can also be viewed as a quiver representation of the Hasse quiver (as defined in example 2.1.3) associated with T.

Example 1.3.10. The three types of Hasse quivers we will work with are the following. If $T = \{0, 1, 2, 3, 4, 5\}$ with the usual ordering, it has as its Hasse quiver the following quiver. It is an example of a finite quiver.

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] & 5. \end{tikzcd}$$

If $T = \mathbb{N}$, we have the following Hasse quiver

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & \cdots . \end{tikzcd}$$

If $T = \mathbb{Z}$, we have the following Hasse quiver

$$\begin{tikzcd} \cdots \arrow[r] & -2 \arrow[r] & -1 \arrow[r] & 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \cdots . \end{tikzcd}$$

Therefore, we can see persistence modules from different points of view depending on the situation or need: we can see them as graded modules, as functors, as a quiver representation, etc.

2 | Quiver Representation

Quivers are very simple objects, although they form an ubiquitous tool in Mathematics, especially representation theory. For example, persistence modules indexed by a finite set can be seen as a representation of a quiver, which is why we are interested in them in this master thesis. In this chapter, we will only present the foundation of these quivers and focus on results that we will link to persistence modules, mainly based on [53] and [24]. We will first define quivers and their representations. Afterwards, we will present their path algebra and link this algebra with the representations. We will also define projective representations and prove that the global dimension of the category of the representations of a (finite, connected, acyclic) quiver is at most 1. Finally, we will prove the main theorem of this chapter: Gabriel's theorem, which relates the indecomposable representations to roots of a quadratic form and allows us to describe the quivers of finite representation type. It will later allow us to state some properties regarding the decomposition of persistence modules.

In particular, we mainly focus on an algebraic approach to the subject, even though it has several deep links with geometry. We will also not dive into the subject of quivers with relations, despite it offering some profound results and a way to see every basic, connected, finite-dimensional **k**-algebra, see, for example, Chapter 4 of [4]. Recently, quivers have also been studied from a deformation theoretic point of view using the notion of Leavitt path algebra, for example, bringing several new results both for quivers and non-commutative algebra [19, 20].

2.1 Definitions and First Properties

Definition 2.1.1. A *quiver* Q is a 4-uple $Q = (Q_0, Q_1, s, t)$ consisting of

- Q_0 a set of vertices,
- Q_1 a set of arrows,
- $s\colon Q_1 \to Q_0$ a map sending each arrow to its starting point,
- $t\colon Q_1 \to Q_0$ a map sending each arrow to its terminal point.

A quiver is *finite* if both Q_0 and Q_1 are finite.

In other words, a quiver is a multi-graph. Moreover, unless specifically mentioned, we will only work with finite quivers.

If the arrow $\alpha \in Q_1$ is such that $s(\alpha) = i$ and $t(\alpha) = j$, we write it as $\alpha\colon i \to j$.

A quiver can also be defined categorically as a functor between the Kronecker category

$$\begin{tikzcd} E \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & V \end{tikzcd}$$

and the category **Set**.

Example 2.1.2. The 4-uple

$$\begin{aligned} Q = \;& (Q_0 = \{1,2,3\}, Q_1 = \{\alpha, \beta, \gamma, \lambda, \mu\}, \\ & s\colon (\alpha, \beta, \gamma, \lambda, \mu) \mapsto (1,1,3,3,3), t\colon (\alpha, \beta, \gamma, \lambda, \mu) \mapsto (1,2,2,2,1)) \end{aligned}$$

is a quiver and can be represented graphically by

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & 2 \\ & 3. \arrow[ul, "\mu"] \arrow[u, shift left, "\gamma"] \arrow[u, shift right, "\lambda"'] \end{tikzcd}$$

Example 2.1.3. A useful example of quivers is the *Hasse*[1] *quiver* as it allows us to see some poset from a different perspective. We use the definition of [57] and [55]. Let $(X, \leq)$ be a poset and $x, y \in X$. We say that x *covers* y if $x \neq y$, $y \leq x$, and there is no element $z \in X \backslash \{x, y\}$ such that $y \leq z \leq x$.

The *Hasse quiver* associated to $(X, \leq)$ is the quiver whose vertices are the elements of X and for which there is an arrow $x \to y$ if and only if y covers x.

Definition 2.1.4. A *representation* $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of a quiver Q is a collection of $\mathbf{k}$ vector spaces M_i for each vertex and a collection of linear maps $\varphi_\alpha\colon M_{s(\alpha)} \to M_{t(\alpha)}$ for each arrow $\alpha \in \mathbb{Q}_1$.

Such a representation is called *finite-dimensional* if the vector spaces M_i of the representation are all of finite dimension. Unless specifically mentioned, we will suppose that all the representations of quivers we work with are finite-dimensional.

Example 2.1.5. The following diagram

$$\begin{tikzcd} k^2 \arrow[loop left, "\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}"] & k^2 \\ & k \arrow[ul, "\begin{pmatrix} 1 \\ 1 \end{pmatrix}"] \arrow[u, shift left, "0"] \arrow[u, shift right, "\begin{pmatrix} 1 \\ -1 \end{pmatrix}"'] \end{tikzcd}$$

---

[1] Helmut Hasse (1898–1979) was a German mathematician who worked in algebraic number theory. He contributed to class field theory and Diophantine geometry by expressing the Hasse principle.

is a representation of the quiver Q. Note that the diagram is not commutative and does not need to be.

As always in Mathematics, after defining some objects, it is interesting to define morphisms between those objects.

Definition 2.1.6. Let Q be a quiver and $M = (M_i, \varphi_\alpha), M' = (M'_i, \varphi'_\alpha)$ be two representations of this quiver. A *morphism* between these representations is a collection of $\mathbf{k}$-linear maps $(f_i)_{i \in Q_0}$ such that for each arrow $i \xrightarrow{\alpha} j$,

$$f_j \circ \varphi_\alpha = \varphi'_\alpha \circ f_i.$$

In other words, the following square commutes

$$\begin{tikzcd} M_i \arrow[r, "\varphi_\alpha"] \arrow[d, "f_i"'] & M_j \arrow[d, "f_j"] \\ M'_i \arrow[r, "\varphi'_\alpha"'] & M'_j. \end{tikzcd}$$

Having defined the representations of a quiver and morphisms between those representations, we can define the category $\operatorname{Rep}(Q)$ of finite-dimensional representations of the quiver Q. The objects of this category are the finite-dimensional representations of Q, and the Hom-set $\operatorname{Hom}_{\operatorname{Rep}(Q)}(V, W)$ between two representations is the set of morphisms between representations.

We will denote by $\widetilde{\operatorname{Rep}}(Q)$ the category of not necessarily finite-dimensional representations of Q.

Definition 2.1.7. Let Q be a quiver and M a representation of Q. A *subrepresentation* of M is a representation M' such that for all $i \in Q_0$, we have $M'_i \subset M_i$ and for all $\alpha \colon i \to j \in Q_1$, $\varphi'_\alpha = (\varphi_\alpha)|_{M'_i}$.

We say that M is a *simple representation* of Q if there are no subrepresentations M' of M different from the zero representation and M itself.

Simple representations must not be confused with indecomposable representations, which are the following.

Definition 2.1.8. A representation M of the quiver Q is *indecomposable* if it can not be written as $M = M' \oplus M''$ with M', M'' non-zero representations of Q.

Note, however, that simple representations are indecomposable.

Definition 2.1.9. If $i \in Q_0$ is a vertex of Q, we denote by $S(i)$ the representation defined by

$$S(i)_j = \begin{cases} \mathbf{k} & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

and $\varphi_\alpha = 0$ for all $\alpha \in Q_1$. This representation is called the *simple representation associated to the vertex i*.

It is direct to see that $S(i)$ is a simple representation and that all simple representations of a quiver are of the form $S(i)$ for some $i \in Q_0$.

2.2 Path Algebra

A concept arising when studying quivers is one of paths: a way to go from one vertex to another by following arrows. A formal definition is the following.

Definition 2.2.1. Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $i, j \in Q_0$ be two vertices. A *path* c from i to j of length l is given by

$$c = (i|\alpha_1, \alpha_2, \cdots, \alpha_l|j)$$

such that

1. $\alpha_k \in Q_1$ for all $k \in \{1, 2, \cdots, l\}$
2. $s(\alpha_1) = i$
3. $t(\alpha_k) = s(\alpha_{k+1})$ for all $k \in \{1, 2, \cdots, l-1\}$
4. $t(\alpha_l) = j$

The path c can also be denoted by $\alpha_l \cdots \alpha_2\alpha_1$ (note the similarity with the composition of functions). We further denote by $s(p) = s(\alpha_1)$ and $t(p) = t(\alpha_l)$ the starting and terminal vertex of p respectively. We also denote by Q_l the set of paths of length l.

We can define for all vertices the *lazy path* starting at i as the path $(i||i)$ of length 0. We denote it by ϵ_i.

Definition 2.2.2. Let $c = (i|\alpha_1, \cdots, \alpha_l|j)$ and $c' = (i'|\beta_1, \cdots, \beta_k|j')$ be two paths. The *composition* of c followed by c' is

$$c' \circ c = \begin{cases} (i|\alpha_1, \cdots, \alpha_l, \beta_1, \cdots, \beta_k|j') & \text{if } j = i', \\ 0 & \text{else} \end{cases}$$

where 0 represents the empty path. We also define $p\epsilon_{s(p)} = p = \epsilon_{t(p)}p$.

Definition 2.2.3. Given a quiver Q, a *loop* is a non-trivial path (i.e. a path of length greater than or equal to 1) such that its starting and terminal points are the same. If the quiver Q has no loop, we say that it is *acyclic*.

Example 2.2.4. On the quiver Q presented in the last examples, $(3|\mu, \alpha, \alpha, \beta|2) = \beta\alpha\alpha\mu$, $(3|\gamma|2)$ and $(3||3) = \epsilon_3$ are all different paths. The path α is a loop.

Given a representation $M = (M_i, \varphi_\alpha)$ and a path $p = \alpha_l \cdots \alpha_1$ of the quiver Q, we can define the following $\mathbf{k}$-linear map from $M_{s(p)}$ to $M_{t(p)}$

$$\varphi_p := \varphi_{\alpha_l} \circ \cdots \circ \varphi_{\alpha_1}.$$

Note that, by going from commutative square to commutative square, if $f\colon M \to M'$ is a morphism of representation, then we have $\varphi'_p f_{s(p)} = f_{t(p)}\varphi_p$.

Using this notion of paths, we can define an algebra that is closely linked with the representations of the quiver Q.

DEFINITION 2.2.5. The *path algebra* $\mathbf{k}Q$ is a $\mathbf{k}$-algebra with basis all the paths of Q. Denote by $\langle p \rangle$ the element of $\mathbf{k}Q$ corresponding to the path p in Q. The multiplication of base elements is defined as the element corresponding to the composition of the paths:

$$\langle p \rangle \cdot \langle q \rangle = \begin{cases} \langle pq \rangle & \text{if } s(p) = t(q), \\ 0 & \text{else.} \end{cases}$$

In other words, elements of the path algebra are formal linear combinations of paths.

In the following, we will drop the $\langle \cdot \rangle$ to refer directly to p instead of $\langle p \rangle$. As the composition of paths is clearly associative, so is the multiplication of the path algebra.

The unit of the algebra is given by

$$1_{\mathbf{k}Q} = \sum_{i \in Q_0} \epsilon_i$$

as $p \cdot 1_{\mathbf{k}Q} = p \sum_{i \in Q_0} \epsilon_i = \sum_{i \in Q_0} p\epsilon_i = p\epsilon_{s(p)} = p$.

If A is an associative algebra with unit, we denote by A-mod the category of finite-dimensional left A-modules and by $\widetilde{A\text{-mod}}$ the category of left A-modules.

Let us explore the link between path algebra and quiver representations. This is given by the following proposition, whose proof (inspired by [24]) is not per se difficult, although pretty tedious to write as there are a lot of conditions to check. In the proof, we use the notation $V = (V_i, \varphi_\alpha)$ to denote a representation instead of M in order to make a clearer distinction between quiver representation and modules. Recall the definition of equivalence of categories (Definition B.1.10).

THEOREM 2.2.6. The categories $\mathrm{Rep}(Q)$ and $\mathbf{k}Q$-mod are equivalent. Similarly, the categories $\widetilde{\mathrm{Rep}(Q)}$ and $\widetilde{\mathbf{k}Q\text{-mod}}$ are equivalent.

Proof. Let us first prove the possibly infinite-dimensional case; the finite-dimensional case will be a direct consequence.

Let us first construct the functor $F\colon \widetilde{\mathbf{k}Q\text{-mod}} \to \widetilde{\mathrm{Rep}(Q)}$. Let M be a $\mathbf{k}Q$ left module. We define

$$F(M) = \left(V_i = \epsilon_i \cdot M, \varphi_\alpha\colon \epsilon_i M \to \epsilon_j M, \epsilon_i m \mapsto \epsilon_j \alpha \epsilon_i m \ (\alpha\colon i \to j \in Q_1)\right).$$

It is a representation of Q as each $\epsilon_i M$ are $\mathbf{k}$-vector spaces and φ_α is a linear map. Let $f\colon M \to M'$ be a morphism of left $\mathbf{k}Q$-modules. We define $F(f)$ by $F(f)_i = f|_{\epsilon_i M}$. It is well defined as $f(\epsilon_i m) = \epsilon_i f(m) \in \epsilon_i M'$, where the equality is due to the fact that f is a morphism of $\mathbf{k}Q$-modules. It is furthermore a morphism of representations as each $F(f)_i$ are **k**-linear maps and if $\alpha\colon i \to j \in Q_1$, we have

$$\begin{aligned}F(f)_j \circ \varphi_\alpha(\epsilon_i m) &= F(f)_j(\epsilon_j \alpha \epsilon_i m)\\ &= f(\epsilon_j \alpha \epsilon_i m)\\ &= \epsilon_j \alpha \epsilon_i f(m)\\ &= \varphi'_\alpha(\epsilon_i f(m))\\ &= \varphi'_\alpha(f(\epsilon_i m))\\ &= \varphi'_\alpha \circ F(f)_i(\epsilon_i m)\end{aligned}$$

which proves that it is a morphism of representation. Moreover, using the definition of the image by F of the morphism of $\mathbf{k}Q$-modules using restriction, we directly have that $F(f \circ g) = F(f) \circ F(g)$ and $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$. Therefore, F is a functor.

Let us now construct the functor $G\colon \widetilde{\mathrm{Rep}(Q)} \to \widetilde{\mathbf{k}Q\text{-mod}}$. Let $V = (V_i, \varphi_\alpha)$ be a representation of M. We define

$$G(V) = M := \bigoplus_{i \in Q_0} V_i.$$

It forms an abelian group with the addition defined term by term in the direct sum as all the V_i are **k**-vector spaces. We endow it with the operation $\cdot\colon \mathbf{k}Q \times M \to M$ defined on the basis elements by $\epsilon_j \cdot m = i_j \pi_j(m)$ and

$$p \cdot m = i_{t(p)} \varphi_p \pi_{s(p)}(m)$$

where $i_j\colon V_j \to \bigoplus_{i \in Q_0} V_i$ is the canonical injection and $\pi_j\colon \bigoplus_{i \in Q_0} V_i \to V_j$ is the canonical projection. The operation is then extended by **k**-linearity to $\mathbf{k}Q$. Let us check the conditions of the definition of modules (Definition 1.3.3). From the definition of the operations, it is clear that the conditions M1 and M2 are satisfied as all morphisms are **k**-linear. Let us check that the conditions M3 and M4 are respected. It suffices to show the relation on the elements of the base. Let p, q be two paths and $m \in M$. Suppose first that $s(q) = t(p)$, it implies that we have $\pi_{s(p)} i_{t(p)} = \mathrm{id}_{V_{t(p)}}$ and we have

$$\begin{aligned}(qp) \cdot m =& i_{t(qp)} \varphi_{qp} \pi_{s(qp)}(m)\\ =& i_{t(q)} \varphi_q \varphi_p \pi_{s(p)}(m)\\ =& i_{t(q)} \varphi_q \pi_{s(q)} i_{t(p)} \varphi_p \pi_{s(p)}(m)\\ =& i_{t(q)} \varphi_q \pi_{s(q)}(p \cdot m)\\ =& q \cdot (p \cdot m).\end{aligned}$$

In the case where $s(q) \neq t(p)$, $qp = 0$ as well as $\pi_{s(q)} i_{t(p)} = 0$, therefore,

$$(qp) \cdot m = 0 = i_{t(q)} \varphi_q \pi_{s(q)} i_{t(p)} \varphi(p) \pi_{s(p)}(m) = q \cdot (p \cdot m).$$

For condition M4, we have

$$1 \cdot m = \left(\sum_{j \in Q_0} \epsilon_j \right) \cdot m = \sum_{j \in Q_0} (\epsilon_j \cdot m) = \sum_{j \in Q_0} i_j \pi_j(m) = m.$$

It implies that $G(V)$ is a $\mathbf{k}Q$ left module. Let $f\colon V \to V'$ be a morphism of representations. We define

$$G(f) = \bigoplus_{j\in Q_0} f_j = \sum_{j\in Q_0} i_j f_j \pi_j \colon M \to M'.$$

We need to check that it is a morphism of modules. First, we have

$$G(f)(\epsilon_j{\cdot}m) = \sum_{k\in Q_0} i_k f_k \pi_k(i_j\pi_j(m)) = i_j f_j \pi_j(m) = i_j\pi_j\left(\sum_{k\in Q_0} i_k f_k \pi_k(m)\right) = \epsilon_j{\cdot}G(f)(m)$$

and similarly,

$$\begin{aligned}
G(f)(p\cdot m) &= \sum_{k\in Q_0} i_k f_k \pi_k(i_{t(p)}\varphi_p \pi_{s(p)}(m))\\
&= i_{t(p)} f_{t(p)} \varphi_p \pi_{s(p)}(m)\\
&= i_{t(p)} \varphi'_p f_{s(p)} \pi_{s(p)}(m)\\
&= i_{t(p)} \varphi'_p \pi_{s(p)}\left(\sum_{k\in Q_0} i_k f_k \pi_k(m)\right)\\
&= p\cdot G(f)(m)
\end{aligned}$$

where the third equality comes from the fact that f is a morphism of representations. Similarly to the functor F, it is direct from the definition of $G(f)$ that G is a functor.

It now only remains to show that the compositions of these functors are naturally equivalent to the relevant identity functors.

We have

$$GF(M) = G((\epsilon_j M, \varphi_\alpha)) = \bigoplus_{j\in Q_0} \epsilon_j M.$$

Denote by η_M the isomorphism

$$\eta_M = \bigoplus_{j\in Q_0} \epsilon_j M \to M, (\epsilon_j m_j)_{j\in Q_0} \mapsto \sum_{j\in Q_0} \epsilon_j m_j.$$

In the meantime, if $f\colon M \to M'$, we have

$$GF(f) = G((f|_{\epsilon_j M})_j) = \bigoplus_{j\in Q_0} f|_{\epsilon_j M} = \sum_{j\in Q_0} i_j f|_{\epsilon_j M} \pi_j.$$

It implies that if $(\epsilon_j m_j)_{j\in Q_0} \in GF(M)$, we have

$$\begin{aligned}
\eta_{M'} GF(f)((\epsilon_j m_j)_{j\in Q_0}) &= \eta_{M'}\left(\bigoplus_{k\in Q_0} f|_{\epsilon_k M}\left((\epsilon_j m_j)_{j\in Q_0}\right)\right)\\
&= \eta_{M'}\left((f|_{\epsilon_j M}(\epsilon_j m_j))_{i\in Q_0}\right)\\
&= \sum_{j\in Q_0} f|_{\epsilon_j M}(\epsilon_j m_j)\\
&= f(\sum_{j\in Q_0} \epsilon_j m_j)\\
&= f\eta_M((\epsilon_j m_j)_{j\in Q_0}).
\end{aligned}$$

Therefore, we have that $\eta\colon GF \to \mathrm{id}$ is a natural transformation and as η_M is an isomorphism for all M, we have that GF is naturally equivalent to the identity functor of the category $\mathbf{k}Q$-mod.

On the other hand, we have

$$FG(V) = (V_i, \varphi_\alpha) = F\left(\bigoplus_{i\in Q_0} V_i\right) = \left(\left(\epsilon_i \bigoplus_{k\in Q_0} V_k\right)_{i\in Q_0}, \psi_\alpha\right).$$

with $\psi_\alpha\colon \epsilon_j \bigoplus_{k\in Q_0} V_k \to \epsilon_{j'} \bigoplus_{k\in Q_0} V_k, \epsilon_j \cdot (m_k)_{k\in Q_0} \mapsto \epsilon_{j'} \cdot (\alpha\epsilon_j \cdot m_k)_{k\in Q_0}$ for the arrow $\alpha\colon j \to j' \in Q_1$ (we changed the usual starting and terminal vertices of α to avoid confusion with the map i).

We further have, for $f : V \to V'$,

$$FG(f) = F\left(\bigoplus_{i\in Q_0} f_i\right) = \left(\left(\bigoplus_{i\in Q_0} f_i\right)\Bigg|_{\epsilon_k(\bigoplus_{j\in Q_0} V_j)}\right)_{k\in Q_0}.$$

From the isomorphisms

$$\epsilon_j \cdot \left(\bigoplus_{k\in Q_0} V_k\right) \simeq \bigoplus_{k\in Q_0} \epsilon_j \cdot V_k \simeq \bigoplus_{k\in Q_0} i_j\pi_j V_k \simeq V_j,$$

we define the isomorphism of representations $\eta_V\colon FG(V) \to V$ by

$$(\eta_V)_j(\epsilon_j \cdot (m_k)_{k\in Q_0}) = m_j.$$

It is indeed a morphism of representation as

$$\begin{aligned}
(\eta_V)_{j'}\psi_\alpha(\epsilon_j \cdot (m_k)_{k\in Q_0}) &= (\eta_V)_{j'}(\epsilon_{j'} \cdot (\alpha\epsilon_j \cdot m_k)_{k\in Q_0}) \\
&= (\eta_V)_{j'}(\epsilon_{j'} \cdot (i_{j'}\varphi_\alpha\pi_j(m_k))_{k\in Q_0}) \\
&= (\eta_V)_{j'}(\epsilon_{j'} \cdot (i_{j'}(\varphi_\alpha(m_j)))) \\
&= \varphi_\alpha(m_j) \\
&= \varphi_\alpha(\eta_V)_j(\epsilon_j \cdot (m_k)_{k\in Q_0}).
\end{aligned}$$

The last thing to check is the naturality of η, i.e. that $\eta_{V'}FG(f) = f\eta_V$ for all morphisms of representation $f\colon V \to V'$. It follows from

$$\begin{aligned}
(\eta_{V'}FG(f))_j(\epsilon_j \cdot (m_k)_{k\in Q_0}) &= (\eta_{V'})_j(FG(f))_j(\epsilon_j \cdot (m_k)_{k\in Q_0}) \\
&= (\eta_{V'})_j\left(\left(\bigoplus_{i\in Q_0} f_i\right)|_{\epsilon_j(\bigoplus_{k\in Q_0} V_k)}\right)(\epsilon_j \cdot (m_k)_{k\in Q_0}) \\
&= (\eta_{V'})_j\left(f_k|_{\epsilon_j(\bigoplus_{j\in Q_0} V_j)}(\epsilon_j \cdot (m_k))\right)_{k\in Q_0} \\
&= (\eta_{V'})_j\left(f_k i_j\pi_j(m_k)\right)_{k\in Q_0} \\
&= (\eta_{V'})_j\left(f_k(m_j)\right)_{k\in Q_0} \\
&= (\eta_{V'})_j\left(f_j(m_j)\right)
\end{aligned}$$

$$\begin{aligned}&= (\eta_{V'})_j\,(\epsilon_j \cdot (f_k(m_j))_{k\in Q_0})\\ &= f_j(m_j)\\ &= f_j(\eta_V)_j(\epsilon_j \cdot (m_k)_{k\in Q_0}).\end{aligned}$$

The case of the finite-dimensional representations and left modules is a consequence of the above proof, as, upon direct inspection, the functors F and G send finite-dimensional representations to finite-dimensional left modules and vice versa. □

This theorem implies that we can use tools from non-commutative algebra and module theory to study the representations of quivers.

2.3 Projective Representation

In category theory, a *projective* object of an abelian category $\mathcal{C}$ is an object P such that the functor $\mathrm{Hom}_{\mathcal{C}}(P, -)$ is exact (see Definition B.5.1). In other words, it means that for all epimorphisms $g\colon M \twoheadrightarrow N$ and morphisms $f\colon P \to N$, there is a function $h\colon P \to M$ such that $f = gh$. It can also be described as the following commutative diagram (note that the function h is not necessarily unique).

$$\begin{tikzcd} & P \arrow[ld, dashed, "h"'] \arrow[d, "f"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0. \end{tikzcd}$$

Definition 2.3.1. For a quiver Q and $i \in Q_0$, we define the *projective representation associated to the vertex* i as

$$P(i) = (P(i)_j, \varphi_\alpha)$$

with $P(i)_j$ a $\mathbf{k}$-vector space with basis all the paths from i to j. For all arrows $\alpha\colon j \to k \in Q_1$, $\varphi_\alpha\colon P(i)_j \to P(i)_k$ is a linear map defined on the basis by composing the path from i to j with α.

In other words,
$$\varphi_\alpha(\sum_c \lambda_c \cdot c) = \sum_c \lambda_c \cdot \alpha c.$$

Note that if Q is not acyclic, then $P(i)$ might be infinite-dimensional. In the following, we will work with acyclic quivers (i.e. quivers without loops).

Let us show that $P(i)$ is indeed projective and hence the definition makes sense.

Proposition 2.3.2. The representation $P(i)$ is projective.

Proof. Let $M = (M_j, \psi_\alpha)$ and $N = (N_j, \tau_\alpha)$ be two representations of the quiver Q. Let $f\colon P(i) \to N$ be any morphism of representation and $g\colon M \twoheadrightarrow N$ a surjective morphism of representation. Let us construct $h\colon P(i) \to M$ such that $f = gh$.

$$\begin{tikzcd}
 & P(i) \arrow[d, "f"] \arrow[dl, dashed, "h"'] \\
M \arrow[r, two heads, "g"'] & N.
\end{tikzcd}$$

Let c be a path from i to j. We will construct $h_j(1 \cdot c)$ by induction on the length of c.

If $|c| = 0$, we have $c = \epsilon_i$. We have $f_i(1 \cdot \epsilon_i) \in N_i$. As g is surjective, we have g_i surjective, thus there is $t_{\epsilon_i} \in M_i$ such that $g(t_{\epsilon_i}) = f(1 \cdot \epsilon_i)$. Then define $h_i(1 \cdot \epsilon_i) = t_{\epsilon_i}$.

Now, suppose that for all vertex $j \in Q_0$ and path c of length less than n starting in i and ending in j, we have defined $h_j(1 \cdot c) = t_c \in M_j$ such that $f_j(1 \cdot c) = g_j(t_c)$. Let c be a path of length n from i to j. We have $c = \alpha c'$ with c' a path from i to k of length $n - 1$. By induction, we have defined $h_k(1 \cdot c') = t_{c'} \in M_k$ such that $g_k(t_{c'}) = f_k(1 \cdot c')$. As f is a morphism of representation, we have the following commutative diagram:

$$\begin{tikzcd}
P(i)_k \arrow[r, "\varphi_\alpha"] \arrow[d, "f_k"'] & P(i)_j \arrow[d, "f_j"] \\
N_k \arrow[r, "\tau_\alpha"'] & N_j.
\end{tikzcd}$$

Hence,

$$f_j(1 \cdot c) = f_j(\varphi_\alpha(1 \cdot c')) = \tau_\alpha(f_k(1 \cdot c')).$$

As g is a morphism of representation, we have the diagram

$$\begin{tikzcd}
M_k \arrow[r, "\psi_\alpha"] \arrow[d, "g_k"'] & M_j \arrow[d, "g_j"] \\
N_k \arrow[r, "\tau_\alpha"'] & N_j.
\end{tikzcd}$$

Therefore, $g_i \psi_\alpha(t_{c'}) = \tau_\alpha g_k(t_{c'})$.

Let us define $t_c = h_j(1 \cdot c)$ as $t_c = \psi_\alpha(t_{c'})$. We then have

$$g_j h_j(1 \cdot c) = g_j(\psi_\alpha(t_{c'})) = \tau_\alpha g_k(t_{c'}) = \tau_\alpha(f_k(1 \cdot c')) = f_j(1 \cdot c).$$

Using this definition of h on the basis and extending it by linearity, we have by linearity of f_j and g_j the $f = gh$.

It remains to show that h is a morphism of representation. Let $x = \sum_c \lambda_c \cdot c \in P(i)_j$. We have (where the sum is over all the paths from i to j)

$$h_j(\varphi_\alpha(x)) = \sum_c \lambda_c h_j(\varphi(c)) = \sum_c \lambda_c h_j(\alpha c) = \sum_c \lambda_c t_{\alpha c} = \sum_c \lambda_c \psi_\alpha(t_c) = \psi_\alpha(h_k(x)).$$

Therefore, the following diagram commutes, and we have our result.

$$\begin{tikzcd}
P(i)_k \arrow[r, "\varphi_\alpha"] \arrow[d, "h_k"'] & P(i)_j \arrow[d, "h_j"] \\
M_k \arrow[r, "\psi_\alpha"'] & M_j.
\end{tikzcd}$$

□

We even have the stronger following result.

> **Proposition 2.3.3.** If $M = (M_i, \psi_\alpha)$ is a representation of an acyclic graph Q then for all $i \in Q_0$, we have
>
> $$\mathrm{Hom}(P(i), M) \simeq M_i.$$

Proof. Let ϵ_i be the lazy path at i. As Q is acyclic, a basis for $P(i)_i$ is $\{\epsilon_i\}$. Let us define the map

$$\Psi \colon \mathrm{Hom}(P(i), M) \to M_i, \;\; f = (f_j)_{j \in Q_0} \mapsto f_i(\epsilon_i).$$

It is direct to see that Ψ is linear.

Let us show that Ψ is injective. If $f \in \ker(\Psi)$, it means that $f_i(\epsilon_i) = 0$ and $\epsilon_i \in \ker(f_i)$. As $\{\epsilon_i\}$ is a basis for $P(i)_i$, it implies that $f_i = 0$. We also know that $(P(i))_j$ has as basis all the paths c from i to j. Let $\psi_c \colon V_i \to V_j$ and $\phi_c \colon P(i)_i \to P(i)_j$ be the morphisms associated to the path c in the representations V and $P(i)$, we have $f_j \phi_c = \psi_c f_i$. Moreover, we have that $\phi_c(\epsilon_i) = c$. Hence,

$$f_j(c) = f_j(\phi_c(\epsilon_i)) = \psi_c(f_i(\epsilon_i)) = 0.$$

It then implies that $f = 0$ and Ψ is injective.

It remains to show that Ψ is surjective. Let $m \in M_i$. We can define g_i by $g_i(\epsilon_i) = m$ and extend it by linearity, as $\{\epsilon_i\}$ is a basis for $P(i)_i$, the function is well defined. We then extend g_i to $g \colon P(i) \to M$ by $g_j(c) = \psi_c(m)$. ☐

Projective objects are objects of central interest in homological algebra as they are used to create projective resolutions of objects; those sequences are then used to compute derived functors, the study of which provides crucial information. For more details about homological algebra, see [56]. Some more information about projective objects is given in Section B.5.

We will show that the global dimension (see Definition B.5.10) of the category of representations of quivers is less than or equal to 1. In the proof inspired by Theorem 2.15 of [53], we will call the *degree* of a linear combination of paths as being the length of the longest path with a non-zero coefficient.

> **Proposition 2.3.4.** Let Q be a finite, acyclic, connected quiver. Then the category of finite representation of Q is of global dimension less than 1. In other words, for all M representations of Q, one can construct an exact sequence
>
> $$0 \to P_1 \to P_0 \to M \to 0$$
>
> with P_1 and P_0 projective representations.

After the proof, an example of the construction of the resolution developed in the proof is given. I strongly advise following the proof along with the example, as the definition of the maps can be quite disconcerting.

Proof. Let $M = (M_i, \varphi_i)$. Let us denote $d_i = \dim(M_i)$ for all $i \in Q_0$. Define the representations $P_0 = ((P_0)_i, \psi_\alpha)$ and $P_1 = ((P_1)_i, \tau_\alpha)$ by

$$\begin{array}{rcl} P_1 & = & \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} \cdot P(t(\alpha)) \\ P_0 & = & \bigoplus_{i \in Q_0} d_i \cdot P(i) \end{array}$$

where $d \cdot P(i)$ means the direct sum of d copies of $P(i)$, and ψ_α, τ_α are defined on each component as for the projective representation. By Proposition B.5.3, as P_1 and P_0 are the direct sum of projective resolutions, they are also projective. Let us now construct the different morphisms. To do that, let us first define some basis of the different vector spaces of the representation. Let $\{m_{i1}, \cdots, m_{\mathrm{id}_i}\}$ be a basis for M_i, then define

$$\mathcal{B}'' = \{m_{ij} | i \in Q_0, 1 \leq j \leq d_i\}.$$

In other words, $\mathcal{B}''$ is the union of all the bases of the M_i. It is a basis of M. Take

$$\mathcal{B} = \{c_{ij} | i \in Q_0, c_i \text{ is a path with } s(c_i) = i \text{ and } 1 \leq j \leq d_i\}$$

as a basis for P_0, it means that for each vertex $i \in Q_0$, we take every paths starting from that vertex d_i times. The different maps ψ_α are defined component-wise: $\psi_\alpha(c_{ij}) = (\alpha c_i)_j$ using the convention that if $s(\alpha) \neq t(c_i)$, the composition is null.

Finally, for P_1, we take as basis

$$\mathcal{B}' = \{b_{\alpha j} | \alpha \in Q_1, b_\alpha \text{ a path with } s(b_\alpha) = t(\alpha) \text{ and } 1 \leq j \leq d_{s(\alpha)}\}.$$

Similarly than for ψ_α, we have $\tau_\alpha(b_{\beta i}) = (\alpha b_\beta)_i$.

Note that $B = \{b_\alpha : \alpha \in Q_1\} \subset \{c_i : i \in Q_0\} = C$. When we want to consider an element b_α of B as an element of C, we will denote it by $\widetilde{b_\alpha}$. In particular, we have $\widetilde{\alpha b_\beta} = \alpha \widetilde{b_\beta}$. Note also that there might be several paths starting from the same vertex. therefore the notation c_i and b_α might be used for several elements.

Let us now define the morphisms between those representations. Let

$$g \colon P_0 \to M, \ c_{ij} \mapsto \varphi_{c_i}(m_{ij}).$$

In particular, this function sends the d_i copies of ϵ_i into a basis of M_i.

Let also define

$$f \colon P_1 \to P_0, \ b_{\alpha j} \mapsto (\widetilde{b_\alpha} \alpha)_j - b^M_{\alpha, j}$$

where $\widetilde{b_\alpha}\alpha$ is the path from $s(\alpha)$ to $t(b_\alpha)$ which is the composition of b_α (considered as an element of C) with α. We define $b^M_{\alpha,j} = \sum_{l=1}^{d_t(\alpha)} \theta_{l,j} (\widetilde{b_\alpha})_l$ where $\theta_{l,j}$ is the coefficient of $m_{t(\alpha)l}$ in the decomposition of $\varphi_\alpha(m_{s(\alpha)j})$ in the basis of $M_{t(\alpha)}$. In other words, we have

$$\varphi_\alpha(m_{s(\alpha)j}) = \sum_{l=1}^{d_t(\alpha)} \theta_{l,j} m_{t(\alpha)l}. \tag{2.1}$$

It also means that we consider b_α as a path of Q and $(\widetilde{b_\alpha})_l$ means that we take the l-th copy of that path in the basis $\mathcal{B}$.

Let us first show that f and g are morphisms of representations.

We first need to show that this diagram is commutative.

$$\begin{tikzcd}
(P_0)_i \arrow[r, "g_i"] \arrow[d, "\psi_\alpha"'] & M_i \arrow[d, "\varphi_\alpha"] \\
(P_0)_j \arrow[r, "g_j"'] & M_j.
\end{tikzcd}$$

Let $x \in (P_0)_i$. It is of the form $x = \sum \lambda_{c_{kl}} c_{kl}$ where $t(c_k) = i$. Let us show that the diagram commutes on the element of the basis, we will have the result by linearity of the function. We have

$$g_j \psi_\alpha(c_{kl}) = g_j((\alpha c_k)_l) = \varphi_{\alpha c_k}(m_{kl}) = \varphi_\alpha \varphi_{c_k}(m_{kl}) = \varphi_\alpha g_i(c_{kl}).$$

Therefore, g is a morphism of representations.

Let us now show that f is a morphism of representation. To do that, let us show that the following diagram commutes. As above, we only need to show the equality on the elements of the basis.

$$\begin{tikzcd}
(P_1)_i \arrow[r, "f_i"] \arrow[d, "\tau_\alpha"'] & (P_0)_i \arrow[d, "\psi_\alpha"] \\
(P_1)_j \arrow[r, "f_j"'] & (P_0)_j.
\end{tikzcd}$$

A basis for $(P_1)_i$ is given by the $b_{\beta k}$ with $t(b_{\beta k}) = i$ and we have

$$\begin{aligned}
f_j \tau_\alpha(b_{\beta k}) =& f_j((\alpha b_\beta)_k) \\
=& (\widetilde{\alpha b_\beta \beta})_k - \sum_l \theta_{l,k} (\widetilde{\alpha b_\beta})_l \\
=& \psi_\alpha((\widetilde{b_\beta}\beta)_k) - \psi_\alpha(\sum_l \theta_{l,k} (\widetilde{b_\beta})_l) \\
=& \psi_\alpha f_i(b_{\beta k}).
\end{aligned}$$

We will now show that the sequence is exact.

- g is surjective: for all basis vector m_{ij} of M, we have $g(\epsilon_{ij}) = \varphi_{\epsilon_i}(m_{ij}) = m_{ij}$. We obtain the image of the other elements by linearity.

- $\ker(g) \supset \operatorname{im}(f)$: in order to show that, it suffice to show $g \circ f = 0$. It suffices to show the equality on the basis elements $\mathcal{B}'$.

$$\begin{aligned}
g(f(b_{\alpha j})) =& \; g((\widetilde{b_\alpha}\alpha)_j - b^M_{\alpha,j}) \\
=& \; g((\widetilde{b_\alpha}\alpha)_j) - g\left(\textstyle\sum_{l=1}^{d_t(\alpha)} \theta_{l,j} (\widetilde{b_\alpha})_l\right) \\
=& \; \varphi_{\widetilde{b_\alpha}\alpha}(m_{s(\alpha)j}) - \varphi_{\widetilde{b_\alpha}} \left(\textstyle\sum_{l=1}^{d_t(\alpha)} \theta_{l,j} m_{t(\alpha)l}\right) \\
=& \; \varphi_{\widetilde{b_\alpha}} \left(\varphi_\alpha(m_{s(\alpha)j}) - \textstyle\sum_{l=1}^{d_t(\alpha)} \theta_{l,j} m_{t(\alpha)l}\right) \\
=& \; 0
\end{aligned}$$

where the last equality comes from Equation 2.1.

-ker$(g) \subset$ im(f): Let $x \in P_0$, it can be written as a linear combination of elements of $\mathcal{B}$:

$$x = \sum_{c_{ij} \in \mathcal{B}} \lambda_{c_{ij}} c_{ij} = x_0 + \sum_{c_{ij} \in \mathcal{B} \setminus \mathcal{B}_0} \lambda_{c_{ij}} c_{ij}$$

where $\mathcal{B}_0 = \{\epsilon_{ij} | i \in Q_0, 1 \leq j \leq d_i\}$ contains all the lazy paths and the element x_0 is given by $x_0 = \sum_{c_{ij} \in \mathcal{B}_0} \lambda_{c_{ij}} c_{ij}$.

Moreover, any non-lazy path can be decomposed as an arrow followed by a path. In the meantime, recall that $(\widetilde{b_\alpha}\alpha)_j = f(b_{\alpha j}) + b^M_{\alpha,j}$. Therefore,

$$x = x_0 + \sum_{c_{ij} : c_i = \widetilde{b_\alpha}\alpha} \lambda_{c_{ij}} (\widetilde{b_\alpha}\alpha)_j = x_0 + \sum_{c_{ij} : c_i = b_\alpha \alpha} \lambda_{c_{ij}} (f(b_{\alpha j}) + b^M_{\alpha,j}).$$

Let $x_1 = x_0 + \sum_{c_{ij} : c_i = b_\alpha \alpha} \lambda_{c_{ij}} b^M_\alpha$. We directly have that the difference $x - x_1$ is in the image of f: $x - x_1 = \sum_{c_{ij}} \lambda_{c_{ij}} f(b_{\alpha j}) \in \operatorname{im} f$. Moreover, we have that the degree (the length of the longest path with non-zero coefficient) of x_1 is strictly less than the one of x and $\deg x_0 = 0$.

Suppose now that $x \in \ker g$, we then have $0 = g(x) = g(x_1)$ as $g \circ f = 0$. Hence, $x_1 \in \ker g$. We can then repeat the same argument, replacing x by x_1.

There is some $x_h \in \ker g, x - x_h \in \operatorname{im} f$ and $\deg x_h = 0$. As the degree of x_h is zero, it means that x_h is a linear combination of lazy paths: $x_h = \sum_{i,j} \mu_{ij} \epsilon_{ij}$. As x_h is in the kernel of g, we have $0 = g(x_h) = \sum_{ij} \mu_{ij} m_{ij}$ and as m_{ij} is a basis, all the $\mu_{ij} = 0$, therefore $x_h = 0$ and $x \in \operatorname{im} f$.

- f is injective. Suppose that $0 = f(\sum \lambda_{b_{\alpha j}} b_{\alpha j}) = \sum \lambda_{b_{\alpha j}} ((\widetilde{b_\alpha}\alpha)_j - b^M_{\alpha,j})$. We then have

$$\sum \lambda_{b_{\alpha j}} (\widetilde{b_\alpha}\alpha)_j = \sum \lambda_{b_{\alpha j}} b^M_\alpha = \sum \lambda_{b_{\alpha j}} \sum_l \theta_{l,j} (\widetilde{b_\alpha})_l.$$

As Q is finite and has no oriented cycle, M has a source i_0, i.e. it has a vertex i_0 such that there is no arrow $j \to i_0$ with $d_j \neq 0$. As b_α is a path starting at $t(\alpha)$ for all arrows in Q_1, it cannot go through i_0. Therefore, on the right-hand side of the equality, all the $(\widetilde{b_\alpha})_l$ are paths not going through i_0. It means that the coefficients of the paths starting in i_0 on the left-hand side must be 0. In other words, $\lambda_{b_{\alpha j}} = 0$ for all $\alpha \in Q_1$ such that $s(\alpha) = i_0$. Now, let i_1 be a source in $M \setminus \{i_0\}$. As $\lambda_{b_{\alpha j}} = 0$ for all paths α with $s(\alpha) = i_0$, it means that there is no path $\widetilde{b_{\alpha j}}$ with $\lambda_{b_{\alpha j}} \neq 0$ that goes through i_1. We can therefore repeat the argument. As Q has a finite number of arrows, we have $\lambda_{b_{\alpha j}} = 0$ for all α and j, and the result follows. $\square$

The projective resolution constructed above is called the *canonical projective resolution* of Q. Note that this resolution is not unique nor necessarily the smallest one.

Example 2.3.5. Consider the (finite, acyclic, connected) quiver $Q =$

$$\begin{tikzcd} & 1 \arrow[d, "\alpha"] & \\ & 2 \arrow[dl, "\beta"'] \arrow[dr, "\gamma"] & \\ 3 & & 4 \end{tikzcd}$$

.

Consider its representation M :

$$\begin{tikzcd}
& \mathbf{k} \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"] & \\
& \mathbf{k}^2 \arrow[dl, "\begin{pmatrix}0&1\end{pmatrix}"'] \arrow[dr, "\begin{pmatrix}1&1\end{pmatrix}"] & \\
\mathbf{k} & & \mathbf{k}.
\end{tikzcd}$$

Let us construct its canonical projective resolution.
We have

$$\begin{tikzcd}
& & \mathbf{k} \arrow[d, "1"] & & & & 0 \arrow[d, "0"] & & & & 0 \arrow[d, "0"] & & & & 0 \arrow[d, "0"] & & & & 0 \arrow[d, "0"] & \\
P_0 = & & \mathbf{k} \arrow[dl, "1"'] \arrow[dr, "1"] & & \bigoplus\Bigg(& & \mathbf{k} \arrow[dl, "1"'] \arrow[dr, "1"] & & \bigoplus & & \mathbf{k} \arrow[dl, "1"'] \arrow[dr, "1"] & & \Bigg)\bigoplus & & 0 \arrow[dl, "1"'] \arrow[dr, "0"] & & \bigoplus & & 0 \arrow[dl, "0"'] \arrow[dr, "1"] & \\
& \mathbf{k} & & \mathbf{k} & & \mathbf{k} & & \mathbf{k} & & \mathbf{k} & & \mathbf{k} & & \mathbf{k} & & 0 & & 0 & & \mathbf{k}.
\end{tikzcd}$$

The basis are

$$\begin{aligned}
(P_0)_1 &= \{\epsilon_1\} \\
(P_0)_2 &= \{\alpha, \epsilon_{2,1}, \epsilon_{2,2}\} \\
(P_0)_3 &= \{\beta\alpha, \beta_1, \beta_2, \epsilon_3\} \\
(P_0)_4 &= \{\gamma\alpha, \gamma_1, \gamma_2, \epsilon_4\}.
\end{aligned}$$

Let us compute the value of g on the element of the basis. Recall that $g(c_{ij}) = \varphi_{c_i}(m_{ij})$ We have

$$\begin{aligned}
g(\epsilon_1) &= \varphi_{\epsilon_1}(1) = 1_1, \\
g(\alpha) &= \varphi_\alpha(1) = \begin{pmatrix}1\\0\end{pmatrix}(1) = \begin{pmatrix}1\\0\end{pmatrix}, \\
g(\epsilon_{2,1}) &= \varphi_{\epsilon_2}(\begin{pmatrix}1\\0\end{pmatrix}) = \begin{pmatrix}1\\0\end{pmatrix}, \\
g(\epsilon_{2,2}) &= \varphi_{\epsilon_2}(\begin{pmatrix}0\\1\end{pmatrix}) = \begin{pmatrix}0\\1\end{pmatrix}, \\
g(\beta\alpha) &= \varphi_{\beta\alpha}(1) = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}(1) = 0, \\
g(\beta_1) &= \varphi_\beta(\begin{pmatrix}1\\0\end{pmatrix}) = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0, \\
g(\beta_2) &= \varphi_\beta(\begin{pmatrix}0\\1\end{pmatrix}) = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 1_3, \\
g(\epsilon_3) &= 1_3, \\
g(\gamma\alpha) &= \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}(1) = 1_4,
\end{aligned}$$

$$\begin{aligned} g(\gamma_1) &= \varphi_\gamma(\begin{pmatrix}1\\0\end{pmatrix}) = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1_4, \\ g(\gamma_2) &= \varphi_\gamma(\begin{pmatrix}0\\1\end{pmatrix}) = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 1_4, \\ g(\epsilon_4) &= 1_4. \end{aligned}$$

Here, the subscript of the 1 represents in which vector space $\mathbf{k}$ it is the unit. For instance, 1_3 represents the unit of the vector space related to the vertex 3 in the representation M. We can see that the basis elements of each vector space are in the image of g; therefore, g is indeed surjective.

Let us now construct P_1 and the map f. We have

$$P_1 = \begin{tikzcd}[column sep=small] & 0 \arrow[d, "0"] & \\ & \mathbf{k} \arrow[dl, "1"'] \arrow[dr, "1"] & \\ \mathbf{k} & & \mathbf{k} \end{tikzcd} \bigoplus \left(\begin{tikzcd}[column sep=small] & 0 \arrow[d, "0"] & \\ & 0 \arrow[dl, "1"'] \arrow[dr, "0"] & \\ \mathbf{k} & & 0 \end{tikzcd} \bigoplus \begin{tikzcd}[column sep=small] & 0 \arrow[d, "0"] & \\ & 0 \arrow[dl, "1"'] \arrow[dr, "0"] & \\ \mathbf{k} & & 0 \end{tikzcd} \right) \bigoplus \left(\begin{tikzcd}[column sep=small] & 0 \arrow[d, "0"] & \\ & 0 \arrow[dl, "0"'] \arrow[dr, "1"] & \\ 0 & & \mathbf{k} \end{tikzcd} \bigoplus \begin{tikzcd}[column sep=small] & 0 \arrow[d, "0"] & \\ & 0 \arrow[dl, "0"'] \arrow[dr, "1"] & \\ 0 & & \mathbf{k} \end{tikzcd} \right).$$

The basis are

$$\begin{aligned} (P_1)_1 &= \{0\} \\ (P_1)_2 &= \{\epsilon_2\} \\ (P_1)_3 &= \{\beta, \epsilon_{3,1}, \epsilon_{3,1}\} \\ (P_1)_4 &= \{\gamma, \epsilon_{4,1}, \epsilon_{4,1}\}. \end{aligned}$$

Note that the basis elements related to α are $\epsilon_2, \beta, \gamma$, the ones related to β are $\epsilon_{3,1}, \epsilon_{3,1}$ and the ones related to γ are $\epsilon_{4,1}, \epsilon_{4,1}$.

In order to compute f, let us first compute the different $b^M_{\alpha,l} = \sum_{l=1}^{d_t(\alpha)} \theta_{l,j}(\widetilde{b_\alpha})_l$ with $\varphi_\alpha(m_{s(\alpha)j}) = \sum_{l=1}^{d_t(\alpha)} \theta_{l,j} m_{t(\alpha)l}$.

We have $\varphi_\alpha(1_1) = 1 \cdot \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + 0 \cdot \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ Therefore, $(\epsilon_2)^M = 1 \cdot \epsilon_{2,1} + 0 \cdot \epsilon_{2,2}$, $\beta^M = \beta_1$ and $\gamma^M = \gamma_1$.

Similarly, $\varphi_\beta\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = 0$ and $\varphi_\beta\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = 1$. Therefore, $(\epsilon_{3,1})^M = 0$ and $(\epsilon_{3,2})^M = \epsilon_3$.

Finally, $\varphi_\gamma\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = 1$ and $\varphi_\gamma\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = 1$. Therefore, $(\epsilon_{4,1})^M = \epsilon_4$ and $(\epsilon_{4,2})^M = \epsilon_4$.

It means that we have

$$\begin{aligned} f(\epsilon_2) &= \alpha - \epsilon_{2,1}, \\ f(\beta) &= \beta\alpha, \\ f(\epsilon_{3,1}) &= \beta_1, \\ f(\epsilon_{3,2}) &= \beta_2 - \epsilon_3, \\ f(\gamma) &= \gamma\alpha, \\ f(\epsilon_{4,1}) &= \beta_1 - \epsilon_4, \\ f(\epsilon_{4,2}) &= \beta_2 - \epsilon_4. \end{aligned}$$

If we represent f in matrix form, it is easy to compute the rank of that matrix, which is 7 and which implies that f is injective. Moreover, computing fg on the elements of the basis of P_1 directly gives that the composition of the two functions is null. Therefore, we have $\ker(g) \supset \operatorname{im}(f)$. It remains to check the other inclusion.

Let $x \in \ker(g)$. We can split it into its part x_0 of degree 0 and the rest.

$$x = x_0 + \sum_{c_{ij}:c_i=\widetilde{b_\alpha\alpha}} \lambda_{c_{ij}} (\widetilde{b_\alpha\alpha})_j = x_0 + \sum_{c_{ij}:c_i=b_\alpha\alpha} \lambda_{c_{ij}} (f(b_{\alpha j}) + b^M_{\alpha,j}).$$

We have

$$\begin{aligned} x_1 =& x_0 + \sum_{c_{ij}:c_i=b_\alpha\alpha} \lambda_{c_{ij}} b^M_{\alpha,j} \\ =& x_0 + \left(\lambda_\alpha \epsilon_2^M + \lambda_{\beta\alpha}\beta^M + \lambda_{\gamma\alpha}\gamma^M\right) + \left(\lambda_{\beta_1}\epsilon^M_{3,1} + \lambda_{\beta_2}\epsilon^M_{3,2}\right) + \left(\lambda_{\gamma_1}\epsilon^M_{4,1} + \lambda_{\gamma_2}\epsilon^M_{4,2}\right) \\ =& x_0 + (\lambda_\alpha\epsilon_{2,1} + \lambda_{\beta\alpha}\beta_1 + \lambda_{\gamma\alpha}\gamma_1) + (0 + \lambda_{\beta_2}\epsilon_3) + (\lambda_{\gamma_1}\epsilon_4 + \lambda_{\gamma_2}\epsilon_4) \end{aligned}$$

We have that the degree of x_1 is at most one, $x - x_1 \in \operatorname{im} f$, and $0 = g(x) = g(x_1)$. We can write x_1 as x_0' its degree zero part and the rest, we will have

$$x_1 = x_0' + \lambda_{\beta\alpha}\beta_1 + \lambda_{\gamma\alpha}\gamma_1.$$

Define

$$x_2 = x_0' + \lambda_{\beta\alpha}\epsilon^M_{3,1} + \lambda_{\gamma\alpha}\epsilon^M_{4,1} = x_0' + \lambda_{\gamma\alpha}\epsilon_4.$$

We have x_2 of degree 0 and such that $x - x_2 \in \operatorname{im} f$. Moreover, $g(x_2) = 0$, which implies that $x_2 = 0$ as g is a bijection on the vector space spanned by $\{\epsilon_1, \epsilon_{2,1}, \epsilon_{2,2}, \epsilon_3, \epsilon_4\}$. Therefore, $x \in \operatorname{im} f$.

2.4 Gabriel's Theorem

In this section, we will only consider finite acyclic quivers (recall Definition 2.2.3). We denote by n the number of vertices of that quiver. The proof is greatly inspired by Chapter 8 of [53].

Definition 2.4.1. A quiver Q is of *finite representation type* if there are only a finite number of isoclasses of indecomposable representations of Q.

Recall from Definition 2.1.8 that an indecomposable representation is a representation that cannot be written as the direct sum of two other non-trivial representations.

Example 2.4.2. Not all the quivers are of finite representation type. For instance, consider the following quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2. \end{tikzcd}$$

and for all $s \in \mathbf{k}$, consider the representation V_s given by

$$\begin{tikzcd} \mathbf{k} \arrow[r, shift left, "\mathrm{id}_{\mathbf{k}}"] \arrow[r, shift right, "s\cdot"'] & \mathbf{k}. \end{tikzcd}$$

The second map is given by the multiplication by s. The representation V_s is indecomposable. Indeed, $\mathbf{k}$ is indecomposable, therefore, the only possible decomposition would be of the form $\mathbf{k} \underset{f_2}{\overset{f_1}{\rightrightarrows}} 0 \oplus 0 \underset{g_2}{\overset{g_1}{\rightrightarrows}} \mathbf{k}$ and we must have $f_1 = f_2 = g_1 = g_2 = 0$. Then, the map $\mathrm{id}_{\mathbf{k}}$ does not decompose.

Moreover, if $s \neq t$, then $V_s \not\simeq V_t$. Indeed, if $V_s \simeq V_t$, then we have φ_1 and φ_2 $\mathbf{k}$-automorphisms such that the two following diagrams commute. Furthermore, $\mathbf{k}$-automorphisms can be seen as multiplication by a non-zero constant.

$$\begin{tikzcd} \mathbf{k} \arrow[r, "\mathrm{id}_{\mathbf{k}}"] \arrow[d, "c_1\cdot"'] & \mathbf{k} \arrow[d, "c_2\cdot"] \\ \mathbf{k} \arrow[r, "\mathrm{id}_{\mathbf{k}}"'] & \mathbf{k} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{k} \arrow[r, "s\cdot"] \arrow[d, "c_1\cdot"'] & \mathbf{k} \arrow[d, "c_2\cdot"] \\ \mathbf{k} \arrow[r, "t\cdot"'] & \mathbf{k}. \end{tikzcd}$$

The first diagram implies that $c_1 = c_2$ and the second implies that $t \cdot_1 = s \cdot c_2 = s \cdot c_1$ and therefore $t = s$. In particular, it means that Q has an infinite number of indecomposable representations.

The goal of this section is to prove that the only quivers of finite representation types are the ones that are a disjoint union of simply laced Dynkin quivers. We will start by presenting those quivers, then apply some algebraic geometric arguments to get some first results. Afterwards, we will define a quadratic form on a quiver before linking everything together.

2.4.1 Simply-Laced Dynkin Quivers

Dynkin quivers are quivers whose underlying (non-oriented multi-)graph is a Dynkin[2] diagram. These types of graphs are used in the classification of semisimple Lie algebras. There are 7 types of such diagrams, which are the following.

$$A_n : \quad 1 \;\relbar\!\relbar\!\relbar\; 2 \;\relbar\!\relbar\!\relbar\; 3 \;\relbar\!\relbar\!\relbar\; \cdots \;\relbar\!\relbar\!\relbar\; (n-1) \;\relbar\!\relbar\!\relbar\; n$$

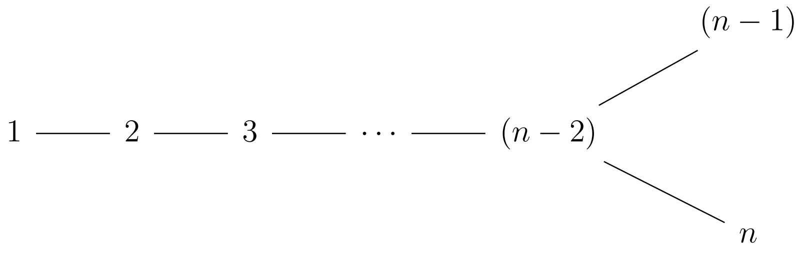
$$B_n : \quad 1 \;=\!=\!=\; 2 \;\relbar\!\relbar\!\relbar\; 3 \;\relbar\!\relbar\!\relbar\; \cdots \;\relbar\!\relbar\!\relbar\; (n-1) \;\relbar\!\relbar\!\relbar\; n$$

$$C_n : \quad 1 \;\relbar\!\relbar\!\relbar\; 2 \;\relbar\!\relbar\!\relbar\; 3 \;\relbar\!\relbar\!\relbar\; \cdots \;\relbar\!\relbar\!\relbar\; (n-1) \;=\!=\!=\; n$$

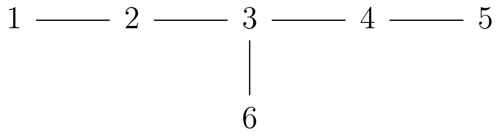
---

[2] Eugene Dynkin (1924-2014) was a Soviet and American mathematician who worked in Algebra and Probability.

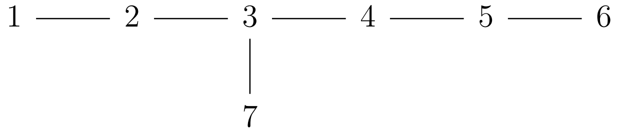
D_n :



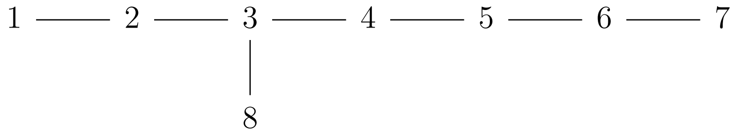
E_6 :



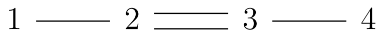
E_7 :



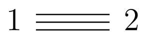
E_8 :



F_4 :



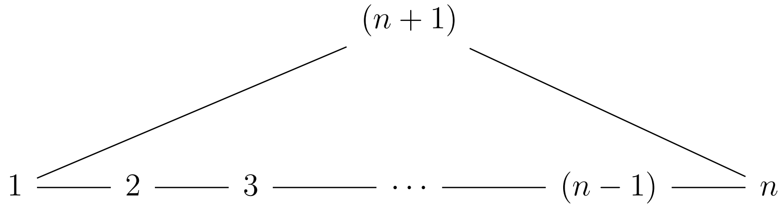
G_2 :



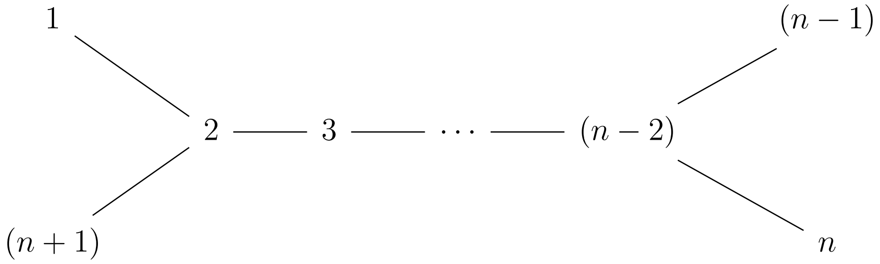
Simply laced Dynkin quivers, also called Dynkin type ADE, are quivers whose underlying graph is Dynkin with no parallel edges. They are exactly the above graphs of type A_n, D_n, E_6, E_7 and E_8. In the following, unless otherwise specified, we will use the above numbering of the vertices. Note that to avoid repetitions, we take $n \geq 4$ for D_n as $D_1 = A_1$, $D_2 = A_2$ and $D_3 = A_3$.

Also related to Dynkin quivers are the Euclidean or extended Dynkin quivers. Those are quivers whose underlying graph is a Dynkin diagram with an extra vertex and such that, if any vertex is removed, the diagram becomes a disjoint union of Dynkin diagrams. We call such diagrams *extended Dynkin* or *Euclidean diagrams.* Here we will only focus on the extended simply laced Dynkin diagrams. They are the following.

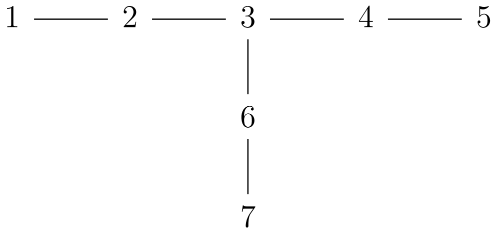
$\tilde{A}_n$:



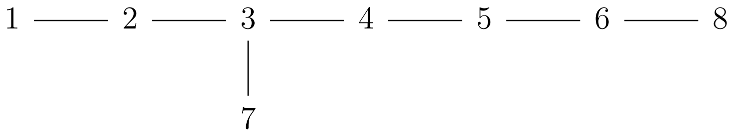
$\tilde{D}_n$:



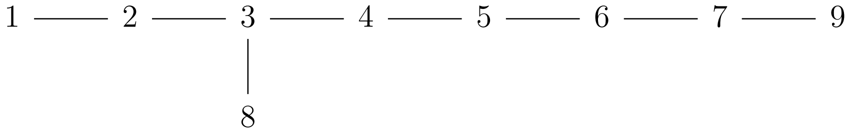
$\tilde{E}_6$:



$\tilde{E}_7$:



$\tilde{E}_8$:

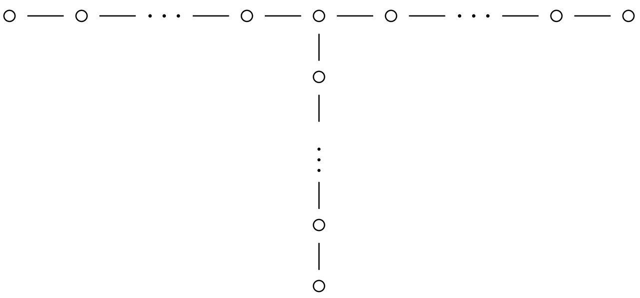


The vertex with the highest numbering is the added vertex compared to the Dynkin version. Once again, unless specified, we will use the above numbering of vertices.

The following easy result comes from [1].

Proposition 2.4.3. Let Q be a finite, connected and acyclic quiver such that its underlying graph $\overline{Q}$ is not a simply laced Dynkin graph. Then $\overline{Q}$ contains a Euclidean graph as a subgraph.

Proof. Suppose that $\overline{Q}$ contains no Euclidean subgraph, we will show that it is simply laced Dynkin. As $\overline{Q}$ does not contain $\tilde{A}_1$, it means that every pair of vertices is linked with at most one edge. As $\overline{Q}$ does not contain $\tilde{A}_n$ $(n \geq 2)$, it means that it has no cycle and thus it must be a tree. Furthermore, as $\tilde{D}_4$ is not a subgraph, no vertex has more than 3 neighbours, and the exclusion of $\tilde{D}_n$ for $n \geq 5$ implies that at most one point has more than 2 neighbours. Therefore, $\overline{Q}$ is of the form



Denote by l the number of vertices left to the central one, r the number of vertices right to it and b the number of vertices below it. Without loss of generality, we can assume $b \leq l \leq r$. As $\tilde{E}_6$ is not a subgraph of $\overline{Q}$, we have $b \leq 1$. If $b = 0$, we have $\overline{Q} = A_{l+r+1}$, which is simply laced Dynkin. Suppose $b = 1$. The exclusion of $\tilde{E}_7$ gives $1 \leq l \leq 2$. If $l = 1$, we get $\overline{Q} = D_{r+3}$. Assume $l = 2$. The exclusion of $\tilde{E}_8$ implies that $2 \leq r \leq 4$, which implies that $\overline{Q}$ is either E_6, E_7 or E_8. ☐

2.4.2 Quiver Varieties

Let $d \in \mathbb{Z}^{Q_0}$ such that $d_i \geq 0$ for all $i \in Q_0$ be a dimension vector and let R_d be the space of representations M of the quiver Q of dimension vector d, i.e. the space of representations M such that $M_i \simeq \mathbf{k}^{d_i}$ for all $i \in Q_0$.

It means that the vector spaces M_i are fixed and the representations of R_d are completely determined by their linear maps φ_α. Therefore,

$$R_d \simeq \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_{\mathbf{k}}(\mathbf{k}^{d_{s(\alpha)}}, \mathbf{k}^{d_{t(\alpha)}}).$$

Moreover, each $\mathrm{Hom}_{\mathbf{k}}(\mathbf{k}^{d_{s(\alpha)}}, \mathbf{k}^{d_{t(\alpha)}})$ is isomorphic to the space of matrices of size $d_{t(\alpha)} \cdot d_{s(\alpha)}$. It implies that R_d is a $\mathbf{k}$-vector space of dimension $N = \sum_{\alpha \in Q_1} d_{s(\alpha)} d_{t(\alpha)}$. We can see it as an affine variety $\mathbf{k}^N$.

Let $G_d = \prod_{i \in Q_0} \mathrm{GL}_{d_i}(\mathbf{k})$ where $\mathrm{GL}_n(\mathbf{k})$ denote the group of invertible automorphism of $\mathbf{k}^n$. Moreover, we can see G_d as an algebraic variety, and the multiplication as well as the inversion of elements of G_d are morphisms of algebraic varieties. Therefore, G_d is an algebraic group (see Definition A.3.1). The algebraic group G_d acts on R_d by conjugation: if $g = (g_i)_{i \in Q_0} \in G_d$, $M = (M_i, \varphi_\alpha) \in R_d$ and $\alpha \colon i \to j \in Q_1$, then we can define the group action by $(g \cdot \varphi)_\alpha = g_j \varphi_\alpha g_i^{-1}$. It is immediate to see that it is a group action. Moreover, the group action is a morphism of algebraic varieties and thus we can see R_d as a G_d-space.

Let $\mathcal{O}_M = \{g \cdot M | g \in G_d\}$ be the orbit of M under the action. As discussed in Proposition A.3.3, the $\mathcal{O}_M$ are varieties and locally closed in R_d.

Lemma 2.4.4. The orbit of M can also be expressed as

$$\mathcal{O}_M = \{M' \in \mathrm{Rep}\, Q | M \simeq M'\}.$$

Proof. For the inclusion $\mathcal{O}_M \subset \{M' \in \mathrm{Rep}\, Q | M \simeq M'\}$, we can suppose that M and $M' = (M'_i, \varphi'_\alpha)$ are in the same orbit. It means that there is an element $g \in G_d$ such that $g \cdot M = M'$. It implies that for all $\alpha \colon i \to j$ in Q_1, we have the following commutative diagram:

$$\begin{tikzcd}
M_i \arrow[r, "\varphi_\alpha"] \arrow[d, "g_i"'] & M_j \arrow[d, "g_j"] \\
M'_i \arrow[r, "\varphi'_\alpha"'] & M'_j.
\end{tikzcd}$$

In other words, g is a morphism of representations. Moreover, as each $g_i \in \mathrm{GL}_{d_i}(\mathbf{k})$, the morphism g is an isomorphism and $M \simeq M'$.

For the converse inclusion, suppose $g : M \to M'$ is an isomorphism of representations. Then each g_i is an element of $\mathrm{GL}_{d_i}(\mathbf{k})$ and therefore $M' = g(M) = g \cdot M$. ☐

Similarly to general group actions, we can define the stabilisers of an element,

$$\operatorname{Stab} M = \{g \in G_d : g \cdot M = M\}.$$

Using the previous lemma, it is direct to see that it corresponds to the set of automorphisms of M.

Proposition 2.4.5. Let d be a representation vector. Then,

1. For all representation M, the dimension of the varieties $\mathcal{O}_M, G_d$ and $\operatorname{Aut} M$ satisfy
$$\dim \mathcal{O}_M = \dim G_d - \dim \operatorname{Aut} M.$$
2. There is at most one orbit $\mathcal{O}$ of codimension zero in R_d.

Proof. 1. By the first theorem of isomorphism, we have $G_d / \operatorname{Stab} M \xrightarrow{\sim} \mathcal{O}_M$, $\overline{g} \mapsto g \cdot M$. Therefore we have the equality of dimensions

$$\dim \mathcal{O}_M = \dim(G_d / \operatorname{Stab} M) = \dim G_d - \dim \operatorname{Stab} M = \dim G_d - \dim \operatorname{Aut} M.$$

2. It is a well-known fact of algebraic geometry that R_d is an irreducible algebraic variety as it is isomorphic to $\mathbf{k}^N$ for some $N \in \mathbb{N}$. Therefore, any non-empty open subset is dense. Let us show that a codimension zero orbit is open in R_d. By Proposition A.3.3, $\mathcal{O}_M$ is locally closed. As $\mathcal{O}_M$ is of codimension 0, its closure is R_d, hence $\mathcal{O}_M$ is open in R_d. Therefore, if $\mathcal{O}'$ is another orbit of codimension 0, it also contains a non-empty open subset of R_d, which implies that $\mathcal{O}_M \cap \mathcal{O}' \neq \emptyset$; therefore, they are equal. □

The following proposition is inspired by Theorem 2.3.1 of [10]. Recall the equivalent definition of split exact sequences in Proposition B.4.4.

Proposition 2.4.6. A short exact sequence of representations of the quiver Q

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is split exact if and only if $M \simeq L \oplus N$.

Proof. One direction is direct as one condition for being split exact is to have an isomorphism $\varphi : M \to L \oplus N$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "1_L"'] & M \arrow[r, "g"] \arrow[d, "\varphi"'] & N \arrow[r] \arrow[d, "1_N"'] & 0 \\
0 \arrow[r] & L \arrow[r, "i_N"'] & l \oplus N \arrow[r, "\pi_N"'] & N \arrow[r] & 0.
\end{tikzcd}$$

In particular, we have $M \simeq L \oplus N$.

Let us show the other implication. Let us apply the left exact functor $\operatorname{Hom}(N, -) : \operatorname{Rep} Q \to \mathbf{Vect_k}$ to our exact sequence. We get the following exact sequence

$$0 \to \operatorname{Hom}(N, L) \xrightarrow{f\circ} \operatorname{Hom}(N, M) \xrightarrow{g\circ} \operatorname{Hom}(N, N).$$

We can extend it by

$$0 \to \mathrm{Hom}(N, L) \xrightarrow{f\circ} \mathrm{Hom}(N, M) \xrightarrow{g\circ} \mathrm{Hom}(N, N) \to C \to 0.$$

with $C = \mathrm{coker}(\mathrm{Hom}(N, g))$. It is still exact.

In the meantime, as $M \simeq L \oplus N$, $\mathrm{Hom}(N, M) \simeq \mathrm{Hom}(N, L \oplus N)$. Moreover, if $\psi \in \mathrm{Hom}(N, L \oplus N)$, we can see ψ as (ψ_1, ψ_2) with $\psi_1 \in \mathrm{Hom}(N, L)$ and $\psi_2 \in \mathrm{Hom}(N, N)$. In particular, we have $\dim(\mathrm{Hom}(N, M)) = \dim(\mathrm{Hom}(N, L)) + \dim(\mathrm{Hom}(N, N))$.

Using the rank-nullity theorem on the extended exact sequence, we get

$$0 = \dim(\mathrm{Hom}(N, L)) - \dim(\mathrm{Hom}(N, M)) + \dim(\mathrm{Hom}(N, N)) - \dim(C) = -\dim(C).$$

It implies that $C = 0$ and that the sequence

$$0 \to \mathrm{Hom}(N, L) \xrightarrow{f\circ} \mathrm{Hom}(N, M) \xrightarrow{g\circ} \mathrm{Hom}(N, N) \to 0$$

is exact. In particular, the post-composition by g is surjective. Therefore, there is $h \in \mathrm{Hom}(N, M)$ such that $g \circ h = \mathrm{id}_N$. It means that g has a splitting and that the short exact sequence splits. ☐

Proposition 2.4.7. If the sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is an exact sequence of representations (not necessarily of the same dimension) that is non-split, then

$$\dim(\mathcal{O}_{L\oplus N}) < \dim(\mathcal{O}_M).$$

Proof. Let $L = (L_i, \psi_\alpha)$, $M = (M_i, \varphi_\alpha)$ and $N = (N_i, \tau_\alpha)$. For each vertex $i \in Q_0$, we choose a basis $\mathcal{B}'_i$ of the vector space L_i. As f_i is a monomorphism, the set $f_i(\mathcal{B}'_i)$ is still free and can be completed into a basis $\mathcal{B}_i$ of M_i. As g_i is an epimorphism, the set $g_i(\mathcal{B}_i)$ is still a spanning set, and we can take a subset of it to form a basis $\mathcal{B}''_i$ of N_i. After possibly reordering the elements, we have that the functions f_i and g_i are represented in those bases by

$$f_i = \left(\begin{array}{c} I_{d'_i} \\ \hline 0 \end{array}\right) \quad \text{and} \quad g_i = \left(\begin{array}{c|c} 0 & I_{d''_i} \end{array}\right)$$

where d'_i is the dimension of L_i and d''_i is the dimension of N_i. Moreover, as the sequence is exact, we have $d_i = d'_i + d''_i$.

As f and g are morphisms of representations, for $\alpha\colon i \to j$ is Q_1, we have $\varphi_\alpha f_i = f_j \psi_\alpha$ and $g_j \varphi_\alpha = \tau_\alpha g_i$. It gives that

$$\varphi_\alpha = \left(\begin{array}{c|c} \psi_\alpha & \xi_\alpha \\ \hline 0 & \tau_\alpha \end{array}\right)$$

where ξ_α is a matrix of size $(d_j - d''_j) \times (d_i - d'_i)$. Indeed, we can write the matrix form of φ_α as $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ with A of dimension $d'_j \times d'_i$, B of dimension $d'_j \times d''_i$, C of dimension $d''_j \times d'_i$ and D of dimension $d''_j \times d''_i$. The equality $\varphi_\alpha f_i = f_j \psi_\alpha$ implies

$$\left(\begin{array}{c} A \\ \hline C \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \left(\begin{array}{c} I_{d'_i} \\ \hline 0 \end{array}\right) = \left(\begin{array}{c} I_{d'_j} \\ \hline 0 \end{array}\right) (\psi_\alpha) = \left(\begin{array}{c} \psi_\alpha \\ \hline 0 \end{array}\right).$$

The equality $g_j \varphi_\alpha = \tau_\alpha g_i$ gives

$$\left(\begin{array}{c|c} C & D \end{array}\right) = \left(\begin{array}{c|c} 0 & I_{d''_j} \end{array}\right) \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = (\tau_\alpha) \left(\begin{array}{c|c} 0 & I_{d''_i} \end{array}\right) = \left(\begin{array}{c|c} 0 & \tau_\alpha \end{array}\right)$$

Moreover, by the last lemma, we have that the sequence splits if and only if $M \simeq L \oplus N$ if and only if $\xi_\alpha = 0$ for all $\alpha \in Q_1$. Suppose it is non-split, in particular, $M \not\simeq L \oplus N$. We have $\alpha \colon i \to j$ such that $\xi_\alpha \neq 0$. Then for all $t \in \mathbf{k}$, $t \neq 0$, the map

$$t \cdot \varphi_\alpha = \left(\begin{array}{c|c} \psi_\alpha & t \cdot \xi_\alpha \\ \hline 0 & \tau_\alpha \end{array}\right)$$

defines a representation $t \cdot M = (M_i, t \cdot \varphi_\alpha)$ that is isomorphic to M, the isomorphism being $f_i = \left(\begin{array}{c|c} t \cdot I_{d'_i} & 0 \\ \hline 0 & I_{d''_i} \end{array}\right)$. It is indeed direct to check that $f_j(\varphi_\alpha) = (t \cdot \varphi_\alpha)(f_i)$.

We therefore have in particular that $N \oplus L \in \overline{\mathcal{O}_M}$. As the conjugation by an element $g \in G_d$ is a continuous map, for every subset $A \subset R_d$, we have that $g \cdot \overline{A} \subset \overline{g \cdot A}$, which further gives

$$G_d \cdot \overline{A} = \bigcup_{g \in G_d} g \cdot \overline{A} \subset \bigcup_{g \in G_d} \overline{g \cdot A} \subset \overline{\bigcup_{g \in G_d} g \cdot A} = \overline{G_d \cdot A}.$$

In our case, it implies that

$$\mathcal{O}_{L \oplus N} = G_d \cdot (L \oplus N) \subset G_d \cdot \overline{\mathcal{O}_M} \subset \overline{G_d \cdot \mathcal{O}_M} = \overline{\mathcal{O}_M}.$$

It further implies that $\overline{\mathcal{O}_M} \supset \overline{\mathcal{O}_{L \oplus N}}$. By Chevalley's lemma (Theorem A.2.4), $\mathcal{O}_M$ is open in $\overline{\mathcal{O}_M}$ and $\mathcal{O}_{L \oplus N}$ is open in $\overline{\mathcal{O}_{L \oplus N}}$. It then gives that $\dim(\mathcal{O}_M) = \dim(\overline{\mathcal{O}_M}) \geq \dim(\mathcal{O}_{L \oplus N}) = \dim(\overline{\mathcal{O}_{L \oplus N}})$. Suppose that we have the equality of dimensions. It therefore gives that $\overline{\mathcal{O}_{L \oplus N}} = \overline{\mathcal{O}_M}$ as they are two closed subsets, one included in the other with the same dimension, therefore they must be equal. It means that $\mathcal{O}_{L \oplus N}$ and $\mathcal{O}_M$ are both open in $\overline{\mathcal{O}_{L \oplus N}}$, therefore, they must be of non-empty intersection as they are both dense open subsets of $\overline{\mathcal{O}_{L \oplus N}}$. Therefore, we must have $M \simeq L \oplus N$, which is a contradiction with the previous lemma.

We therefore have $\dim(\mathcal{O}_{L \oplus N}) < \dim(\mathcal{O}_M)$. □

2.4.3 Tits Forms

To a quiver, we associate a quadratic form that we call its Tits[3] form. The study of this form will provide precious information on the quiver and its representations.

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[3] Jacques Tits (1930-2021) was a Belgian mathematician that worked on group theory.

Definition 2.4.8. Let Q be a quiver. The *Tits form* of the quiver Q is

$$q\colon \mathbb{Q}^n \to \mathbb{Q}, x \mapsto \sum_{i\in Q_0} x_i^2 - \sum_{\alpha\in Q_1} x_{s(\alpha)}x_{t(\alpha)}.$$

It is direct to see that it is a quadratic form and that this form does not depend on the orientation of the quiver.

Recall from Section B.5.3 the definition of the Ext functor and its properties.

Proposition 2.4.9. For any representation M of degree vector d, we have

$$q(d) = \dim \operatorname{Hom}(M,M) - \dim \operatorname{Ext}^1(M,M).$$

Proof. Take $P_1 = \bigoplus_{\alpha\in Q_1} d_{s(\alpha)} \cdot P(t(\alpha))$ and $P_0 = \bigoplus_{i\in Q_0} d_i \cdot P(i)$. By Proposition 2.3.4, they are projective representations, and we have that

$$0 \to P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \to 0$$

form a projective resolution of M. Applying the contravariant functor $\operatorname{Hom}(-,M)$ to the exact sequence gives, by Proposition B.5.12, a long exact sequence

$$\begin{aligned}0 \to \operatorname{Hom}(M,M) \to \bigoplus_{i\in Q_0} d_i \cdot \operatorname{Hom}(P(i),M) \to \bigoplus_{\alpha\in Q_1} d_{s(\alpha)} \cdot \operatorname{Hom}(P(t(\alpha)),M)\\ \to \operatorname{Ext}^1(M,M) \to 0.\end{aligned}$$

The sequence stops as all $P(i)$ are projective, therefore $\operatorname{Ext}^1(P(i),M) = 0$. Moreover, by Proposition 2.3.3, we have $\operatorname{Hom}(P(i),M) \simeq M_i$. It implies that we have

$$\dim(\operatorname{Hom}(M,M)) - \sum_{i\in Q_0} d_i \dim(M_i) + \sum_{\alpha\in Q_1} d_{s(\alpha)} \dim(P(t(\alpha)) - \dim \operatorname{Ext}^1(M,M) = 0.$$

Therefore,

$$q(d) = \sum_{i\in Q_0} d_i^2 - \sum_{\alpha\in Q_1} d_{s(\alpha)}d_{t(\alpha)} = \dim \operatorname{Hom}(M,M) - \dim \operatorname{Ext}^1(M,M).$$

□

Recall that from a quadratic form q, one can define a *bilinear form* $(x,y) = q(x+y) - q(x) - q(y)$. Recall also the following definition, which is widely used in linear algebra.

Definition 2.4.10. Let q' be a quadratic form on a vector space F.

1. q' is *positive definite* if $q'(x) > 0$ for all $x \neq 0$ in F.
2. q' is *positive semi-definite* if $q'(x) \geq 0$ for all $x \in F$.

Proposition 2.4.11. Let Q be a connected quiver with no directed cycle and n vertices. Let $d \in \mathbb{N}^n \setminus \{0\}$ be such that the bilinear form associated to the Tits quadratic form q satisfies $(d,x) = q(d+x) - q(d) - q(x) = 0$ for all $x \in \mathbb{Q}^n$. Then

1. The Tits form q is positive semi-definite.
2. Each component d_i of d is non zero.
3. $q(x) = 0 \Leftrightarrow x = \frac{a}{b}d$ with $a, b \in \mathbb{Z}$.

Proof. Denote by n_{ij} the number of edges between i and j (without considering their orientation). Label the vertices from 1 to n. We then have

$$q(x) = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} \sum_{j:j<i} n_{ij} x_i x_j \tag{2.2}$$

and

$$(x,y) = \sum_{i=1}^{n} 2x_i y_i - \sum_{i=1}^{n} \sum_{j:j\neq i} n_{ij} x_i y_j. \tag{2.3}$$

2. Suppose $d_i = 0$. Let e_i be the ith canonical unit vector. The condition on d gives

$$0 = (d, e_i) = 2d_i - \sum_{j:j\neq i} n_{ij} d_j. \tag{2.4}$$

So that $0 = 2d_i = \sum_{j:j\neq i} n_{ij} d_j$. As $n_{ij} \geq 0$ for all j, we must have $d_j = 0$ for all the vertices neighbouring i. As the quiver is connected, it means that all the $d_j = 0$, so that $d = 0$, which is a contradiction.

1. Using Equation 2.4, we have $1 = \sum_{j:j\neq i} \frac{n_{ij} d_j}{2d_i}$, it then gives

$$\begin{aligned}
\sum_{i=1}^{n} x_i^2 &= \sum_{i=1}^{n} \sum_{j:j\neq i} \frac{n_{ij} d_j}{2} \frac{x_i^2}{d_i} \\
&= \sum_{i=1}^{n} \sum_{j:j<i} \left(\frac{n_{ij} d_j}{2} \frac{x_i^2}{d_i} + \frac{n_{ij} d_i}{2} \frac{x_j^2}{d_j} \right) \\
&= \sum_{i=1}^{n} \sum_{j:j<i} \frac{n_{ij} d_j d_i}{2} \left(\frac{x_i^2}{d_i^2} + \frac{x_j^2}{d_j^2} \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
q(x) &= \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} \sum_{j:j<i} n_{ij} x_i x_j \\
&= \sum_{i=1}^{n} \sum_{j:j<i} \frac{n_{ij} d_j d_i}{2} \left(\frac{x_i^2}{d_i^2} + \frac{x_j^2}{d_j^2} - 2\frac{x_i x_j}{d_i d_j} \right) \\
&= \sum_{i=1}^{n} \sum_{j:j<i} \frac{n_{ij} d_j d_i}{2} \left(\frac{x_i}{d_i} - \frac{x_j}{d_j} \right)^2 \geq 0.
\end{aligned}$$

We have the inequality as $d_i, d_j > 0$ and $n_{ij} \geq 0$.

3. Using the above equality, we have $q(x) = 0 \Leftrightarrow \frac{x_i}{d_i} = \frac{x_j}{d_j}$ whenever $n_{ij} \neq 0$. As the quiver is connected, going from one vertex to its neighbours, one gets the equality for all i, j, and the result follows. ☐

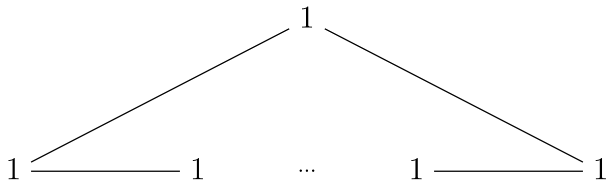
Note that if d satisfies the conditions of the proposition, then in particular, we have $q(d) = (d, d) = 0$.

The following lemma is rather technical and its proof consists only of straightforward computations. For the sake of being complete, they are included, but those computations add nothing to the understanding of the subject. The result is summarised after the proof.

Lemma 2.4.12. For each Euclidean quiver, one can find a vector $\delta \in \mathbb{N}^n$, $\delta \neq 0$ such that $(\delta, x) = 0$ for all $x \in \mathbb{Q}^n$.

Proof. Recall that $(\delta, x) = q(\delta + x) - q(x) - q(\delta)$

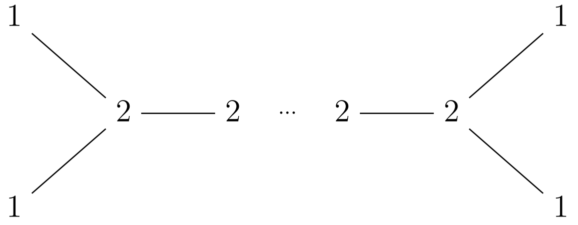
For $\tilde{A}_n$, we can take



Suppose that the vertices are numbered such that there is an edge between the vertex j and $j+1$ as well as 1 and $n+1$. Then, for all $x \in \mathbb{Z}^{n+1}$, we have

$$\begin{aligned}
(\delta, x) =& \left(\sum_{i=1}^{n+1} (x_i + 1)^2 - \sum_{i=1}^{n} (x_i + 1)(x_{i+1} + 1) - (x_1 + 1)(x_{n+1} + 1) \right) \\
&- \left(\sum_{i=1}^{n+1} (x_i)^2 - \sum_{i=1}^{n} x_i x_{i+1} - x_1 x_{n+1} \right) \\
&- \left(\sum_{i=1}^{n+1} (1)^2 - \sum_{i=1}^{n} 1 - 1 \right) \\
=& \sum_{i=1}^{n+1} (x_i^2 + 2x_i + 1) - \sum_{i=1}^{n} (x_i x_{i+1} + x_i + x_{i+1} + 1) - x_1 x_{n+1} - x_1 - x_{n+1} - 1 \\
&- \left(\sum_{i=1}^{n+1} (x_i)^2 - \sum_{i=1}^{n} x_i x_{i+1} - x_1 x_{n+1} \right) \\
=& \left(\sum_{i=1}^{n+1} (2x_i + 1) - \sum_{i=1}^{n} (x_i + x_{i+1} + 1) - (x_1 + x_{n+1} + 1) \right) \\
=& 0
\end{aligned}$$

For $\tilde{D}_n$, we can take

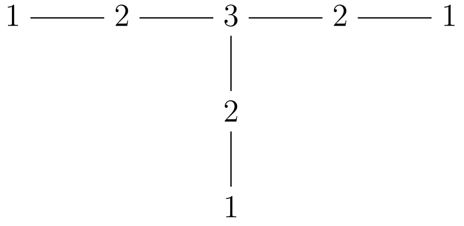


If we number the vertices as in the definition of $\tilde{D}_n$, we have

$$\begin{aligned}
(\delta, x) = \Bigg(& (x_1+1)^2 + (x_2+1)^2 + \sum_{i=3}^{n-1}(x_i+2)^2 + (x_n+1)^2 + (x_{n+1}+1)^2 \\
& - (x_1+1)(x_3+2) - (x_2+1)(x_3+2) - \sum_{i=3}^{n-2}(x_i+2)(x_{i+1}+2) \\
& -(x_{n-1}+2)(x_n+1) - (x_{n-1}+2)(x_{n+1}+1) \Bigg) \\
& - \left(\sum_{i=1}^{n+1} x_i^2 - x_1x_3 - x_2x_3 - \sum_{i=3}^{n-2} x_ix_{i+1} - x_{n-1}x_n - x_{n-1}x_{n+1} \right) \\
& - \big(4 \cdot 1^2 + (n-3) \cdot 4 - (4 \cdot (2 \cdot 1) + (n-4) \cdot (2 \cdot 2))\big)
\end{aligned}$$

$$\begin{aligned}
= & \Bigg(\sum_{i=1}^{n+1} x_i^2 + 2(x_1 + x_2 + x_n + x_{n+1}) + 4\sum_{i=3}^{n-1} x_i + 4 + 4(n-3) \\
& - (x_1x_3 + x_2x_3 + x_{n-1}x_n + x_{n-1}x_{n+1}) \\
& - 2(x_1 + x_2 + x_3 + x_{n-1} + x_n + x_{n+1}) - 4 \cdot 2 \\
& - \sum_{i=3}^{n-2}(x_ix_{i+1} + 2x_i + 2x_{i+1} + 4) \Bigg) \\
& - \left(\sum_{i=1}^{n+1} x_i^2 - x_1x_3 - x_2x_3 - \sum_{i=3}^{n-2} x_ix_{i+1} - x_{n-1}x_n - x_{n-1}x_{n+1} \right) \\
& = 4\sum_{i=3}^{n-1} x_i + 4(n-2) - 2x_3 - 2x_{n-1} - 4 \cdot 2 - \sum_{i=3}^{n-2}(2x_i + 2x_{i+1} + 4) \\
= & 0
\end{aligned}$$

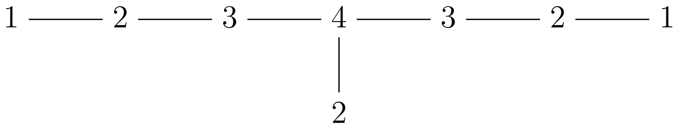
For $\tilde{E}_6$, we can take δ as



Then, we have

$$\begin{aligned}(\delta, x) =&(x_1+1)^2+(x_2+2)^2+(x_3+3)^2+(x_4+2)^2\\ &+(x_5+1)^2+(x_6+2)^2+(x_7+1)^2\\ &-((x_1+1)(x_2+2)+(x_2+2)(x_3+3)+(x_3+3)(x_4+2))\\ &-((x_4+2)(x_5+1)+(x_3+3)(x_6+2)+(x_6+2)(x_7+1))\\ &-\left(\sum_{i=1}^{7} x_i^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_4x_5 - x_3x_6 - x_6x_7\right)\\ &-(1+4+9+4+1+4+1-(2+6+6+2+6+2))\\ =&(2x_1+4x_2+6x_3+4x_4+2x_5+4x_6+2x_7)\\ &-(2x_1+x_2+3x_2+2x_3+2x_3+3x_4+x_4+2x_5+2x_3+3x_6+x_6+2x_7)\\ =&0.\end{aligned}$$

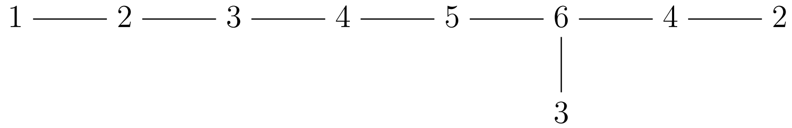
For $\tilde{E}_7$, δ is



Then we have

$$\begin{aligned}(\delta, x) =&(x_1+1)^2+(x_2+2)^2+(x_3+3)^2+(x_4+4)^2+(x_5+3)^2+(x_6+2)^2\\ &+(x_7+1)^2+(x_8+2)^2-(x_1+1)(x_2+2)-(x_2+2)(x_3+3)\\ &-(x_3+3)(x_4+4)-(x_4+4)(x_5+3)-(x_5+3)(x_6+2)\\ &-(x_6+2)(x_7+1)-(x_4+4)(x_8+2)-\left(\sum_{i=1}^{8} x_1^2-\sum_{i=1}^{6} x_ix_{i+1}-x_4x_8\right)\\ &-((1+4+9+16+9+4+1+4)-(2+6+12+12+6+2+8))\\ =&2x_1+4x_2+6x_3+8x_4+6x_5+4x_6+2x_7+4x_8+(1+4+9+16+9\\ &+4+1+4)-(2x_1+x_2+3x_2+2x_3+4x_3+3x_4+3x_4+4x_5+2x_5\\ &+3x_6+x_6+2x_7+4x_8+2x_4)-(2+6+12+12+6+2+8)\\ =&0\end{aligned}$$

Finally, for $\tilde{E}_8$, take δ to be



We then have

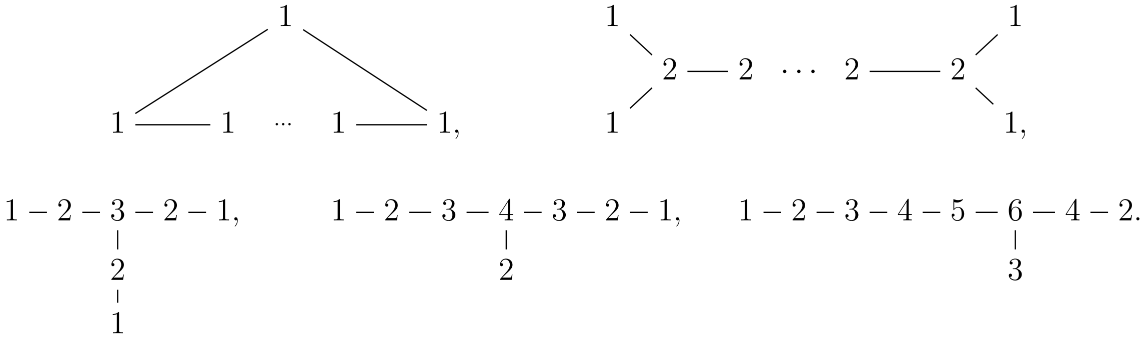
$$\begin{aligned}(\delta, x) =&2x_1+4x_2+6x_3+8x_4+10x_5+12x_6+8x_7+4x_8+6x_9\\ &-(2x_1+x_2+3x_2+2x_3+4x_3+3x_4+5x_4+4x_5+6x_5+5x_6+4x_6\\ &+6x_7+2x_7+4x_8+3x_6+6x_9)\\ =&0\end{aligned}$$

□

PROPOSITION 2.4.13. Let Q be a connected quiver. Then

1. q is positive definite if and only if Q is simply laced Dynkin.
2. q is positive demi-definite if and only if Q is of Euclidean type $\tilde{A}, \tilde{B}, \tilde{E}$ (or if Q is simply laced dynkin).

Proof. 2. Using Proposition 2.4.11, to prove that the condition is sufficient, it suffices to find a vector δ for each Euclidean diagram such that $(\delta, x) = 0$. By the last lemma, we have those δ's. Recall that they are given by



To show that the condition is necessary, suppose by contraposition that Q is neither Euclidean nor Dynkin. Then, by Proposition 2.4.3, Q contains a proper subquiver Q' of Euclidean type. Denote by q' its quadratic form and δ the corresponding dimension vector given above.

If Q and Q' have the same set of vertices, then Q has more arrows than Q' and then $0 = q'(\delta) < q(\delta)$.

If Q has more vertices than Q', then we can choose a vertex i_0 in $Q \backslash Q'$ which is connected by an edge to a vertex j_0 of Q'. Taking x such that $x_i = 2\delta_i$ for all $i \in Q'$, $x_{i_0} = 1$ and $x_j = 0$ for the rest yields $q(x) = q'(2\delta) + 1 - 2\delta_{j_0} = 1 - 2\delta_{j_0} < 0$.

In both cases, q is not positive semi-definite.

1. The condition is necessary as for each Euclidean diagram, one gets $q(\delta) = 0$ so that q is not positive definite.

It remains to show that the condition is sufficient. Extend the quiver Q by a vertex, labelled $(n+1)$ and denote $\overline{Q}$ the Euclidean quiver obtained. Denote also by $\overline{q}$ the Tits form of $\overline{Q}$. By contradiction, suppose that there is $x \in \mathbb{Q}^n$, $x \neq 0$ such that $q(x) \leq 0$. Define $\overline{x} \in \mathbb{Q}^{n+1}$ by $\overline{x_i} = x_i$ for $i \leq n$ and $\overline{x_{n+1}} = 0$. Then one gets $\overline{q}(\overline{x}) = q(x) \leq 0$. It implies that $\overline{q}(\overline{x}) = 0$ as $\overline{q}$ is positive semi-definite. It also implies, by the third point of Proposition 2.4.11, that $\overline{x} = \frac{a}{b}\delta$. As $\overline{x_{n+1}} = 0$, we have $a = 0$, which then implies $x = 0$, a contradiction. □

DEFINITION 2.4.14. For a quadratic form $q' \colon \mathbb{Q}^n \to \mathbb{Q}$, we say that the element $x \in \mathbb{Z}^n \backslash \{0\}$ is

1. a *real root* of q' if $q'(x) = 1$.
2. an *imaginary root* of q' if $q'(x) = 0$.

Moreover, we denote Φ the set of all roots of the form (both imaginary and real).

Let $\{e_i : 1 \leq i \leq n\}$ be the standard basis for $\mathbb{Z}^n$ and x_i the decomposition of the root x. We say that the root is *positive* if $x_i \geq 0$ for all i and we say that it is *negative* if $x_i \leq 0$ for all i.

We denote by Φ_+ the set of positive roots and by Φ_- the set of negative roots.

Note that we only consider the integer roots. It will have a crucial importance in the following propositions.

Proposition 2.4.15. Let Q be a quiver with n vertices and q its Tits form. Then

1. e_i is a real root of q for all $1 \leq i \leq n$.
2. If α is a root, then so is $-\alpha$.
3. If Q is Euclidean and α is a root different from δ (that we defined in Lemma 2.4.12), then $\alpha - \delta$ is also a root.
4. If q is positive semi-definite, each root is either positive or negative. Therefore, $\Phi = \Phi_- \cup \Phi_+$ and $\Phi_- = -\Phi_+$.

Proof. The 2 first points are direct as $q(e_i) = 1$ and $q(-\alpha) = (-1)^2 q(\alpha) = q(\alpha)$. The 3rd point follows from $q(\alpha-\delta) = q(\alpha)+q(-\delta)+(\alpha, -\delta) = q(\alpha)$ as $q(\delta) = 0 = (\alpha, -\delta)$.

For the last point, let a_i be the decomposition of α in the standard basis. Let $\beta = \sum_{i:a_i>0} a_i e_i := \sum_i b_i e_i$ and similarly $\gamma = \sum_{i:a_i<0} a_i e_i := \sum_i c_i e_i$. We have $\alpha = \beta + \gamma$. Moreover, we have $b_i c_i = 0$ and $b_i c_j \leq 0$ for $i \neq j$. Suppose, by contradiction, that β and γ are non-zero. It implies that $(\beta, \gamma) = \sum_{i=1}^n \beta_i \gamma_i - \sum_{\alpha \in Q_1} \beta_{s(\alpha)} \gamma_{t(\alpha)} > 0$. Therefore, $q(\alpha) = q(\beta+\gamma) = q(\beta)+q(\gamma)+(\beta, \gamma)$ so that $q(\alpha) > q(\beta)+q(\gamma)$. However, α is a root, so $q(\alpha)$ is less than or equal to 1. It means that $q(\beta) + q(\gamma) \leq 0$. As q is positive semi-definite, it means that $q(\beta) = q(\gamma) = 0$. Using the 3rd point of Proposition 2.4.11, it means that $\beta = \frac{a}{b}\delta$ for a, b non-zero integers as β is non-zero. It implies that $b_i \neq 0$ for all i, which means that $\gamma = 0$, a contradiction. □

Proposition 2.4.16. If Q is of Dynkin type, then there is a finite number of roots.

Proof. There are no imaginary roots as q is positive definite. Let α be a root of q. Let $\overline{Q}$ be an Euclidean quiver obtained by extending Q. Denote the new vertex i_0 and by $\overline{q}$ the Tits form of $\overline{Q}$. Extend α to $\mathbb{Z}^{n+1}$ by setting $\alpha_{i_0} = 0$. By the 3rd point of Proposition 2.4.15, $\alpha - \delta$ is also a root of $\overline{q}$. This root is negative at vertex i_0 and, by point 4 of Proposition 2.4.15, it is a negative root. Therefore, for all $i \in Q_0$, we have $\alpha_i \leq \delta_i$, which gives a finite number of possibilities. □

2.4.4 List of the Positive Real Roots of Dynkin Quivers

In this section, we will briefly list the positive roots for the different quivers, i.e. the n-uples x such that $x_i \geq 0$ and $\sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)} = 1$. For that, we will use the results that all the positive roots must be smaller than the imaginary root δ

of the corresponding extended Dynkin quiver (defined in Lemma 2.4.12), as proven in the proposition above. We will only provide a proof of the results for the case A_n and D_n, the others can be checked through a simple program computing the value of the Tits form given the dimension.

Positive Roots in A_n

All positive roots x must be such that $x_i \leq 1$. The arrow α between i and $i+1$ will give the value 1 in the computation of the tits form if and only if $x_i = x_{i+1} = 1$, otherwise it will give the value 0. Therefore, $q(x)$ is given by $k - l$ where k is the number of vertices i such that $x_i = 1$ and l is the number of edges such that both $x_{s(\alpha)} = 1$ and $x_{s(\alpha)} = 1$.

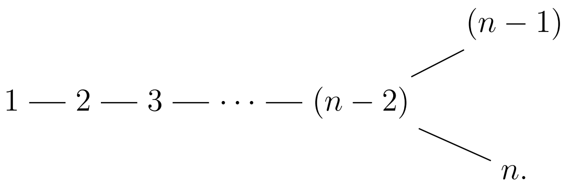
If $I = \{i, i+1, \cdots, j\}$ is such that for all $n \in I$, $x_n = 1$ and $x_{i-1} = x_{j+1} = 0$. Then there are $j - i + 1$ vertices and $j - i$ arrows in this connected component of A. In particular, it means that for each such connected component of vertices of dimension 1, the difference between k and l increases by one. It means in particular that in this case, $q(x)$ denotes the number of connected components with all vertices of dimension 1. Therefore, x is a real positive root if and only if all the vertices of degree 1 form a connected component of A_n.

In particular, there are $\frac{n(n+1)}{2}$ such roots as they are 1 connected component of legnth n, 2 connected components of length $n-2$, ..., n connected components of length 1.

> **Example 2.4.17.** The real positive roots of A_4 are
>
> $$1000, 0100, 0010, 0001, 1100, 0110, 0011, 1110, 0111, 1111.$$

Positive Roots in D_n

Recall that D_n is the following quiver (we will use the number of the vertices to denote those vertices)



All the roots x must be with $x_i \leq 2$ except the 3 outer vertices $x_1, x_{n-1}, x_n \leq 1$. Taking $x_n = 0$, we can find all the same roots as for the type A_{n-1} quiver (there are $\frac{(n-1)n}{2}$ such roots). Taking $x_{n-1} = 0$ and $x_n = 1$, we can find the other roots "of type A_n" (and there are $n-1$ of them).

Let us now focus on the case where $x_{n-1} = x_n = 1$. Then it means that $x_{n-2} = 1$, otherwise it would mean that the subquiver Q of vertices up until $n-3$ form a quiver of type A_{n-3}, with Tits form $q(x')$ positive and we would have

$$q(x) = q(x') + x_{n+1}^2 + x_n^2 \geq 2.$$

Note that, in general in D_n, if a vertex i is such that $x_i = 0$ (with $i < n-1$), then the Tits form would be the sum of the Tits forms of the quivers with vertices from 1 to $i-1$ and the one with vertices from i to n. Moreover, the first quiver will always be of type A, so its Tits form will be strictly positive if all the vertices are not of degree 0. Therefore, we can focus on the other part.

As for the quiver of type A_n, if all the vertices are of degree at most 1, then the Tits form represent the number of connected components of vertices of degree 1, therefore we only want one of such and we take x of the form $\begin{array}{r}1\\0\cdots01\cdots1\\1\end{array}$. There are $n-2$ of those roots.

Let us now investigate the case where some vertices are of degree 2. First of all, we can assume that $x_1 = 1$, otherwise we fall back into the roots of a quiver of type D_{n-1}. Moreover, as shown above, all the vertices of strictly positive degree must be connected; therefore, we can assume that all the vertices are of degree greater than or equal to 1. If we compute $q(2\text{—}2)$, we get the same result than just computing $q(2) = 4$. Therefore, we can group together all the connected vertices of dimension 2 and consider them as only one vertex to simplify the computation. Similarly, as $q(1\text{—}1) = q(1) = 1$, we can also group the connected vertices of dimension 1.

Therefore, we can investigate the following cases: $\begin{array}{r}1\\1212\cdots121\\1\end{array}$ or $\begin{array}{r}1\\1212\cdots212\\1\end{array}$.

Let us proceed by induction on the number of occurrences of 2 to show that only $\begin{array}{r}1\\12\\1\end{array}$ works. It is clear to see that it works. On the other hand, we have $q\left(\begin{array}{r}1\\121\\1\end{array}\right) = q\left(\begin{array}{r}1\\1212\\1\end{array}\right) = q\left(\begin{array}{r}1\\2\\1\end{array}\right) = 2$. Moreover, if $x = 12x'$ and suppose that the vertex of x' connected to the vertex of degree 2 is of degree 1, we have $q(x) = 1 + 4 + q(x') - 2 - 2 = q(x') + 1 > 1$ and therefore x is not a real root. It means that the real roots with at least a vertex of degree 2 are of the form $\begin{array}{r}1\\0\cdots01\cdots12\cdots2\\1\end{array}$ with at least on vertex of degree 1. There are $\frac{(n-3)(n-2)}{2}$ of those as there are $n-3$ roots with one vertex of degree 2, $n-2$ roots with 2 vertices of degree 2, etc.

In total, there are $\frac{(n-1)n}{2} + n - 1 + n - 2 + \frac{(n-3)(n-2)}{2} = n(n-1)$ of such roots.

Example 2.4.18. The 20 roots of D_5 are:

$$\begin{array}{r}0\\100\\0\end{array},\ \begin{array}{r}0\\010\\0\end{array},\ \begin{array}{r}0\\001\\0\end{array},\ \begin{array}{r}1\\000\\0\end{array},\ \begin{array}{r}0\\000\\1\end{array},\ \begin{array}{r}0\\110\\0\end{array}\ \begin{array}{r}0\\011\\0\end{array},\ \begin{array}{r}1\\001\\0\end{array},\ \begin{array}{r}0\\001\\1\end{array},\ \begin{array}{r}0\\111\\0\end{array},$$

$$\begin{array}{r}1\\011\\0\end{array},\ \begin{array}{r}0\\011\\1\end{array},\ \begin{array}{r}1\\001\\1\end{array},\ \begin{array}{r}1\\111\\0\end{array},\ \begin{array}{r}0\\111\\1\end{array},\ \begin{array}{r}1\\011\\1\end{array},\ \begin{array}{r}1\\111\\1\end{array},\ \begin{array}{r}1\\012\\1\end{array}\ \begin{array}{r}1\\112\\1\end{array},\ \begin{array}{r}1\\122\\1\end{array}.$$

Positive Roots in E_6

All the roots x are smaller (component-wise) than $\begin{matrix}12321\\2\end{matrix}$. There are 36 of those. Other than the ones related to quivers of type A and D, we have the following:

$$\begin{matrix}11111\\1\end{matrix},\ \begin{matrix}11211\\1\end{matrix},\ \begin{matrix}12211\\1\end{matrix},\ \begin{matrix}11221\\1\end{matrix},\ \begin{matrix}12221\\1\end{matrix},\ \begin{matrix}12321\\1\end{matrix},\ \begin{matrix}12321\\2\end{matrix}.$$

Positive Roots in E_7

All the roots x are smaller (component-wise) than $\begin{matrix}234321\\2\end{matrix}$. There are 63 of those. Other than the ones related to quivers of type A, D and E_6, we have the following:

$$\begin{matrix}111111\\1\end{matrix},\begin{matrix}112111\\1\end{matrix},\begin{matrix}112211\\1\end{matrix},\begin{matrix}122111\\1\end{matrix},\begin{matrix}122211\\1\end{matrix},\begin{matrix}112221\\1\end{matrix},\begin{matrix}122221\\1\end{matrix},\begin{matrix}123211\\1\end{matrix},$$

$$\begin{matrix}123221\\1\end{matrix},\begin{matrix}123211\\2\end{matrix},\begin{matrix}123321\\1\end{matrix},\begin{matrix}123221\\2\end{matrix},\begin{matrix}123321\\2\end{matrix},\begin{matrix}124321\\2\end{matrix},\begin{matrix}134321\\2\end{matrix},\begin{matrix}234321\\2\end{matrix}.$$

Positive Roots in E_8

All the roots x are smaller (component-wise) than $\begin{matrix}2465432\\3\end{matrix}$. There are 120 of those. Other than the ones related to quivers of type A, D, E_6 and E_7, we have the following:

$$\begin{matrix}1111111\\1\end{matrix},\begin{matrix}1121111\\1\end{matrix},\begin{matrix}1122111\\1\end{matrix},\begin{matrix}1221111\\1\end{matrix},\begin{matrix}1222111\\1\end{matrix},\begin{matrix}1122211\\1\end{matrix},\begin{matrix}1232111\\1\end{matrix},$$

$$\begin{matrix}1222211\\1\end{matrix},\begin{matrix}1122221\\1\end{matrix},\begin{matrix}1232111\\2\end{matrix},\begin{matrix}1232211\\1\end{matrix},\begin{matrix}1222221\\1\end{matrix},\begin{matrix}1232211\\2\end{matrix},\begin{matrix}1232211\\1\end{matrix}.$$

2.4.5 Linking Roots of Tits Forms and Representations

Now that we have studied the roots of Tits form, we will focus on the link between those roots and the representation of quivers.

> **Proposition 2.4.19.** Let Q be a connected quiver. Let M be a representation of Q with dimension vector d. Then we have
>
> $$\operatorname{codim}(\mathcal{O}_M) = \dim \operatorname{End}(M) - q(d) = \dim \operatorname{Ext}^1(M, M).$$

Proof. By Proposition 2.4.5, we know that $\dim(\mathcal{O}_M) = \dim G_d - \dim \operatorname{Aut}(M)$. The group of automorphisms of M is an open subgroup of the group of endomorphisms for the Zariski topology as $\operatorname{Aut}(M) = \{f \in \operatorname{End}(M) : \det(f_i) \neq 0\ \forall i \in Q_0\}$, therefore $\dim \operatorname{Aut}(M) = \dim \operatorname{End}(M)$. Each GL_d is of dimension d_i^2, so $\dim G_d = \sum_{i\in Q_0} d_i^2$.

Recall that R_d is the space of representations of Q of dimension d. Recall also that $\dim(R_d) = \sum_{\alpha \in Q_1} d_{s(\alpha)} d_{t\alpha}$ Therefore,

$$\operatorname{codim} \mathcal{O}_M = \dim R_d - dim\mathcal{O}_M = \left(\sum_{\alpha \in Q_1} d_{s(\alpha)} d_{t(\alpha)} \right) - \left(\sum_{i \in Q_0} d_i^2 - \dim \operatorname{End} M \right).$$

Using Proposition 2.4.9, we have $q(d) = \dim \operatorname{Hom}(M, M) - \dim \operatorname{Ext}^1(M, M)$ and we get the second equality. ☐

Proposition 2.4.20. If we have a dimension vector d such that $q(d) \leq 0$, then there are infinitely many isoclasses of representations of Q of dimension vector d.

Proof. Let $d \in \mathbb{N}^n$ be such that $q(d) \leq 0$, M a representation of Q with dimension vector d. Using the previous result, we have that $\operatorname{codim} \mathcal{O}_M \geq \dim \operatorname{End}(M) \geq 1$. Therefore, the dimension of R_d is strictly greater than the dimension of the orbits of M for all M, which means that the number of orbits is infinite. ☐

Before proving Gabriel's theorem, it only remains to show a technical lemma from algebra.

Lemma 2.4.21. Let $\mathbf{k}$ be an algebraically closed field. Let A be a $\mathbf{k}$-algebra. Let M be a finite-dimensional A-module and let $\operatorname{End} M$ be its endomorphism algebra. If M is indecomposable, then every endomorphism $f \in \operatorname{End}(M)$ is of the form

$$f = \lambda 1_M + g$$

with $\lambda \in \mathbf{k}$ and $g \in \operatorname{End} M$ be nilpotent.

Proof. Let f be an endomorphism of M. In particular, $f \colon M \to M$ is a $\mathbf{k}$-linear map between finite-dimensional $\mathbf{k}$-vector spaces. We can therefore compute the characteristic polynomial of f as being $\chi_f(x) = \prod_{i=1}^{t} (x - \lambda_i)^{\nu_i}$ where the λ_i are the eigenvalues of f. Moreover, there is a basis $\mathcal{B}$ of M such that

$$f = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_t \end{pmatrix}.$$

Let $M_i = \ker(f - \lambda_i 1_M)^{\nu_i}$. We know that $\dim M_i = \nu_i$ and $M = M_1 \oplus \cdots \oplus M_t$ as vector spaces. Let us show that we also have this decomposition as modules.

Let $h_i = (f - \lambda_i 1_M)^{\nu_i}$. These functions are polynomials in f:

$$h_i = f^{\nu_i} + a_{\nu_i - 1} f^{\nu_i - 1} + \cdots + a_1 f + a_0$$

for some $a_j \in \mathbf{k}$. Therefore, as $f \in \operatorname{End} M$, so does h_i. It implies that its kernel M_i is an A-module and we have the decomposition as A-modules.

As M is indecomposable, we have $t = 1$ so f has only one eigenvalue and the matrix representing f in the basis $\mathcal{B}$ is a triangular one whose diagonal entries are λ_1. Hence $f = \lambda_1 1_M + g$ with g represented as the upper triangular matrix with zero coefficients on the diagonal, which implies that g is nilpotent. ☐

We can now prove the desired results, following the proof of Theorem 8.12 in [53].

Theorem 2.4.22 (Gabriel). Let Q be a connected quiver. Then

1. Q is of finite representation type if and only if Q is simply laced Dynkin.
2. If Q is a simply laced Dynkin quiver, then the dimension vector induces a bijection ψ from the isoclasses of indecomposable representations of Q to the set of positive roots of the Tits form:

$$\psi \colon \operatorname{ind} Q \xrightarrow{\sim} \Phi_+, \ M \mapsto \dim M.$$

Proof. 2. Let us start by showing that ψ is well-defined. In order to do that, we need $q(\dim(M)) = 1$ for all M indecomposable, let us show it by showing that $\mathrm{End}(M) \simeq \mathbf{k}$ and $\mathrm{Ext}^1(M, M) \simeq 0$ as in that case, we have by Proposition 2.4.9 that $q(d) = \dim(\mathrm{End}(M)) - \dim \mathrm{Ext}^1(M, M) = 1 - 0 = 1$.

We show the isomorphisms $\mathrm{End}(M) \simeq \mathbf{k}$ and $\mathrm{Ext}^1(M, M) \simeq 0$ by induction on the dimension $\sum_{i \in \mathbb{Q}_0} d_1$ of M. If M is of dimension one, it is simple and the result is immediate. Now, suppose that $\dim M > 1$ and that for all indecomposable representations L such that $\dim L < \dim M$, we have $\mathrm{End}\, L \simeq \mathbf{k}$ and $\mathrm{Ext}(L, L) \simeq 0$. Using Theorem 2.2.6, we can see M as a $\mathbf{k}Q$-modules. As $\mathbf{k}Q$ is a $\mathbf{k}$-algebra and M is an indecomposable, finite-dimensional $\mathbf{k}Q$-module, by Lemma 2.4.21, every endomorphism of M is of the form $\lambda I_M + g$ for $\lambda \in \mathbf{k}$ and g nilpotent. Suppose by contradiction that $\mathrm{End}(M) \not\simeq \mathbf{k}$. It implies that there is a non-zero idempotent g of M. There is a $m \geq 2$, such that $g^m = 0$ and $g^{m-1} \neq 0$, replacing eventually g by g^{m-1}, we can assume that $m = 2$. Among the non-zero endomorphisms whose square is zero, we take the one whose image is of smallest dimension. As $g^2 = 0$, $\mathrm{im}(g) \subset \ker g$, it implies that there is L an indecomposable summand of $\ker g$ such that $\mathrm{im}\, g \cap L \neq 0$. In particular, we have $\ker g \simeq L \oplus L'$. Define the morphism i to be the composition $i \colon \mathrm{im}\, g \hookrightarrow \ker g \xrightarrow{\pi} \mathrm{L}$. We then have that $M \xrightarrow{g} \mathrm{im}(g) \xrightarrow{i} L \hookrightarrow M$ is a non-zero endomorphism of M whose square is zero and whose image is $i(\mathrm{im}(g))$. By minimality of g, we have that $\dim(\mathrm{im}(g)) \leq \dim(i(\mathrm{im}\, g))$, which implies that i is injective. We get the short exact sequence

$$0 \to \mathrm{im}(g) \overset{i}{\hookrightarrow} L \twoheadrightarrow \operatorname{coker} i \to 0.$$

Applying the covariant functor $\mathrm{Hom}(-, L)$ and using the long homological sequence (Proposition B.5.12), we get

$$\mathrm{Ext}^1(L, L) \to \mathrm{Ext}^1(\mathrm{im}\, g, L) \to \mathrm{Ext}^2(\operatorname{coker} i, L).$$

Moreover, by Proposition 2.3.4, we have that the global dimension of $\mathrm{Rep}(Q)$ is at most one and using Proposition B.5.14, we get that $\mathrm{Ext}^2(\operatorname{coker} i, L) = 0$. By induction, we have that $\mathrm{Ext}^1(L, L) = 0$ and therefore $\mathrm{Ext}^1(\mathrm{im}\, g, L) = 0$. Let u be the inclusion $\ker g \hookrightarrow M$ and let (X, j_1, j_2) be the pushout of u, π as defined in

Section B.2.2. By definition of the pushout, we have the following diagram:

$$\begin{tikzcd}
\ker g \arrow[r, "u"] \arrow[d, "\pi"'] & M \arrow[d, "j_2"] \arrow[rdd, bend left, "g"] & \\
L \arrow[r, "j_1"'] \arrow[rrd, bend right, "0"'] & X \arrow[rd, "\exists g'"'] \arrow[ul, phantom, "\lrcorner", very near start] & \\
 & & \operatorname{im} g.
\end{tikzcd}$$

In particular, $g'j_1 = 0$. Moreover, we can see X as being $(M \sqcup L)/\sim$ where $m \sim l$ if and only if there is some $x \in \ker g$ such that $u(x) = \pi(x)$. The canonical morphisms are $j_1 \colon L \to X, l \mapsto [(l, 0)]$ and $j_2 \colon M \to X, m \to [(0, m)]$. Moreover, in this case, we have $g'[(m, l)] = g(m)$ Therefore, if we suppose $[(m, l)]$ in the kernel of g', then $g(m) = 0$. It implies that there is $x \in \ker g$ such that $u(x) = m$. It means that $[(m, l)] = [(u(x), l)] = [(0, l - \pi(x))] \in \operatorname{im} j_1$.

Furthermore, if $j_1(l) = [(0, l)] = [(0, 0)]$, it means that $(0, b) = (0, 0) + (u(x), -\pi(x))$ for some $x \in \ker g$. As u is injective, it implies that $x = 0$ and therefore $b = 0$. Moreover, as g is surjective and $g = g' \circ j_2$, we have g' surjective. In total, it means that we have the following diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \ker g \arrow[r, "u"] \arrow[d, "\pi"'] & M \arrow[r, "g"] \arrow[d, "j_2"] & \operatorname{im} g \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r, "j_1"] & X \arrow[r] \arrow[ul, phantom, "\lrcorner", very near start] & \operatorname{im} g \arrow[r] & 0.
\end{tikzcd}$$

As $\operatorname{Ext}^1(\operatorname{im} g, L) = 0$, applying $\operatorname{Hom}(-, L)$ and taking the long exact cohomological sequence, we get

$$0 \to \operatorname{Hom}(\operatorname{im} g, L) \to \operatorname{Hom}(X, L) \xrightarrow{\circ j_1} \operatorname{Hom}(L, L) \to 0.$$

It means that the pre-composition by j_1 is surjective. Therefore, there is $h \colon X \to L$ such that $hj_1 = 1_L$ and the bottom row splits. Let $\nu \colon L \hookrightarrow \ker g$ be the inclusion of the direct summand in the decomposition $\ker g = L \oplus L'$, we have $\pi\nu = 1_L$. The morphisms $hj_2 \colon M \to L$ and $u\nu \colon L \to M$ are such that

$$hj_2u\nu = hj_1\pi\nu = 1_L 1_L = 1_L.$$

It means that L is a direct summand of M. However, as M is indecomposable, we have $L \simeq M$ or $L \simeq 0$. The former is impossible as $L \subset \ker g \neq M$ as g is non-zero, and the latter is also impossible as $L \cap \operatorname{im} g \neq 0$. Therefore, we have a contradiction. It means that there are no nilpotent endomorphisms of M and that $\dim(\operatorname{End}(M)) = 1$. Moreover, as q is positive definite, we have $0 \leq \dim(\operatorname{Ext}^1(M, M)) = \dim(\operatorname{End}(M)) - q(d) \leq 1 - 1 = 0$.

Let us now show that ψ is injective. Let M, M' be indecomposable representations with the same dimension vector. We then have

$$\operatorname{codim}(\mathcal{O}_M) = \dim \operatorname{Ext}^1(M, M) = 0 = \dim \operatorname{Ext}^1(M', M') = \operatorname{codim}(\mathcal{O}_{M'}).$$

By the second point of Lemma 2.4.5, it means that $\mathcal{O}_M = \mathcal{O}_{M'}$ hence $M \simeq M'$.

It remains to show that ϕ is surjective. Let Q be simply laced Dynkin and d a positive root of the Tits form. Let M be a representation of Q with degree vector d such that $\mathcal{O}_M$ is of maximal dimension in R_d. Let us show that M is indecomposable. Suppose $M = M_1 \oplus M_2$ and let us first show $\operatorname{Ext}^1(M_1, M_2) = 0 = \operatorname{Ext}^1(M_2, M_1)$. Suppose $\operatorname{Ext}^1(M_1, M_2) \neq 0$, it means that there is a non split exact sequence

$$0 \to M_2 \to E \to M_1 \to 0.$$

We have that $\dim E = \dim M$ and using Proposition 2.4.7, we get that $\dim \mathcal{O}_M < \dim \mathcal{O}_E$, a contradiction with the maximality of $\mathcal{O}_M$. Moreover, using Proposition B.5.15 and, as $M_1 \oplus M_2 \simeq M_1 \times M_2$, we have that $\operatorname{Ext}^1(M_1 \oplus M_2, X) \simeq \operatorname{Ext}^1(M_1, X) \oplus \operatorname{Ext}^1(M_2, X)$ and $\operatorname{Ext}^1(X, M_1 \oplus M_2) \simeq \operatorname{Ext}^1(X, M_1) \oplus \operatorname{Ext}^1(X, M_2)$. It means that

$$\begin{aligned} \dim \left(\operatorname{Ext}^1(M_1 \oplus M_2, M_1 \oplus M_2)\right) =& \dim \operatorname{Ext}^1(M_1, M_1) + \dim \operatorname{Ext}^1(M_1, M_2) \\ &+ \dim \operatorname{Ext}^1(M_2, M_1) + \dim \operatorname{Ext}^1(M_2, M_2) = 0. \end{aligned}$$

Moreover, if M_1 and M_2 are non-zero, then $\operatorname{End}(M_1 \oplus M_2) \supset \operatorname{End}(M_1) \times \operatorname{End}(M_2)$. Therefore,

$$\begin{aligned} 1 = q(d) =& \dim\left(\operatorname{End}(M_1 \oplus M_2)\right) - \dim\left(\operatorname{Ext}^1(M_1 \oplus M_2, M_1 \oplus M_2)\right) \\ =& \dim\left(\operatorname{End}(M_1 \oplus M_2)\right) \\ \geq& \dim(\operatorname{End}(M_1)) + \dim(\operatorname{End}(M_2)) \geq 2, \end{aligned}$$

which yields a contradiction.

1. The fact that Q is of finite representation type if Q is simply laced Dynkin follows immediately from point 2. For the converse, assume Q is not simply laced Dynkin. It means that the tits form associate to Q is not positive definite. There is some non-zero $d \in \mathbb{Z}^n$ such that $q(d) \leq 0$. Using Proposition 2.4.19, we have $\operatorname{codim}(\mathcal{O}_M) = \dim(\operatorname{End}(M)) - q(d) \geq \dim(\operatorname{End}(M)) > 0$ for all representation M of dimension d as $\operatorname{End}(M)$ contains the maps $t \cdot \mathrm{id}_M$ for all $t \in \mathbf{k}$, which is of dimension 1. Therefore, there is an infinite number of isoclasses of representations of dimension d. Otherwise, let $M_1, \cdots M_n$ be the different isoclasses of representations, we would have $R_d = \bigcup_{i=1}^n \mathcal{O}_{M_i}$, which would implies that at least one of the $\mathcal{O}_{M_i}$ is of codimension 0. As each of the representations of dimension d is a finite direct sum of indecomposable representations, it implies that the number of indecomposable representations is infinite. □

3 | Decomposition of Persistence Modules

Recall that the module M defined in Equation 1.1 as

$$M = H_p(\mathbb{X}^0) \oplus H_p(\mathbb{X}^1) \oplus \cdots \oplus H_p(\mathbb{X}^m)$$

on the ring $\mathbf{k}[t]$ with an action by t being

$$t^k \alpha = \begin{cases} \varphi_p^{i,i+k}(\alpha) & \text{if } i+k \leq m \\ 0 & \text{else.} \end{cases}$$

for each $\alpha \in H_p(\mathbb{X}^i)$ is a finitely generated module over a graded PID. The well-known theorem structure of finitely generated modules over a PID (for example, given and proven in Section 3.8 of [36]) gives that, in the case of $\mathbf{k}[t]$ as base ring, it decomposes uniquely into the form

$$M \simeq \left(\bigoplus_{i=1}^{r} t^{a_i} \mathbf{k}[t] \right) \oplus \left(\bigoplus_{i=1}^{s} t^{b_i} \mathbf{k}[t]/(t^{(b_i+c_i)}) \right).$$

In the context of persistent homology, the first terms represent homology classes that are born at $\mathbb{X}^{a_i}$ and never die whereas the last terms represent homology classes that are born at $\mathbb{X}^{b_i}$ and die entering $\mathbb{X}^{b_i+c_i}$.

Indeed, the first definition we had of an element $\alpha \in H_p(\mathbb{X}^s)$ being born at j is that $\alpha \in \operatorname{im}(\varphi_p^{j,s})$ but $\alpha \notin \operatorname{im}(\varphi_p^{j-1,s})$. It means that there is no $\beta \in H_p(\mathbb{X}^{j-1})$ such that $t^{s-j+1} \cdot \beta = \alpha$ but that there is a $\gamma \in H_p(\mathbb{X}^{j+1})$ such that $t^{s-j} \cdot \gamma = \alpha$. We can even write γ as $t^j \cdot \delta$ to see more easily each element generated by γ as an element of $t^j \mathbf{k}[t]\delta$ (note that the elements $t^i\delta$ are not defined for $i < j$).

Similarly, if the element $\alpha = t^s \cdot \delta \in H_p(\mathbb{X}^s)$ is born at j and dies at k, then $t^{k-s} \cdot \alpha = t^k \delta \in \operatorname{im}(\varphi_p^{j-1,k})$, if we choose δ carefully (recall Remark 1.2.12), we can have $t^k \delta = 0$, which gives that δ is a generator of $t^j \mathbf{k}[t]/(t^{(k)})$.

Therefore, in the case of tame functions, the decomposition of persistence modules is equivalent to the barcode. Studying how different persistence modules decompose provides crucial information about the structure of those modules. The goal of this chapter will be to generalise the result of the decomposition of persistence modules related to tame functions to more persistence modules.

We will first present the interval modules and show that they are indecomposable. Afterwards, we will show that if a persistence module admits a decomposition into indecomposable modules, then this decomposition is essentially unique. Then, we will show 2 instances in which a persistence module admits a decomposition into interval submodules. To finish the section, we will give a definition of the barcode and persistent diagram.

Definition 3.0.1. A persistence module $\mathbb{V}$ is *indecomposable* if the only decompositions $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$ are the trivial decompositions $\mathbb{V} \oplus 0$ and $0 \oplus \mathbb{V}$.

Example 3.0.2. The persistence module $\mathbb{V}$ defined by $V_0 = \mathbf{k}$, $V_j = 0$ for $j \neq 0$ and $v_i^j = 0$ for all $i, j \in \mathbb{R}$ is indecomposable.

Indeed, if $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$, we would have in particular $U_j = W_j = 0$ for all $k \neq 0$. Moreover, the vector space $V_0 = \mathbf{k}$ is indecomposable, so either U_0 or W_0 is null; therefore, either $\mathbb{U}$ or $\mathbb{V}$ is null.

Let T be an ordered set and $J \subset T$ an *interval*, i.e. a subset of T such that for all $x, y \in J$, and $z \in T$ such that $x \leq z \leq y$, we have $z \in J$. To this interval, we associate the *interval module* $\mathbb{I}^J$ with vector spaces

$$I_t = \begin{cases} \mathbf{k} & \text{if } t \in J \\ 0 & \text{else} \end{cases}$$

and maps

$$v_i^j = \begin{cases} 1 & \text{if } i, j \in J \\ 0 & \text{else.} \end{cases}$$

These interval modules are of major importance in the study of persistence modules as they are indecomposable and provide an easy interpretation of the module: they describe a feature and its "lifetime".

Proposition 3.0.3. Interval modules are indecomposable.

Proof. Let $\mathbb{I}$ be the interval module associated to the interval $J \subset T$. First of all, the set of endomorphisms of $\mathbb{I}$ is isomorphic to $\mathbf{k}$. Indeed, let $t \in J$, then an endomorphism of $\mathbb{I}$ acts on $I_t = \mathbf{k}$ by a scalar multiplication by c_t. Moreover, we know that if φ is a morphism of persistence modules, then we have $(v')_t^s \circ \varphi_t = \varphi_s \circ v_t^s$ where v, v' are the morphisms between the different vector spaces and $t \geq s$. So in our case, it implies that $c_t = c_s$ and the set of endomorphisms is isomorphic to $\mathbf{k}$ by taking the constant of the multiplication.

Now, let us suppose $\mathbb{I} = \mathbb{U} \oplus \mathbb{V}$. Then the projection map from $\mathbb{I}$ to $\mathbb{U}$ is an idempotent in $\mathrm{End}(\mathbb{I})$. However, the only idempotents of $\mathbf{k}$ are 0 and 1, so the projection is either the null function or the identity and $\mathbb{U}$ is either 0 or $\mathbb{I}$, the latter case implies that $\mathbb{V} = 0$. □

3.1 Uniqueness of Decomposition

Now that we have proven that interval modules are indecomposable, let us show that interval decompositions are unique up to isomorphism. In order to do that, we will show a more general result for modules proven in [3].

Let M be an R-module and N its *endomorphism ring*[1]. Suppose M has a decomposition as a direct sum (finite or infinite) of indecomposable submodules[2]:

$$M = \bigoplus_{\mu \in L} M_\mu.$$

Then, for all $\mu \in L$, the projection

$$e_\mu = i_\mu \circ \pi_\mu \colon M \to M, x = (x_\lambda)_{\lambda \in L} \mapsto x_\mu$$

is idempotent in N and such that $e_\mu(M) = M_\mu$. Moreover, it is orthogonal with the other e_ν (for $\nu \neq \mu : e_\mu \circ e_\nu = 0 = e_\nu \circ e_\mu$).

We will first prove two results from algebra: one technical lemma and the other closely related to the desired result, then we will only need to translate those results in our case.

Lemma 3.1.1. Let $M = \bigoplus_{\mu \in L} M_\mu$ be a decomposition of the module M into indecomposable submodules. Suppose that, for all $\mu \in L$, the sum of 2 endomorphisms of M_μ which are not isomorphisms is not an isomorphism. Let $a, b \in N$ be such that $a + b = 1$. Then, for any $\{\mu_1, \cdots, \mu_s\} \subset L$, finite subset of L, we can find submodules $\overline{M_1}, \cdots, \overline{M_s}$ such that $\overline{M_i} \simeq M_{\mu_i}$ with the isomorphism given by the restriction of a or the restriction of b. Moreover, we have

$$M = \overline{M_1} \oplus \cdots \oplus \overline{M_s} \oplus \bigoplus_{\mu \in L \setminus \{\mu_1, \cdots, \mu_s\}} M_\mu.$$

Note that the interesting part of the lemma is that the isomorphism between the 2 submodules is induced by either a or b.

Proof. The morphisms $e_{\mu_1} a$ and $e_{\mu_1} b$ induce on M_{μ_1} two endomorphisms whose sum is the identity as $e_{\mu_1} a + e_{\mu_1} b = e_{\mu_1}(a + b) = e_{\mu_1}$ which restricts to the identity on M_{μ_1}. By our assumption, it implies that either $e_{\mu_1} a$ or $e_{\mu_1} b$ is an automorphism on M_{μ_1}. Without loss of generalities, suppose that $e_{\mu_1} a$ is an isomorphism on M_{μ_1}. It means that we have

$$M_{\mu_1} \underset{a}{\overset{\sim}{\to}} \overline{M_1} \underset{e_{\mu_1}}{\overset{\sim}{\longrightarrow}} M_{\mu_1}$$

with $a(M_{\mu_1}) = \overline{M_1}$ a submodule of M as the image of a module by a module homomorphism is always a module. For such a module, we have that $e_{\mu_1} : \overline{M_1} \overset{\sim}{\to} M_{\mu_1}$ is an isomorphism between the two modules.

We have $M \simeq \overline{M_1} + \bigoplus_{\mu \neq \mu_j} M_\mu$ as the map $x = (x_1, x_2, \cdots) \mapsto (e_{\mu_1}(x), x_2, \cdots)$ is an isomorphism. Let us show that the sum is direct. Let $x \in \overline{M_1} \cap \bigoplus_{\mu \neq \mu_j} M_\mu$. As the kernel of e_{μ_1} is $\bigoplus_{\mu \neq \mu_1} M_\mu$, we have $e_{\mu_1}(x) = 0$ and as e_{μ_1} is isomorphic on $\overline{M_1}$, we have that $x = 0$. In particular, we have

$$M \simeq \overline{M_1} \oplus \bigoplus_{\mu \neq \mu_j} M_\mu.$$

---

[1] It is the set of morphisms of modules from M to M endowed with the addition defined as $f + g \colon x \mapsto f(x) + g(x)$ and the composition of functions. It is routine to check that it forms a ring.

[2] In the same sense than for a persistence module: M is *indecomposable* if the only decomposition $M = N_1 \oplus N_2$ are the trivial decompositions $M \oplus 0$ and $0 \oplus M$.

As $\overline{M_1}$ is isomorphic to M_{μ_1}, the sum of 2 non-isomorphic endomorphisms of $\overline{M_1}$ is not isomorphic and the above decomposition still satisfies the conditions of the lemma, we can apply the same reasoning with μ_2 on it. By induction, we have the result. ☐

This technical lemma allows us to show the following result, which is closely related to the result we are looking for.

Proposition 3.1.2. Let $M = \bigoplus_{\mu \in L} M_\mu$ such that, for all $\mu \in L$, the sum of 2 endomorphisms of M_μ which are not isomorphisms is not an isomorphism. Then (using the same notations as above),

1. For any non-zero idempotent element f of N. There is M_μ such that f is an isomorphism on M_μ and the isomorphic image of M_μ by f is an indecomposable direct summand of M.

   In particular, every indecomposable direct summand of M is isomorphic to one of M_μ.

2. Given a second decomposition $M = \bigoplus_{\nu \in L'} M'_\nu$, there exists a bijection $\sigma\colon L \to L'$ such that $M_\mu \simeq M'_\nu$.

   In other words, the decomposition of M into indecomposable components is unique up to isomorphism and permutations.

Proof. 1. Denote by $f' = 1 - f$. It is idempotent and orthogonal to f. Indeed,

$$f'f' = (1-f)(1-f) = 1 - f - f + ff = 1 - f - f + f = 1 - f = f'$$

and

$$ff' = f(1-f) = f - ff = f - f = 0$$

(and similarly, $f'f = 0$). We have

$$M = f(M) \oplus f'(M).$$

First, the sum is direct as if $m \in f(M) \cap f'(M)$, we have $x, y \in M$ such that $m = f(x) = y - f(y)$. Thus, by linearity of f, $f(x+y) = y$ and, applying f and using its idempotence, we get $f(x+y) = f(y)$ and therefore, $m = f(x) = 0$. It only remains to check the inclusion $M \subset f(M) + f'(M)$ as the other is direct. Let $m \in M$, then $m = f(m) + m - f(m) = f(m) + f'(m)$.

Now, let v be a non-zero element of $f(M)$ (such an element exists as f is non-zero) and let $\mu_1, \cdots, \mu_s$ be the finite set of indices such that $e_{\mu_i}(v) \neq 0$ for all $1 \leq i \leq s$ and $e_\mu(v) = 0$ for the other indices. Note that it amounts to taking the different non-zero elements in the decomposition of v into the elements of the direct sum: $v = \mu_1(v) + \cdots + \mu_s(v)$. Hence, the set of indices is indeed finite by definition of the direct sum.

By the previous lemma with $a = f$, $b = f'$ and $\mu_1, \cdots, \mu_s$, there are $\overline{M_1}, \cdots, \overline{M_s}$ with $\overline{M_i} \simeq M_{\mu_i}$ induced by either f or f'. If one of the $\overline{M_i} \simeq M_{\mu_i}$ is induced by f, then we have the required result.

By contradiction, suppose that for all i, the morphism $\overline{M_i} \simeq M_{\mu_i}$ is induced by f'. It implies that f' maps $M_{\mu_1} \oplus \cdots \oplus M_{\mu_s}$ isomorphically into $\overline{M_1} \oplus \cdots \oplus \overline{M_s}$. However,

$$0 \neq v = \mu_1(v) + \cdots + \mu_s(v) \in M_{\mu_1} \oplus \cdots \oplus M_{\mu_s}$$

and

$$f'(v) = (1 - f)(v) = v - f(v) = v - v = 0.$$

The last equality is because f is idempotent and v is already in the image of f. We therefore have a contradiction as we assumed that f' was an isomorphism between $M_{\mu_1} \oplus \cdots \oplus M_{\mu_s}$ and $\overline{M_1} \oplus \cdots \oplus \overline{M_s}$, so $f'(v) = 0$ implies $v = 0$ but we defined v to be non-zero.

The particular remark comes from the fact that if A is an indecomposable direct summand of M, then $e_A = i_A \circ \pi_A$ is a non-zero idempotent. Therefore, there is an M_μ such that e_A is an isomorphism on M_μ. It gives that $e_A(M_\mu) \simeq A'$ with A' a non-zero submodule of A. As A is indecomposable, it implies that $A' = A$.

2. By the first part of the proposition, each M'_ν is isomorphic to some M_μ, so the sum of 2 endomorphisms of M'_ν which are not isomorphisms is not an isomorphism. Moreover, by interchanging the decompositions, we also have that each M_μ is isomorphic to some M'_ν. Intuitively, it means that we have "the same factors" and it only remains to show that they "appear the same number of times". More formally, let us denote by $M(\mu) \subset L$ the set of indices $\lambda \in L$ such that $M_\mu \simeq M_\lambda$. Similarly, denote by $M'(\nu) \subset L'$ the set of indices $\lambda \in L'$ such that $M'_\nu \simeq M'_\lambda$. To prove the claimed result, it suffices to show that if $M_\alpha \simeq M'_\beta$, then $|M(\alpha)| = |M'(\beta)|$. By symmetry, let us show $|M(\alpha)| \geq |M'(\beta)|$.

For all $\nu \in L'$, let $f_\nu = i_\nu \circ \pi_\nu$ be the primitive idempotent associated to M'_ν (where i_ν and π_ν are the canonical injection and projection with respect to M'_ν and the second decomposition) (we keep the notation e_μ for the idempotent associated to the M_μ of the first decomposition). Consider any $\nu_0 \in M'(\beta)$. By applying the first part to $f = f_{\nu_0}$, as $f_{\nu_0}(M) = M'_{\nu_0} \simeq M'_\beta$ is indecomposable, there is $\mu_0 \in M(\alpha)$ such that M_{μ_0} is carried isomorphically into M'_{ν_0} by f_{ν_0}. Therefore, we have $M = M_{\mu_0} \oplus \left(\bigoplus_{\nu \neq \nu_0} M'_\nu\right)$ and we have $\bigoplus_{\nu \neq \nu_0} M'_\nu \simeq \bigoplus_{\mu \neq \mu_0} M_\mu$.

Suppose first that $M'(\beta)$ is finite, let $\nu_1, \cdots \nu_s$ those indices. After repeating the above argument s times, there are $\mu_1, \cdots, \mu_s \in M(\alpha)$ for which we have $M = M_{\mu_1} \oplus \cdots \oplus M_{\mu_s} \oplus \bigoplus_{\nu \notin \{\nu_1, \cdots, \nu_s\}} M'_\nu$ and $M_{\mu_k} \simeq M'_{\nu_0}$. Therefore, if $M'(\beta)$ is finite, then $|M(\alpha)| \geq |M'(\beta)|$.

Assume now that $|M'(\beta)|$ is infinite. Let $\mu \in M(\alpha)$. For any $0 \neq u \in M_\mu$, the M'_ν component $f_\nu(u)$ is zero for all ν except a finite number. Hence, there exists only finitely many indices $\nu \in M'(\mu)$ such that M_μ is mapped isomorphically into M'_ν by f_ν. Let us denote this finite subset of $M'(\beta)$ by $F(\mu)$. If μ runs through all the indices in $M(\alpha)$, the corresponding $F(\mu)$ exhaust $M'(\beta)$. We thus have $\bigcup_{\mu \in M(\alpha)} F(\mu) = M'(\beta)$. Since $M'(\beta)$ is infinite and all the $F(\mu)$ are finite, we have $M(\alpha)$ infinite too. □

Now that we have the algebraic results, let us apply them to our case. We only need to check the condition on the sum of non-isomorphisms still being non-isomorphic.

Theorem 3.1.3 (Krull-Remak-Schmidt-Azumaya). Suppose that a persistence module $\mathbb{V}$ over T can be expressed as a direct sum of interval modules in two different ways:

$$\mathbb{V} = \bigoplus_{l \in L} \mathbb{I}^{J_l} = \bigoplus_{m \in M} \mathbb{I}^{K_m}.$$

Then there exists a bijection $\sigma \colon L \to M$ such that $J_l = K_{\sigma(l)}$ for all l

Proof. As the endomorphism ring of every interval module is isomorphic to the field $\mathbf{k}$, the only non-isomorphic map is the zero map, and we have the condition of the previous lemma and thus the result. ☐

In other words, if it exists, an interval decomposition is unique up to isomorphism and permutation. We will therefore refer to it as the interval decomposition of the persistence module.

Let us now focus on asserting whether or not such decomposition exists.

3.2 Existence of a Decomposition

Using Gabriel's theorem (Theorem 2.4.22), we have a decomposition if the persistence module is indexed by a finite set and each vector space is finite-dimensional. Indeed, if the persistence module is of this form, then it can also be seen as a finite representation of a quiver of the type A_n. Gabriel's theorem then state that the indecomposable representations of A_n are in bijection with the real zeroes x of the Tits form, which, as proven in Section 2.4.4, are such that the set of vertices with $x_i = 1$ form a connect subquiver of A_n, i.e. it forms an interval persistence module. Moreover, by a simple induction on the sum of the degree of each vector space $N = \sum_{i=1}^{n} d_i$, each persistence module with finite N admits a decomposition into indecomposable submodules, i.e. into interval modules.

We will now strengthen this result in 2 directions: firstly, by allowing infinite-dimensional vector spaces with finite index sets, and secondly, by allowing infinite index sets.

Remark 3.2.1. We cannot say in general that infinite-dimensional vector spaces indexed by infinite sets are decomposable as shown in the next example.

Example 3.2.2. Let $\mathbb{V}$ be the persistence module define by

$$V_0 = \{x = (x_1, x_2, \cdots) \in \mathbb{R}^{\mathbb{N}}\},$$
$$V_{-n} = \{x \in \mathbb{R}^{\mathbb{N}} : x_1 = \cdots = x_n = 0\}$$

for $n \geq 0$ and v_{-m}^{-n} is the inclusion $V_{-m} \subset V_{-n}$ (for $n \leq m$).

By contradiction, suppose that $\mathbb{V}$ admits an interval decomposition. As $V_k = 0$ for $k > 0$, all the intervals of the decomposition must be subsets of $(-\infty, 0]$ As v_{-n-1}^{-n} is injective for each $n \geq 0$, there are no intervals of the form $[a, b]$ or $(-\infty, b]$ for $b < 0$, hence all the intervals of the decomposition must be of the form $[-n, 0]$ or $(-\infty, 0]$. Furthermore, $\dim(V_{-n}/V_{-n-1}) = 1$, so that each interval

$[-n, 0]$ occurs with multiplicity 1. Moreover, since $\bigcap_{n\in\mathbb{N}} V_{-n} = \{0\}$, it means that the interval $(-\infty, 0]$ does not occur.

It therefore implies that $\mathbb{V} \simeq \bigoplus_{n\in\mathbb{N}} \mathbb{I}^{[-n,0]}$, which implies that the dimension of V_0 is countable, a contradiction.

3.2.1 Finite Index Set

Let us first focus on the case where the index set is finite. Then, the persistence module can be seen as the (not necessarily finite) representation of a quiver of underlying graph of type A_n.

Let Q be a quiver. By Theorem 2.2.6, we have an equivalence between the category of representations of Q and the one of $\mathbf{k}Q$-modules. Therefore, the question of whether the representations of Q admit a decomposition into indecomposable representations is equivalent to asking whether the $\mathbf{k}Q$-modules admit a decomposition into indecomposable representations. Then we can use Corollary 4.8 of Auslander [2] that states the following.

Fact 3.2.3. If Λ is an artin ring of finite representation type, then every Λ-module is a (direct) sum of finitely generated indecomposable Λ-modules.

Here, an *artin*[3] *ring* (also called *artinian ring*) is a ring that satisfies the descending chain condition on left and right ideals: every sequence of left (respectively right) ideals $I_1 \supset I_2 \supset I_3 \supset \cdots$ eventually stabilises, i.e. there is a $n \in \mathbb{N}$ such that $I_k = I_n$ for all $k \geq n$.

Similar to Definition 2.4.1, a ring Λ is *of finite representation type* if there are only finitely many finite-dimensional indecomposable Λ-modules, up to isomorphism. In particular, using Theorem 2.2.6, we know that if Q is a simply-laced Dynkin quiver, then $\mathbf{k}Q$ is a ring of finite representation type. Moreover, the quiver being finite and without any loop, $\mathbf{k}Q$ is an artin ring as it is finitely generated over $\mathbf{k}$, a field (which it therefore is artin).

Therefore, using the fact from Auslander, every $\mathbf{k}Q$-module admits a decomposition into finitely generated indecomposable $\mathbf{k}Q$-modules. Using again the equivalence of category of Theorem 2.2.6, we have that every representation (possibly infinite dimensional) of a Dynkin quiver of type ADE admits a decomposition into finite-dimensional indecomposable representations. In the case of persistence modules, it implies that it admits a decomposition into interval modules.

Unfortunately, the proof of the result of Auslander is pretty tedious and requires several results outside the scope of this master thesis. However, using the particular structure of the quivers of type A_n, we can prove the result for persistence modules in a more direct manner, only using results from linear algebra. We will follow the proof of [50].

It Q' is a subquiver of Q, we denote by $V(Q')$ the representation of Q defined as

$$V(Q')_i = \begin{cases} k & \text{if } i \in Q'_0, \\ 0 & \text{otherwise} \end{cases}$$

---

[3] Emil Artin (1898-1962) was an Austrian mathematician who mainly worked on algebraic number theory. He is considered as one of the fathers of modern abstract algebra.

and

$$\varphi_\alpha = \begin{cases} 1 & \text{if } \alpha \in Q_1', \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if $Q_0' = \{i, i+1, \cdots, j\}$ is such that if $\alpha\colon k \to k' \in Q_1$ with $k, k' \in \{i, i+1, \cdots j\}$ implies that $\alpha \in Q_1'$, then we denote $V(Q')$ by $V([i,j])$ and call it an *interval representation*.

Our goal is to prove that all the representations of quivers whose underlying graph is of type A_n admit a decomposition into a direct sum of interval representations. We first start with a technical lemma from linear algebra (note that we do assume the axiom of choice).

Lemma 3.2.4. Let A, B be two subspaces of a $\mathbf{k}$-vector space L. Then one can find a base $\mathcal{B}$ of L such that $\mathcal{B} \cap A$ and $\mathcal{B} \cap B$ are bases for A and B respectively.

We call such a basis *compatible*.

Proof. Let $C = A \cap B$; it is a vector space. Let $\Lambda = \{\lambda_i : i \in I\}$ be a basis of C. In particular, it is a free set in A and B, therefore it can be completed to $\Lambda \cup \{\mu_j : j \in J\}$ and $\Lambda \cup \{\nu_h : h \in H\}$ to form basis for A and B respectively.

Let us show that the set $\Lambda \cup \{\mu_j : j \in J\} \cup \{\nu_h : h \in H\}$ is a free set of L. Suppose that there are some c_i, a_j and b_h for $i \in I, j \in J, h \in H$ elements of the field $\mathbf{k}$ such that

$$\sum_{i\in I} c_i\lambda_i + \sum_{j\in J} a_j\mu_j + \sum_{h\in H} b_h\nu_h = 0.$$

It is equivalent to $-\sum_{h\in H} b_h\nu_h = \sum_{i\in I} c_i\lambda_i + \sum_{j\in J} a_j\mu_j$, which then must be an element of $A \cap B$. It implies that

$$-\sum_{h\in H} b_h\nu_h = \sum_{i\in I} c_i'\lambda_i$$

for some $c_i' \in \mathbf{k}$ as Λ is a basis of $A \cap B$. Therefore,

$$0 = \sum_{h\in H} b_h\nu_h + \sum_{i\in I} c_i'\lambda_i$$

and $b_h = c_i' = 0$ for all $i \in I$ and $h \in H$ as $\Lambda \cup \{\nu_h : h \in H\}$ is a basis of B. Therefore, we get $0 = \sum_{i\in I} c_i\lambda_i + \sum_{j\in J} a_j\mu_j$, which is an element of A and, as $\Lambda \cup \{\mu_j : j \in J\}$ is a basis of A, all the coefficients are zeros. It shows our claim.

As $\Lambda \cup \{\mu_j : j \in J\} \cup \{\nu_h : h \in H\}$ is a free set of L, it can be extended into a basis of L. By construction, such a base will satisfy the condition of the lemma. □

Now that this technical lemma is taken care of, we can prove the base case of our result, i.e. the representations of the A_3 quivers.

Lemma 3.2.5. Any representation V of an A_3 quiver is the direct sum of interval representations.

Proof. Up to reflection, we have 3 different quivers whose underlying graph is of type A_3:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3, \qquad 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3, \qquad 1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3.$$

Case 1: If Q is given by the left quiver, we have

$$V = (\bigoplus_{n\in N} S(1)) \oplus (\bigoplus_{m\in M} S(3)) \oplus V'$$

with V' not containing any direct summands isomorphic to $S(1)$ nor $S(3)$ (recall from Definition 2.1.9. that $S(1)$ and $S(3)$ are the simple representations associated to the vertex 1 and 3 respectively). In particular, it implies that φ'_α is injective. Otherwise,

$$\ker(\varphi'_\alpha) \to 0 \to 0$$

would be a direct summand of V' which is isomorphic to $\bigoplus_{s\in S} S(1)$. Similarly, φ'_β is surjective. Otherwise,

$$0 \to 0 \to \operatorname{coker}(\varphi'_\beta) = V'_3/\operatorname{im}(\varphi'_3)$$

would be a direct summand of V' isomorphic to $\bigoplus_{s\in S} S(3)$.

Let $L = V'_2$, and let A be the kernel of φ'_β. As φ'_β is surjective, we can then assume that φ'_β is the projection $L \to L/A$. Let $B = V'_1$, as φ'_α is injective, we can assume that $\varphi'_\alpha\colon B \to L$ is the inclusion. In other words, we can consider that V' is the representation

$$V'\colon B \to L \to L/A.$$

By the previous lemma, there is a basis $\mathcal{B}$ which is compatible with A and B. Therefore, V' is isomorphic to the representation

$$\bigoplus_{b\in\mathcal{B}\cap B} \mathbf{k}b \to \bigoplus_{b\in\mathcal{B}} \mathbf{k}b \to \bigoplus_{b\in\mathcal{B}\setminus A} \mathbf{k}b.$$

Then V' is the direct sum of copies of the representations:

$$\begin{array}{ll} \mathbf{k} \to \mathbf{k} \to \mathbf{k} & \text{if } b \in (\mathcal{B}\cap B)\setminus A; \\ \mathbf{k} \to \mathbf{k} \to 0 & \text{if } b \in \mathcal{B}\cap A \cap B; \\ 0 \to \mathbf{k} \to \mathbf{k} & \text{if } b \in \mathcal{B}\setminus(A \cup B); \\ 0 \to \mathbf{k} \to 0 & \text{if } b \in (\mathcal{B}\cap A)\setminus B. \end{array}$$

All of these representations are interval representations.

Case 2: If Q is given by the middle quiver. As above, we have

$$V = \left(\bigoplus_{n\in N} S(1)\right) \oplus \left(\bigoplus_{m\in M} S(3)\right) \oplus V'$$

with V' not containing any direct summands isomorphic to $S(1)$ nor $S(3)$.

Let $L = V'_2$, $A = V'_1$ and $B = V'_3$. By a similar reasoning as above, we can assume that φ'_α is the inclusion $A \to L$ and φ'_β is the inclusion $B \to L$. By the previous lemma, we have a base $\mathcal{B}$ of L which is compatible with A and B. We then get

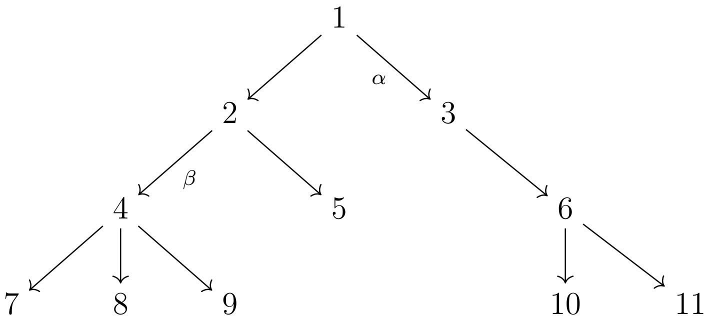
$$V' \simeq \bigoplus_{b\in\mathcal{B}\cap A} \mathbf{k}b \to \bigoplus_{b\in\mathcal{B}} \mathbf{k}b \leftarrow \bigoplus_{b\in\mathcal{B}\cap B} \mathbf{k}b.$$

It admits a decomposition into interval representations.

<u>Case 3:</u> If Q is the right quiver. Then we have, by a similar reasoning as above, that V is a direct sum of copies of $S(1)$, $S(3)$ and a representation isomorphic to $L/A \leftarrow L \to L/B$. Also, by a similar reasoning as above, this representation has a decomposition into a direct sum of interval representations. $\square$

Definition 3.2.6. Let Q be a tree quiver (i.e. a quiver whose underlying graph is a tree). Let c be a vertex of Q. The arrow $\alpha\colon x \to y$ *points to* c if y and c are in the same connected component of $Q\backslash\alpha$.

Example 3.2.7. Let Q be the following quiver.



In Q, we have that α points to the vertices 3 and 10 but not to the vertex 5. On the other hand, the arrow β points to the vertices $7, 8, 9$ but not to the vertices $1, 5, 6$.

Definition 3.2.8. As above, let Q be a tree quiver and c be a vertex of Q. A representation V of Q is *c-conical* if φ_α is injective for all arrows of Q pointing to c and surjective for all the other arrows.

A representation V of A_n is $([i,j],c)$-conical if $c \in [i,j]$ and $V|_{[i,j]}$ is c-conical.

Example 3.2.9. The representation of the subinterval quiver $V([i,j])$ is c-conical if and only if $c \in [i,j]$.

Indeed, the only maps to check are the ones between the vertices $(i-1,i)$ and $(j,j+1)$ as all the other maps are isomorphisms. We have 2 cases for each map between $(i-1)$ and i and between j and $j+1$, depending on the orientation of the arrow. If $c \in [i,j]$, we can have $\alpha\colon (i-1) \to i = 0 \to \mathbf{k}$, which is the zero map which points to c and is injective. In the case where $\alpha\colon i \to (i-1) = \mathbf{k} \to 0$, it does not point to c and it is surjective. Similar consideration for the map between the vertices j and $(j+1)$ confirm that $V([i,j])$ is c-conical.

Reciprocally, suppose $c \notin [i,j]$, suppose $c < i$. Then if $\alpha\colon (i-1) \to i = 0 \to \mathbf{k}$, then α does not point to c but is not surjective. If $\alpha\colon i \to (i-1) = \mathbf{k} \to 0$, then α points to c but is not injective. Similar considerations for $c > j$ give the result.

Example 3.2.10. The direct sum of two c-conical representations of the quiver Q is still a c-conical representation of Q.

It is clear as the direct sum of injective (resp. surjective) morphisms is injective (resp. surjective).

Lemma 3.2.11. Let V be a representation of Q (where the underlying graph of Q is A_n) such that all maps in $V|_{[2,n-1]}$ are isomorphisms. Then the representation V is isomorphic to the representation V' with $V'_k = V_2$ for all $2 \leq k \leq n-1$ and such that for all γ vertex in $Q|_{[2,n-2]}$, $\varphi'_\gamma = \mathrm{id}_{V_2}$.

Proof. The goal is to construct an isomorphism $f\colon V \to V'$ of representations between these two representations (note that there are no arrows on the horizontal lines as we don't know in which direction each arrow goes):

$$\begin{tikzcd}
V : \arrow[d, "f"'] & V_1 \arrow[r, dash, "\varphi_{\alpha_1}"] \arrow[d, "f_1"'] & V_2 \arrow[r, dash, "\varphi_{\alpha_2}"] \arrow[d, "f_2"'] & V_3 \arrow[d, "f_3"'] & \cdots & V_{n-1} \arrow[r, dash, "\varphi_\beta"] \arrow[d, "f_{n-1}"'] & V_n \arrow[d, "f_n"] \\
V' : & V_1 \arrow[r, dash, "\varphi'_{\alpha_1}"'] & V_2 \arrow[r, dash, "\mathrm{id}"'] & V_2 & \cdots & V_2 \arrow[r, dash, "\varphi'_\beta"'] & V_n
\end{tikzcd}$$

We start by defining $f_1\colon V_1 \to V_1$ as id_{V_1} and $f_2 = \mathrm{id}_{V_2}$ as well as $\varphi'_{\alpha_1} = \varphi_{\alpha_1}$. It is thus direct to see that the leftmost square commutes. Now, suppose that we defined $f_k\colon V_k \to V_2$ isomorphic (with $k < n-1$) such that all the squares left to f_k commute. Let us construct f_{k+1} such that the relevant square commutes. We have 2 possibilities depending on the orientation of φ_{α_k}:

$$\begin{tikzcd}
V_k \arrow[r, "\varphi_{\alpha_k}"] \arrow[d, "f_k"'] & V_{k+1} \arrow[d, dashed, "f_k(\varphi_{\alpha_k})^{-1}"] \\
V_2 \arrow[r, "\mathrm{id}"'] & V_2
\end{tikzcd}
\qquad
\begin{tikzcd}
V_k \arrow[d, "f_k"'] & V_{k+1} \arrow[l, "\varphi_{\alpha_k}"'] \arrow[d, dashed, "\varphi_{\alpha_k} f_k"] \\
V_2 & V_2. \arrow[l, "\mathrm{id}"]
\end{tikzcd}$$

In both cases, the square commutes and the map f_{k+1} defined is an isomorphism as both f_k and φ_{α_k} are isomorphisms. It remains to define φ'_β. Once again, we have 2 cases:

$$\begin{tikzcd}
V_{n-1} \arrow[r, "\varphi_\beta"] \arrow[d, "f_{n-1}"'] & V_n \arrow[d, "\mathrm{id}"] \\
V_2 \arrow[r, dashed, "\varphi_\beta(f_{n-1})^{-1}"'] & V_n
\end{tikzcd}
\qquad
\begin{tikzcd}
V_{n-1} \arrow[d, "f_{n-1}"'] & V_n \arrow[l, "\varphi_\beta"'] \arrow[d, "\mathrm{id}"] \\
V_2 & V_n. \arrow[l, dashed, "f_n\varphi_\beta"]
\end{tikzcd}$$

By construction, all the squares commute and the f_k are all isomorphisms. Therefore, we have the expected result. ☐

We will now prove the main result of the section using the proof developed in [50]. We will proceed by induction.

Theorem 3.2.12. Any representation of a quiver Q, whose underlying graph is of type A_n, admits a decomposition into a direct sum of interval representations.

Proof. First, note that the result is trivial in the case A_2: $V \simeq S(1)^n \oplus S(3)^m \oplus P(1)^p$ with $P(1)\colon \mathbf{k} \to \mathbf{k}$ and $n, m, p \in \mathbb{N} \cup \{\infty\}$. Indeed, we can suppose with loss of generalities that $Q = 1 \to 2$. Then, $V = V_1 \xrightarrow{f} V_2$ is such that $V = (\ker_f \to 0) \oplus (V_1 \xrightarrow{\sim} V_1) \oplus (0 \to \operatorname{coker} f)$.

The base case is already proven in Lemma 3.2.5. Suppose now that every representation of quivers whose underlying graph is A_k for $3 \leq k < n$ admits a decomposition. Let Q be a quiver with underlying graph A_n.

Let us first prove that any representation V of Q can be written as the direct sum $V' \oplus V''$ such that V' is $([1, n-1], n-1)$-conical and $V''_{n-1} = V''_n = 0$.

To do that, we denote by $U = V|_{[1,n-1]}$. By induction, U can be written as the direct sum of interval submodules, i.e. as representations of the form $V([s, t])$ with $1 \leq s \leq t \leq n-1$. In particular, $U = U' \oplus U''$ with U' being the direct sum of all the terms of the form $V([s, n-1])$ in the decomposition of U and U'' being the direct sum of the terms of the form $V([s, t])$ with $t \leq n-2$. As $n-1 \in [s, n-1]$, we have that U' is $(n-1)$-conical.

Moreover, $U'_{n-1} = V_{n-1}$ and $U''_{n-1} = 0$. It implies that we can define V' as $V'|_{[0,n-1]} = U'$ and $V'|_{[n-1,n]} = V|_{[n-1,n]}$. Similarly, we define V'' as $V''|_{[0,n-1]} = U''$ and $V''|_{[n-1,n]} = 0$. These representations are well defined thanks to the previous equalities. As required, $V''_{n-1} = V''_n = 0$ and $V'|_{[1,n-1]} = U'$ is $(n-1)$-conical.

Furthermore, by considering the quiver Q^{op}, the quiver with same vertices and were all the arrows are reversed, we can deduce that any representation of Q can be written as $W' \oplus W''$ with W' $([2, n], 2)$-conical and such that $W''_2 = W''_1 = 0$.

In particular, it implies that any representation V of Q can be written as $Z' \oplus Z''$ with $Z' = (Z'_i, \psi_\alpha)$ such that for all γ, arrow in $[2, n-1]$, ψ_γ is an isomorphism and such that Z'' is a direct sum of representations of the form $V([s, t])$ with $s \geq 3$ or $t \leq n-2$.

Indeed, $V = V' \oplus V''$ and, using the 2nd decomposition that we defined ("$W' \oplus W''$"), we get $V' = Z' \oplus \tilde{Z}$ with Z' both $([2, n], 2)$-conical and $([1, n-1], n-1)$-conical, i.e. all the maps in $Z'|_{[2,n-1]}$ are isomorphism. Moreover, we have $\tilde{Z}_1 = \tilde{Z}_2 = 0$ so it admits a decomposition into elements $V([i, j])$ with $i \geq 3$. Similarly, $V''_n = V''_{n-1} = 0$ so it admits a decomposition into elements $V([i, j])$ wiht $j \leq n-2$. It just remains to take $Z'' = V'' \oplus \tilde{Z}$.

Therefore, using Lemma 3.2.11, we have that

$$Z' \simeq \left(Z'_1 \overset{\psi_{\alpha_1}}{\longrightarrow\!\!\!\!\!-} Z'_2 \overset{\mathrm{id}}{\longrightarrow\!\!\!\!\!-} Z'_2 \quad \cdots \quad Z'_2 \xrightarrow{\widetilde{\psi_{\alpha_{n-1}}}} Z'_n \right) \simeq \left(Z'_1 \overset{\psi_{\alpha_1}}{\longrightarrow\!\!\!\!\!-} Z'_2 \xrightarrow{\widetilde{\psi_{\alpha_{n-1}}}} Z'_n \right)$$

As the right side is the representation of an A_3 quiver, by Lemma 3.2.5, it admits a decomposition into interval modules, and Z' admits a decomposition. Therefore, $V \simeq Z' \oplus Z''$ admits a decomposition into interval modules as both Z' and Z'' admit such a decomposition. □

This result is even stronger than stating that any persistence module over a finite chain decomposes into interval modules, as here the direction of the arrows is not all the same. It, in fact, implies that any zigzag module decomposes. See more in Chapter 5.

3.2.2 Locally Finite Persistence Modules

Let us focus on the case where the ordering set is arbitrary and all the vector spaces are finite-dimensional. We say that this persistence module is *pointwise finite-dimensional*. We use [22] as a main source for this section. Although the result is an interesting generalisation, its proof is rather tedious and cumbersome. We will first define the notion of cut and construct some sets related to cuts and transition maps, as well as establish how the transition maps interact with those sets. Then we will relate those sets to intervals. Afterwards, we will link those sets to vector spaces and have some first results about decomposition. It will remain to introduce the technical notion of sections of vector spaces before proving the final result.

First, it is easy to see that being point-wise finite-dimensional is a particular case of the following property. We will show the result in this more general context.

Definition 3.2.13. A persistence module $\mathbb{V}$ has the *descending chain condition on images and kernels*, which we will denote by DCC on im and ker, if:
$\forall t \geq s_1 > s_2 > \cdots$, the chain $V_t \supset \operatorname{im} v^t_{s_1} \supset \operatorname{im} v^t_{s_2} \supset \cdots$ stabilizes
$\forall t \leq \cdots < s_2 < s_1$, the chain $V_t \supset \ker v^{s_1}_t \supset \ker v^{s_2}_t \supset \cdots$ stabilizes.

In the following, we will work with T, a totally ordered separable (with respect to the order topology) set. The condition of being separable seems arbitrary at first glance, but we will need it to apply the Mittag-Leffler condition and have some results on the inductive limit of some inductive system.

Dedekind Cuts

Dedekind cuts are generally introduced to provide a way to complete a topological ordered space. In this case, more than just provide a completion, it will also allow us to consider all the different kinds of intervals all at once (whether it is an interval of the form $(i, j], [i, j], (i, j), (-\infty, j], \cdots$ for $i, j \in T$). Intuitively, the first step of the proof is to formalise the notion of limits of kernels and images of v^r_s and see how these limits interact with the composition with another transition function v^t_r.

Definition 3.2.14. A *cut* for T is a pair $c = (c^-, c^+)$ of subsets of T such that $T = c^- \cup c^+$ and $s < t$ for all $s \in c^-$ and $t \in c^+$.

Notation 3.2.15. Let c be a cut, $s \in c^-$ and $t \in c^+$. We denote by

$$\operatorname{im}^-_{ct} = \bigcup_{r \in c^-} \operatorname{im} v^t_r, \qquad \operatorname{im}^+_{ct} = \bigcap_{\substack{r \in c^+ \\ r \leq t}} \operatorname{im} v^t_r,$$

$$\ker^-_{cs} = \bigcup_{\substack{r \in c^- \\ r \geq s}} \ker v^r_s, \qquad \ker^+_{cs} = \bigcap_{r \in c^+} \ker v^r_s.$$

All of these sets are subsets of V_t. Moreover, if $c^- = \emptyset$, we set $\operatorname{im}^-_{ct} = 0$ and if c^+ is empty, we set $\ker^+_{cs} = V_t$.

Intuitively, we can see a cut as a way of cutting R into 2 disjoint intervals that cover R. We can also see im^-_{ct} as being the "limit" for $r \to \sup(c^-)$ by below of $\operatorname{im} v^t_r$. Similarly, $\ker^-_{cs}$ is the limit for $r \to \sup(c^-)$ (by below) of $\ker v^t_s$, im^+_{ct} is the limit for $r \to \inf(c^+)$ (by above) of $\operatorname{im} v^t_r$, and $\ker^+_{cs}$ is the limit of $\ker v^r_s$ for $r \to \inf(c^+)$ by above.

Using the DCC for ker and im, we have the following result.

Lemma 3.2.16. Let c be a cut.

a) if $t \in c^+$, then $\operatorname{im}^+_{ct} = \operatorname{im} v^t_s$ for some $s \in c^+$, $s \leq t$.

b) if $t \in c^-$ and $c^+ \neq \emptyset$, then $\ker^+_{ct} = \ker v^s_t$ for some $s \in c^+$.

Proof. By contradiction, suppose that $\operatorname{im}^+_{ct} \neq \operatorname{im} v^t_s$ for all $s \in c^+$ with $s \leq t$. Let $s_1 = t$, since $\operatorname{im}^+_{ct} \neq \operatorname{im} v^t_s$, there is $s_2 \in c^+$ such that $\operatorname{im} v^t_{s_2} \subsetneq \operatorname{im} v^t_{s_1}$. Repeating this argument, we get an infinite decreasing sequence, contradicting the DCC for im. The same argument applies to the kernel. □

The following lemma provides a way to see how the transition maps affect the different image and kernel sets.

Lemma 3.2.17. Let c be a cut and $s \leq t$.

a) if $s, t \in c^+$, then $v^t_s(\operatorname{im}^\pm_{cs}) = \operatorname{im}^\pm_{ct}$,

b) if $s, t \in c^-$, then $(v^t_s)^{-1}(\ker^\pm_{ct}) = \ker^\pm_{cs}$, so $v^t_s(\ker^\pm_{cs}) \subset \ker^\pm_{ct}$.

Proof. (a) - $v^t_s(\operatorname{im}^+_{cs}) \subset \operatorname{im}^+_{ct}$ as if $x \in \operatorname{im}^+_{cs} = \bigcap_{\substack{r \in c^+ \\ r \leq t}} \operatorname{im} v^t_r$, then for all $r \in c^+, r \leq s$, $\exists y_r \in V_r$ such that $v^s_r(y_r) = x$. Therefore, if $r \leq s \leq t$, we have that

$$v^t_s(x) = v^t_s(v^s_r(y_r)) = v^t_r(y_r) \in \operatorname{im} v^t_r.$$

If $s \leq r \leq t$, then $v^t_s(x) = v^t_r(v^r_s(x)) \in \operatorname{im} v^t_r$. It implies that $v^t_s \in \bigcap_{\substack{r \in c^+ \\ r \leq t}} \operatorname{im} v^t_r$. For the other inclusion, by Lemma 3.2.16, we have $\operatorname{im}^+_{cs} = \operatorname{im} v^s_r$ for some $r \leq s, r \in c^+$. Therefore, $v^t_s(\operatorname{im}^+_{cs}) = v^t_s(\operatorname{im} v^s_r) = \operatorname{im} v^t_r \supset \operatorname{im}^+_{ct}$.

- $v^t_s(\operatorname{im}^-_{cs}) = \operatorname{im}^-_{ct}$: Let $x \in v^t_s(\operatorname{im}^-_{cs})$. There is $r \in c^-$ and $y \in \operatorname{im} v^s_r$ such that $x = v^t_s(y)$, which implies that $x \in \operatorname{im} v^t_r \subset \bigcup_{r \in c^-} v^t_r$. For the other inclusion, let $x \in \operatorname{im}^-_{ct}$, we have $r \in c^-$ such that $x \in \operatorname{im} v^t_r = \operatorname{im}(v^t_s \circ v^s_r) = v^t_s(\operatorname{im} v^s_r) \subset v^t_s(\operatorname{im}^-_{cs})$. Note that $r < s$ as $r \in c^-$ and $s \in c^+$.

(b) - We have $x \in (v^t_s)^{-1}(\ker^+_{ct}) \Leftrightarrow v^t_s(x) \in \ker^+_{cs} \Leftrightarrow \forall r \in c^+, v^r_t(v^t_s(x)) = 0 = v^r_s(x) \Leftrightarrow x \in \ker^+_{cs}$.

- $(v^t_s)^{-1}(\ker^-_{ct}) = \ker^-_{cs}$: let $x \in (v^t_s)^{-1}(\ker^-_{ct})$. Then, $v^t_s(x) \in \ker^-_{ct}$ implies that there is $r \in c^-, r \geq t$ such that $v^t_s(x) \in \ker v^r_t$. Therefore $v^r_t \circ v^t_s(x) = v^r_s(x) = 0 \Rightarrow x \in \ker v^r_t \subset \ker^-_{ct}$. For the other inclusion, let $x \in \ker^-_{cs}$, then there is $r \in c^-, r \geq s$ such that $v^r_s(x) = 0$. We can assume $r \geq t$ as $\ker^{r_1}_s \subset \ker^{r_2}_s$ if $r_1 \leq r_2$. Therefore, $v^t_s(x)$ is such that $v^r_t \circ v^t_s(x) = v^r_s(x) = 0$, which implies that $v^t_s(x) \in \ker v^r_t \subset \ker^-_{ct}$. □

Intervals and Cuts

Now that we have formalised the notion of limits of kernels and images, we want to formalise the notion of elements whose lifespan is exactly a specific interval and how to compute them.

If we have an interval $I \subset T$, it is uniquely determined by the cuts l, u by the relation $I = l^+ \cap u^-$ if we take the cuts

$$\begin{aligned} l^- &= \{t : t < s \ \forall s \in I\}, & l^+ &= \{t : \ \exists s \in I | t \geq s\}, \\ u^- &= \{t : \ \exists s \in I | t \leq s\}, & u^+ &= \{t : t > s \ \forall s \in I\}. \end{aligned}$$

For such cuts and $t \in I$, we define

$$\begin{aligned} V_{It}^- &= (\mathrm{im}_{lt}^- \cap \ker_{ut}^+) + (\mathrm{im}_{lt}^+ \cap \ker_{ut}^-), \\ V_{It}^+ &= \mathrm{im}_{lt}^+ \cap \ker_{ut}^+ . \end{aligned}$$

It is trivial to check that $V_{It}^- \subset V_{It}^+ \subset V_t$. Intuitively, elements of V_{It}^- are elements of V_t that can be written as the sum of elements that "are born" before the interval and "die" when leaving the interval with elements that "are born" when entering the interval and "die" after the end of the interval whereas elements of V_{It}^+ are elements that "live" throughout the whole interval. Here, similarly to Definition 1.2.7 and using Remark 1.2.12, we say that a non-zero element $x \in V_t$ is *born* at s if $x \in \mathrm{im}\, v_s^t$ and $x \notin \mathrm{im}\, v_r^t$ for all $r < s$. The element x *dies* at s' if $x \in \ker v_t^{s'}$ and $x \notin \ker v_t^{r'}$ for all $t \leq r' < s'$. We further say that the element is *alive* between s and s'.

Intuitively, the next lemma uses the fact that the transition maps don't change the "lifespan" of elements: if an element was "born" before the interval, its image too. It further states that the class of elements whose lifespan includes the whole interval quotiented out by elements that live the whole interval and were born strictly before or die strictly after is independent of the point of the interval where we look at.

Lemma 3.2.18. For $s \leq t$ elements of an interval I, we have

$$v_s^t(V_{Is}^{\pm}) = V_{It}^{\pm}.$$

Moreover, the map

$$\overline{v_s^t}\colon V_{Is}^+/V_{Is}^- \to V_{It}^+/V_{It}^-$$

induced by the map v_s^t is an isomorphism.

Proof. Let us begin by proving that $v_s^t(V_{Is}^-) = V_{It}^-$.

Let $y \in V_{Is}^-$, then there are $a \in \mathrm{im}_{ls}^- \cap \ker_{us}^+$ and $b \in \mathrm{im}_{ls}^+ \cap \ker_{us}^-$ such that $y = a + b$. Therefore, $v_s^t(y) = v_s^t(a) + v_s^t(b)$, moreover, using Lemma 3.2.17 for the last inclusion, we have

$$v_s^t(a) \in v_s^t(\mathrm{im}_{ls}^- \cap \ker_{us}^+) \subset v_s^t(\mathrm{im}_{ls}^-) \cap v_s^t(\ker_{us}^+) \subset \mathrm{im}_{lt}^- \cap \ker_{ut}^+ .$$

Similarly, $v_s^t(b) \in \mathrm{im}_{lt}^+ \cap \ker_{ut}^-$. Therefore, $v_s^t(y) \in V_{It}^-$.

Let us show the other inclusion. Let $x \in V_{It}^-$, we have $a \in \mathrm{im}_{lt}^- \cap \ker_{ut}^+$ and $b \in \mathrm{im}_{lt}^+ \cap \ker_{ut}^-$ such that $x = a + b$. In particular, $a \in \mathrm{im}_{lt}^-$, by Lemma 3.2.17, there is $a' \in \mathrm{im}_{ls}^-$ such that $v_s^t(a)$. As $a \in \ker_{ut}^+$, we have that for all $r \in u^+$,

$v_s^r(a') = v_t^r \circ v_s^t(a') = v_t^r(a) = 0$. Therefore $a' \in \mathrm{im}_{ls}^- \cap \ker_{us}^+$ is such that $v_s^t(a') = a$. Similarly, we can find $b' \in \mathrm{im}_{ls}^+ \cap \ker_{us}^-$ such that $v_s^t(b') = b$. Therefore, we have $y = a' + b' \in V_{Is}^-$ such that $v_s^t(y) = x$.

Using similar arguments, we have $v_s^t(V_{Is}^+) = V_{It}^+$.

Let us now show that $\overline{v_s^t}$ is an isomorphism.

-The function is well-defined as $v_s^t(V_{Is}^-) = V_{It}^-$.

- It is also surjective as $v_s^t(V_{Is}^+) = V_{It}^+$.

It only remains to show that it is injective. To do that, let us show

$$V_{Is}^+ \cap (v_s^t)^{-1}(V_{It}^-) \subset V_{Is}^-.$$

Let $x \in V_{Is}^+ \cap (v_s^t)^{-1}(V_{It}^-)$. In particular, we have $x \in (v_s^t)^{-1}(V_{It}^-)$, so $y = v_s^t(x) \in V_{It}^-$. Therefore, there are $a \in \mathrm{im}_{lt}^- \cap \ker_{ut}^+$ and $b \in \mathrm{im}_{lt}^+ \cap \ker_{ut}^-$ such that $y = a + b$. Following the same reasoning as above, we have $a' \in \mathrm{im}_{ls}^- \cap \ker_{us}^+$ such that $v_s^t(a') = a$. Therefore, $v_s^t(x-a') = v_s^t(x) - a = a+b-a = b \in \ker_{ut}^-$. It implies that $x - a' \in \ker_{us}^-$. Moreover, as $x \in V_{Is}^+ = \mathrm{im}_{ls}^+ \cap \ker_{us}^+$ and $a' \in \mathrm{im}_{ls}^- \cap \ker_{us}^+ \subset \mathrm{im}_{ls}^+ \cap \ker_{us}^+$, we have $x - a' \in \mathrm{im}_{ls}^+$. Thus, $x = a' + (x - a') \in \mathrm{im}_{ls}^- \cap \ker_{us}^+ + \mathrm{im}_{ls}^- \cap \ker_{us}^- = V_{It}^-$. ☐

We have another technical result that we will use in the next lemma. It is the reason why we need T to be separable.

Lemma 3.2.19. Any interval I of T contains a countable subset S which is coinitial, i.e. for which $\forall t \in I, \exists s \in S : s \leq t$.

Proof. If I has a minimal element m, then $S = \{m\}$ works.

Suppose now that S has no minimum. As T is separable, we can find X a countable and separable set. The set $S = I \cap X$ is coinitial in I: if $t \in I$, as I has no minimum, we can find $r, u \in I$ such that $u < r < t$. Therefore

$$(u,t) = \{r \in T : u < r < t\}$$

is not empty. As X is dense, $X \cap (u,t) \neq \emptyset$. We therefore have $s \in S \cap (u,t) = X \cap (u,t)$ with $s < t$. ☐

Let I be an interval. As for $s \leq t$ in I, the function v_s^t induces maps on $V_{Is}^\pm \to V_{It}^\pm$, we can consider

$$V_I^\pm = \varprojlim_{t \in I} V_{It}^\pm.$$

The set V_I^+ is the set of all the features whose lifespan includes all of I, whereas V_I^- is the set of all features whose lifespan includes all of I and that were already alive before I or that are still alive after I. In particular, the quotient V_I^+/V_i^- represents all the features whose lifespan is exactly the interval I.

Recall (more details and information can be found in [9] and we generalize this notion to categories in Section B.2) that a *projective system* of sets relative to an ordered set I is as couple $(E_\alpha, f_{\alpha\beta})$ with E_α being a set for each $\alpha \in I$ and $f_{\alpha\beta} \colon E_\beta \to E_\alpha$ a function for each $\alpha \leq \beta$ in I such that if $\alpha \leq \beta \leq \gamma$, then $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$ and $f_{\alpha\alpha} = \mathrm{id}_{E_\alpha}$.

Given a projective system of sets, we can define its *projective limit* as being $E \subset \prod_{\alpha \in I} E_\alpha$ such that for all $x \in E$, $\pi_\alpha(x) = f_{\alpha\beta}\pi_\beta(x)$. This limit is denoted by $\varprojlim_{\alpha \in I} E_\alpha$.

A projective limit also satisfies the following universal diagram:

$$\begin{tikzcd}
 & & E_\alpha \\
E' \arrow[urr, dashed, "u_\alpha"] \arrow[r, dotted, "\exists ! u"] \arrow[drr, dashed, "u_\beta"'] & \varprojlim_{\alpha \in I} E_\alpha \arrow[ur, "\pi_\alpha"] \arrow[dr, "\pi_\beta"'] & \\
 & & E_\beta . \arrow[uu, "f_{\alpha\beta}"']
\end{tikzcd}$$

It states that $\pi_\alpha x = f_{\alpha\beta}\pi_\beta(x)$ and that if there is E' with $u_\alpha \colon E' \to E_\alpha$ such that $f_{\alpha\beta}u_\beta = u_\alpha$ for each $\alpha \leq \beta$, then there is a unique morphism $u \colon E' \to \varprojlim_{\alpha \in I} E_\alpha$ such that $u_\alpha = \pi_\alpha u$ for each $\alpha \in I$.

Let us also recall the *Mittag-Leffler*[4] *condition*

Let $((A_j)_{j \in I}, (f_{ij} \colon A_j \to A_i)_{i \leq j \in I})$ be a projective system. This system satisfies the Mittag-Leffler condition (also denoted (M-L)) if the range of morphisms is stationary, i.e. if for all $k \in I$, there is $j \geq k$ such that for all $i \geq j$, we have $f_{kj}(A_j) = f_{ki}(A_i)$.

The following fact is the Proposition (13.2.2) from Grothendieck [34].

Fact 3.2.20. Let I be an ordered filtered set with a countable cofinal subset. Let $0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$ be a short exact sequence of projective systems with I as indices. If A_n satisfies the Mittag-Leffler condition, then the sequence

$$0 \to \varprojlim_{n \in I} A_n \xrightarrow{\varprojlim_{n \in I} f_n} \varprojlim_{n \in I} B_n \xrightarrow{\varprojlim_{n \in I} g_n} \varprojlim_{n \in I} C_n \to 0$$

is a short exact sequence.

Note that the convention of [34] uses the opposite ordering on I; therefore, a coinitial subset in our ordering forms a cofinal subset in Grothendieck.

The following lemma states that we can recover the information of all the features that are exactly "alive" during the interval and not more from any point of the interval.

Lemma 3.2.21. For any $t \in I$, let $\pi_t \colon V_I^+ \to V_{It}^+$ be the projection associated to the definition of the projective limit, then the induced map $\overline{\pi_t} \colon V_I^+ / V_I^- \to V_{It}^+ / V_{It}^-$ is an isomorphism.

Proof. If $s \leq t$, then $v_s^t(V_{Is}^-) = V_{It}^-$ by Lemma 3.2.18, therefore $(V_{It}^-)_{t \in I}$ with transition maps v_s^t satisfies (M-L). By Lemma 3.2.19, the conditions of the previous fact are

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[4] Gösta Mittag-Leffler (1846 – 1927) was a Swedish mathematician who worked on what is called today complex analysis. He founded the mathematical journal "Acta Mathematica".

satisfied for the short exact sequence $0 \to V_{It}^- \to V_{It}^+ \to V_{It}^+/V_{It}^- \to 0$ with morphisms the inclusion and projection.

Therefore, we have the short exact sequence

$$0 \to V_I^- \to V_I^+ \to \varprojlim_{t \in I} V_{It}^+/V_{It}^- \to 0.$$

It implies, by the 5-lemma (Proposition B.4.7), that $V_I^+/V_I^- \simeq \varprojlim_{t \in I} V_{It}^+/V_{It}^-$.

By Lemma 3.2.18, $\overline{v_s^t}\colon V_{Is}^+/V_{Is}^- \to V_{It}^+/V_{It}^-$ is an isomorphism for all $s \leq t$. It implies that $\varprojlim_{t \in I} V_{It}^+/V_{It}^- \simeq V_{It}^+/V_{It}^-$ for all $t \in I$. □

In particular, the morphism $\overline{\pi_t}\colon V_I^+/V_I^- \to V_{It}^+/V_{It}^-$ is well defined, and thus $\pi_t(V_I^-) \subset V_{It}^-$.

Vector Spaces and Start of the Decomposition

Now that we know how to compute features whose lifespan is exactly the interval I, we will show that they form a submodule. Furthermore, we will show that this submodule is isomorphic to a direct sum of copies of the interval module $\mathbb{I}^I$.

As $V_I^- \subset V_I^+$ are two vector spaces, we can consider W_I^0 the complement of V_I^- in V_I^+, i.e. a vector space such that

$$V_I^+ = V_I^- \oplus W_I^0.$$

Moreover, for all $t \in I$, the restriction of $\pi_t\colon V_I^+ \to V_{It}^+$ is injective on W_I^0. Indeed, by the last lemma, $\overline{\pi_t}$ is an isomorphism. Let $x \in \ker \pi_t \cap W_I^0$, we have $\overline{\pi_t}(x + V_I^-) \in V_{It}^-$, therefore $x \in V_I^-$ and $x \in W_I^0 \cap V_I^- = \{0\}$.

Using this vector space, we can construct a persistence module easily.

Lemma 3.2.22. The assignment

$$W_{It} = \begin{cases} \pi_t(W_I^0) & (t \in I), \\ 0 & \text{else} \end{cases}$$

with the v_s^t restricted define a submodule $\mathbb{W}_I$ of $\mathbb{V}$.

Proof. If $s \leq t$ in I, then $v_s^t \pi_s = \pi_t$ by the definition of the projective limit. By applying the equality to W_I^0, we get $v_s^t(W_{Is}) = W_{It}$. Moreover, for all $s \leq t$ with $s \in I$ and $t \notin I$, then $t \in u^+$, so $W_{Is} \subset V_{Is}^+ \subset \ker_{us}^+ \subset \ker v_s^t$. It means that the map $v_s^t : W_{Is} \to W_{It} = 0$ is well defined. □

We now check that the persistence submodule defined above is indeed the complement of V_{It}^- in V_{It}^+ for each element of the interval.

Lemma 3.2.23. For all $t \in I$, we have

$$V_{It}^+ = W_{It} \oplus V_{It}^-.$$

Proof. Let us start by showing that the sum is direct. Let $x \in W_{It} \cap V_{It}^-$. $x \in W_{It} = \pi_t(W_I^0)$, so there is $y \in W_I^0$ such that $\pi_t(y) = x$. In other words, $y \in \pi_t^{-1}(x)$. On the other hand, $x \in V_{It}^-$, so $x = 0$ in $V_{It}^+/V_{It}^- \simeq V_I^+/V_I^-$ therefore, $\overline{\pi_t^{-1}(x)} = V_I$ and $y \in \pi_t^{-1}(x) \subset V_I^-$. Combining the 2 parts, we have $y \in W_I^0 \cap V_I^- = 0$ by definition of W_I^0. So $y = 0$ and $x = 0$.

Let us now prove that $V_{It}^+ = W_{It} + V_{It}^-$. The inclusion $V_{It}^+ \supset W_{It} + V_{It}^-$ is direct. Let us show the other. Let $x \in V_{It}^+$, by definition of the projective limit, there is $y \in V_I^+$ such that $\pi_t(y) = x$. By construction of W_I^0, we have $V_I^+ = V_I^- \oplus W_I^0$, therefore, $y = y_1 + y_2$ for some $y_1 \in V_I^-$ and $y_2 \in W_I^0$. It implies that $x = \pi_t(y) = \pi_t(y_1) + \pi_t(y_2)$ with $\pi_t(y_1) \in V_{It}^-$ as $\pi_t(V_I^-) \subset V_{It}^-$ and $\pi_t(y_2) \in W_{It}$. ☐

We can now prove the result we have been hinting at: $\mathbb{W}_I$ represents features whose lifespan is exactly the interval I.

Lemma 3.2.24. The submodule $\mathbb{W}_I$ is isomorphic to a direct sum of copies of the interval module $\mathbb{I}^I$.

Proof. As W_I^0 is a vector space, we can choose a basis $\mathcal{B}$ of this vector space. For each element $b \in \mathcal{B}$, as the map π_t is injective on W_t^0, the elements $b_t = \pi_t(b)$ $(t \in I)$ are non zero and satisfy $v_s^t(b_s) = b_t$ for all $s \leq t$ by the definition of the projective limit. It means that they span a submodule $S(b)$ of $\mathbb{W}_I$ that is isomorphic to $\mathbb{I}^I$. Moreover, $\{b_t : b \in \mathcal{B}\}$ is a basis of $W_{It} = \pi_t(W_I^0)$ for all $t \in I$, so $\mathbb{W}_I = \bigoplus_{b \in \mathcal{B}} S(b)$. ☐

Sections of Vector Spaces

It means that if we prove that $\mathbb{V}$ is the direct sum of submodules $\mathbb{W}_I$ as I runs through all the intervals, we will have the required result. To do that, we will first prove some general results about vector spaces.

Definition 3.2.25. A *section* of a vector space F is a pair of subspaces (F^-, F^+) such that $F^- \subset F^+ \subset F$.

A set of sections $\{(F_\lambda^-, F_\lambda^+) | \lambda \in \Lambda\}$ is said to

- be *disjoint* if for all $\lambda \neq \mu$ in Λ, either $F_\lambda^+ \subset F_\mu^-$ or $F_\mu^+ \subset F_\lambda^-$.
- *cover* F if for all proper subset $X \subsetneq F$, there is an index $\lambda \in \Lambda$ with $X + F_\lambda^- \neq X + F_\lambda^+$.
- *strongly cover* F if for all subsets $Y, Z \subset F$ such that $Z \not\subset Y$, there is an index $\lambda \in \Lambda$ with $Y + (F_\lambda^- \cap Z) \neq Y + (F_\lambda^+ \cap Z)$.

Note that by taking $Y = X$ and $Z = F$, one recovers that strongly covering U implies covering it.

The following lemma will allow us to link the fact of having a set of disjoint and covering sections to the decomposition of the vector space.

Lemma 3.2.26. Suppose that $\{(F_\lambda^-, F_\lambda^+) | \lambda \in \Lambda\}$ is a set of disjoint sections that cover U. For each $\lambda \in \Lambda$, let W_λ be a subspace such that $F_\lambda^+ = W_\lambda \oplus F_\lambda^-$, then

$$U = \bigoplus_{\lambda \in \Lambda} W_\lambda.$$

Proof. Let us first prove that the sum is direct. Let $n \in \mathbb{N}$ and $c_{\lambda_i} \in W_{\lambda_i}$ for all $i \leq n$ be such that $c_{\lambda_1} + \cdots + c_{\lambda_n} = 0$. As the set of sections is disjoint, we can assume without loss of generalities (after possibly reordering) that $F_{\lambda_i}^+ \subset F_{\lambda_1}^-$ for all $i > 1$. It gives

$$c_{\lambda_1} = -\sum_{i>1} c_{\lambda_i} \in F_{\lambda_1}^- \cap W_{\lambda_1},$$

which implies that $c_{\lambda_1} = 0$. Repeating the argument, we get that all the $c_{\lambda_i} = 0$.

Let us denote by $X = \bigoplus_{\lambda \in \Lambda} W_\lambda$ and suppose by contradiction that $X \neq U$. By the covering property, there is $\lambda \in \Lambda$ such that $X + F_\lambda^- \neq X + F_\lambda^+$. Then

$$X + F_\lambda^+ = X + W_\lambda + F_\lambda^- = X + F_\lambda^-,$$

a contradiction. □

The next lemma allows us to construct more sets of disjoint covering sections.

Lemma 3.2.27. If the set $\{(F_\lambda^-, F_\lambda^+) | \lambda \in \Lambda\}$ is a set of disjoint sections that cover U and if the set $\{(G_\sigma^-, G_\sigma^+) | \sigma \in \Sigma\}$ is a set of sections which is disjoint and strongly covers U, then the set

$$\{(F_\lambda^- + G_\sigma^- \cap F_\lambda^+, F_\lambda^- + G_\sigma^+ \cap F_\lambda^+) | (\lambda, \sigma) \in \Lambda \times \Sigma\}$$

is disjoint and covers U.

Proof. Let us first prove that the set of sections is disjoint. Suppose $(\lambda, \sigma) \neq (\lambda', \sigma')$. If $\lambda \neq \lambda'$, then we can suppose $F_\lambda^+ \subset F_{\lambda'}^-$. It implies that

$$F_\lambda^- + G_\sigma^+ \cap F_\lambda^+ \subset F_\lambda^+ \subset F_{\lambda'}^- \subset F_{\lambda'}^- + G_{\sigma'}^- \cap F_{\lambda'}^+.$$

In the case where $\lambda = \lambda'$, we have $\sigma \neq \sigma'$ and we can assume $G_\sigma^+ \subset G_{\sigma'}^-$. It implies that $F_\lambda^- + G_\sigma^+ \cap F_\lambda^+ \subset F_\lambda^- + G_{\sigma'}^- \cap F_\lambda^+$.

Let us now show that it covers U. Let $X \subsetneq U$. As the sections $(F_\lambda^-, F_\lambda^+), \lambda \in \Lambda$ cover U, there is $\lambda \in \Lambda$ such that $X + F_\lambda^- \neq X + F_\lambda^+$. If we let $Y = X + F_\lambda^-$ and $Z = F_\lambda^+$, we have $Z \not\subset Y$, therefore as the set of sections (G_σ^+, G_σ^-) strongly covers U, there is a $\sigma \in \Sigma$ such that $Y + (G_\sigma^- \cap Z) \neq Y + (G_\sigma^+ \cap Z)$. □

Final Steps

Let us now circle back to persistence modules with the DCC on ker and im. This final lemma shows that some of the sets for which we worked with are indeed disjoint and strongly covering.

Recall that we have:

$$\operatorname{im}^-_{ct} = \bigcup_{r \in c^-} \operatorname{im}^t_r, \qquad \operatorname{im}^+_{ct} = \bigcap_{\substack{r \in c^+ \\ r \leq t}} \operatorname{im} v^t_r,$$

$$\ker^-_{cs} = \bigcup_{\substack{r \in c^- \\ r \geq s}} \ker v^r_s, \qquad \ker^+_{cs} = \bigcap_{r \in c^+} \ker v^r_s.$$

Lemma 3.2.28. For $t \in T$, each of the sets of sections

$$\{(\operatorname{im}^-_{ct}, \operatorname{im}^+_{ct}) : c \text{ a cut with } t \in c^+\}$$

and

$$\{(\ker^-_{ct}, \ker^+_{ct}) : c \text{ a cut with } t \in c^-\}$$

are disjoint and strongly covers V_t.

Proof. Let us focus on the set of sections with the image; the one with the kernels follows a similar pattern.

-It is disjoint: let c, d be distinct cuts with c^+ and d^+ containing t. Exchanging c, d if necessary, we can assume $c^+ \cap d^- \neq \emptyset$. Let $s \in c^+ \cap d^-$. We then have $s < t$ and $\operatorname{im}^+_{ct} \subset \operatorname{im} v^t_s \subset \operatorname{im}^-_{dt}$.

-It strongly covers V_t: Suppose $Y, Z \subset V_t$ such that $Z \not\subset Y$. Then

$$c^- = \{s \in R : \operatorname{im} v^t_s \cap Z \subset Y\}$$

and

$$c^+ = \{s \in R : \operatorname{im} v^t_s \cap Z \not\subset Y\}$$

define a cut with $t \in c^+$. Moreover, we have

$$Y + (\operatorname{im}^-_{ct} \cap Z) = Y + (\bigcup_{s \in c^-} \operatorname{im} v^t_s \cap Z) = \bigcup_{s \in c^-} (Y + (\operatorname{im} v^t_s \cap Z)) = Y.$$

However, by Lemma 3.2.16, we have $\operatorname{im}^+_{ct} = \operatorname{im} v^t_s$ for some $s \in c^+, s \leq t$. Therefore, $Y + (\operatorname{im}^+_{ct} \cap Z) = Y + (\operatorname{im} v^t_s \cap Z) \neq Y$ by the definition of c^+. ☐

We now have all the required preliminary results to prove (using the method developed in [22]) the result we wanted.

Recall that given an interval $I \subset T$, it is uniquely determined by the cuts l, u by the relation $I = l^+ \cap u^-$ if we take the cuts

$$l^- = \{t : t < s \ \forall s \in I\}, \qquad l^+ = \{t : \ \exists s \in I | t \geq s\},$$
$$u^- = \{t : \ \exists s \in I | t \leq s\}, \qquad u^+ = \{t : t > s \ \forall s \in I\}.$$

Moreover, for such cuts and $t \in I$, we have

$$V^-_{It} = (\operatorname{im}^-_{lt} \cap \ker^+_{ut}) + (\operatorname{im}^+_{lt} \cap \ker^-_{ut}),$$
$$V^+_{It} = \operatorname{im}^+_{lt} \cap \ker^+_{ut}.$$

THEOREM 3.2.29. Any persistence module with the descending chain condition on images and kernels is a direct sum of interval modules.

Proof. Let I be an interval and $t \in I$. Consider the section (F^-_{It}, F^+_{It}) of V_t defined by

$$F^{\pm}_{It} = \mathrm{im}^-_{lt} + \left(\mathrm{ker}^{\pm}_{ut} \cap \mathrm{im}^+_{lt}\right)$$

where l, u are the cuts determined by I. As I runs through all the intervals containing t, the cuts l and u run through all the cuts with $t \in l^+$ and $t \in u^-$.

By Lemma 3.2.28, the sections $(F^-_{It}, F^+_{It})_{I \ni t}$ respect the conditions of Lemma 3.2.27 and therefore, the set of sections $(F^-_{It}, F^+_{It})_{I \ni t}$ is disjoint and covers V_t.

By Lemma 3.2.23, we have $V^+_{It} = W_{It} \oplus V^-_{It}$ for all $t \in I$. Therefore, using the definition of $V^-_{It} = (\mathrm{im}^-_{lt} \cap \mathrm{ker}^+_{ut}) + (\mathrm{im}^+_{lt} \cap \mathrm{ker}^-_{ut})$ and $V^+_{It} = \mathrm{im}^+_{lt} \cap \mathrm{ker}^+_{ut}$, we get

$$\mathrm{im}^+_{lt} \cap \mathrm{ker}^+_{ut} = W_{It} \oplus \left((\mathrm{im}^-_{lt} \cap \mathrm{ker}^+_{ut}) + (\mathrm{im}^+_{lt} \cap \mathrm{ker}^-_{ut})\right).$$

It implies

$$\mathrm{im}^+_{lt} \cap \mathrm{ker}^+_{ut} + \mathrm{im}^-_{lt} = \left(W_{It} \oplus \left((\mathrm{im}^-_{lt} \cap \mathrm{ker}^+_{ut}) + (\mathrm{im}^+_{lt} \cap \mathrm{ker}^-_{ut})\right)\right) + \mathrm{im}^-_{lt}.$$

We then have

$$F^+_{It} = W_{It} + \left((\mathrm{im}^-_{lt} \cap \mathrm{ker}^+_{ut} + \mathrm{im}^-_{lt}) + (\mathrm{im}^+_{lt} \cap \mathrm{ker}^-_{ut} + \mathrm{im}^-_{lt})\right).$$

As we have $\mathrm{im}^-_{lt} \cap \mathrm{ker}^+_{ut} + \mathrm{im}^-_{lt} = \mathrm{im}^-_{lt} \subset F^-_{It}$ and $\mathrm{im}^+_{lt} \cap \mathrm{ker}^-_{ut} + \mathrm{im}^-_{lt} = F^-_{It}$. It gives that we have $F^+_{It} = W_{It} + F^-_{It}$. Let us show that the sum is direct. For that, it only remains to show that $W_{It} \cap F^-_{It} = \{0\}$.

Let $x \in W_{It} \cap F^-_{It}$. As $x \in F^-_{it}$, it gives that $x = a + b$ with $a \in \mathrm{im}^-_{lt}$ and $b \in \mathrm{ker}^-_{ut} \cap \mathrm{im}^+_{lt} \subset \mathrm{ker}^+_{ut}$. Moreover, $x \in W_{It} \subset V^+_{It} = \mathrm{im}^+_{ut} \cap \mathrm{ker}^+_{ut}$. In particular, $a = x - b \in \mathrm{ker}^+_{ut}$ and then $a \in \mathrm{im}^-_{lt} \cap \mathrm{ker}^+_{ut}$. It means that $x = a + b \in V^-_{It}$ and $x \in V^-_{It} \cap W_{It} = \{0\}$ by definition of W_{It}. We therefore have

$$F^+_{It} = W_{It} \oplus F^-_{It}.$$

By Lemma 3.2.26, $V_t = \bigoplus_{I \ni t} W_{It}$. Moreover, as proven in Lemma 3.2.24, $\mathbb{W}_I$ is a submodule of $\mathbb{V}$. In particular, the restriction of v^t_s to W_{Is} has an image contained in W_{It}. Thus, if we consider v^t_s as the map between $\bigoplus_{I \ni s} W_{Is} \to \bigoplus_{I \ni t} W_{It}$, then the image of $x = (x_I)_{I \ni s}$ by v^t_s is $(x_I)_{I \supset \{s,t\}}$. In particular, it implies that the decomposition given above on the elements is still true at the level of persistence modules. Therefore

$$\mathbb{V} = \bigoplus_I \mathbb{W}_I = \bigoplus_I \bigoplus_{j=0}^{n_j} \mathbb{I}^I,$$

the last equality is given by Lemma 3.2.24. □

Using the remark made at the beginning of the section, we know that every pointwise finite-dimensional dimensional respects the descending chain condition on images and kernels. Therefore, the following corollary is direct.

Corollary 3.2.30. Every pointwise finite-dimensional $\mathbb{R}$-persistence module is decomposable.

3.3 Barcode and Persistence Diagram

Using the decomposition of persistence modules into interval modules, we can define two great visual tools to represent the persistence module: the barcode and the persistence diagram. We will extend the definitions of barcode and persistence diagram of Morse functions that we gave in the introduction.

Those two tools are formally multisets. Intuitively, multisets are sets in which we allow the repetition of elements. More formally, they are defined as follows.

Definition 3.3.1. A *multiset* is the data of (U, m) where U is a set and $m\colon U \to \mathbb{N}_0$ is a function defining the *multiplicity* of elements. The value $m(a)$ for $a \in U$ is interpreted as the number of occurrences of the element a in the multiset.

The *support* of a multiset is the set U.

To define a function between multisets, we will implicitly number each repeating element to distinguish them.

Definition 3.3.2. Let F be a decomposable persistence module such that $\mathbb{V} = \bigoplus_{I \in L}(\mathbb{I}^I)^{k_I}$ with $k_I > 0$, then the *barcode* of $\mathbb{V}$ is the multiset

$$B(\mathbb{V}) = \{(I, k_I) : I \in L\}.$$

Definition 3.3.3. Let F be a decomposable persistence module such that $\mathbb{V} = \bigoplus_{I \in L}(\mathbb{I}^I)^{k_I}$, then the (undecorated) *persistence diagram* of F is the multiset of $\overline{\mathcal{H}} = \{(x, y) \in \overline{\mathbb{R}} : x \geq y\}$ defined by

$$\mathrm{dgm}(\mathbb{V}) = \{((\inf(I), \sup(I)), k_I) : I \in L\} \cup (\Delta, \infty).$$

In other words, it consists of points in the extended upper half plane representing the endpoint of each interval, each with the multiplicity of the multiplicity of that interval in the decomposition. We also add the first diagonal $\Delta = \{(x, x) | x \in \overline{\mathbb{R}}\}$ with infinite multiplicity.

Remark 3.3.4. The undecorated diagram contains a bit less information than the barcode as we lose the information about the "openness" and "closed-ness" of the intervals. One way to keep this information is through *decorated* persistence diagram, where we add some information (a + or $-$) on the numbers depending if they are included in the interval or not. For example $(1^+, 2^+)$ represents the interval $(1, 2]$ while $(1^+, 2^-)$ represents the interval $(1, 2)$. This concept is further developed in [18]. Note, however, that in the next chapter, we will define an extended pseudo-metric on barcode and persistence diagram. This extended pseudo-metric will be zero between decorated persistence diagrams whose underlying undecorated persistence diagrams are the same.

Remark 3.3.5. Still in [18], the concept of persistence diagram is extended to persistence modules $\mathbb{V}$ such that $\mathrm{rank}(v_s^t) < \infty$ for all $s < t$. This extension uses the concept of rectangle measure and defines the persisence diagram as being the (unique) multiset D of $\overline{H} \backslash \Delta$ such that for every rectangle $R = [a, b] \times [c, d] \subset \overline{H} \backslash \Delta$, we have

$$\mathrm{Card}(D|_R) = r_b^c - r_a^c - r_b^d + r_a^d$$

where $r_a^b = \mathrm{rank}(v_a^b)$. Intuitively, we want to be in each rectangle $R = [a, b] \times [c, d]$ the number of intervals modules that appear after a and before b and that disappear after c but before d. Note the similarity of formulae with the one developed in Section 1.2.2.

Although this generalisation is widely used, it is a bit outside the scope of this master thesis as the formalisation of this measure is pretty long and tedious, while not being essential to the present work.

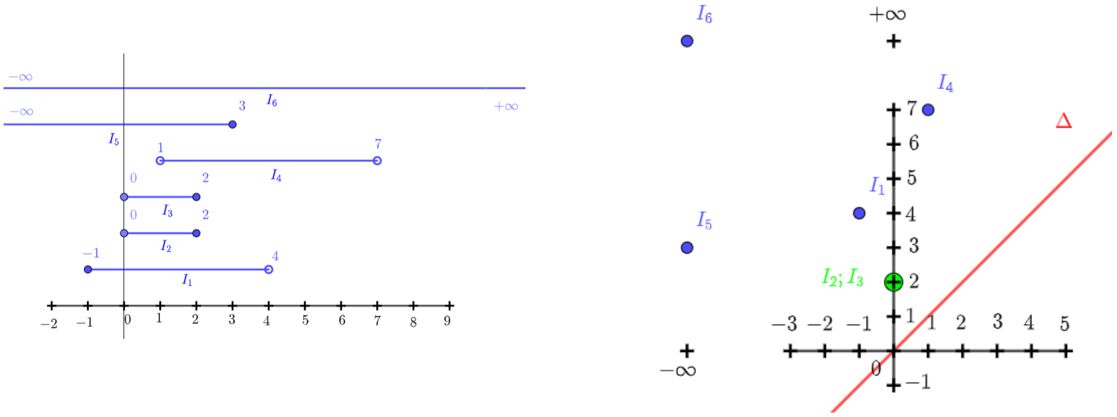

Figure 3.1: Barcode (left) and persistence diagram (right) of the persistence module $\mathbb{V} = \mathbb{I}^{[-1,4)} \oplus (\mathbb{I}^{[0,2]})^2 \oplus \mathbb{I}^{(1,7)} \oplus \mathbb{I}^{(-\infty,3]} \oplus \mathbb{I}^{(-\infty,+\infty)}$. The diagonal is in red as it has infinite multiplicity and the point $(0, 2)$ is green as it has multiplicity 2.

4 | The Stability of Persistence Modules

As mentioned in the introduction, persistent homology and persistence modules are widely used in topological data analysis. One of the main goals of this field is to be able to infer some information on the topology of some manifold underlying a given data set (the "data inference problem"). The idea would be to define some homology thanks to the data given and hope that the homology defined is somewhat close to the homology of the underlying manifold. The goal of this chapter is to introduce a (pseudo-)metric on the persistence modules and their barcode and show that they are linked. We will prove the isometry theorem at the end of this chapter, which will allow us to justify that the idea we had to infer the topology of the manifold is indeed correct.

The main sources for this chapter are [18, 13, 7]. We will use the idea of [13] to use a categorical framework to allow us to be more general while still using the majority of the results (and idea of proofs) of [18].

4.1 Interleaving Distance

In this section, we use the categorical framework. We also focus on persistence modules over $\mathbb{R}$. Recall that we can endow any poset $(T, \leq)$ with the following categorical structure: the objects of $(T, \leq)$ are the elements of T and

$$\operatorname{Hom}(a, a')_{(T,\leq)} = \begin{cases} \{f_a^{a'}\} & \text{if } a \leq a' \\ \emptyset & \text{otherwise.} \end{cases}$$

In particular, we have $f_a^a = \mathrm{id}_a$ for all $a \in T$ and for all $a \leq b \leq c$, we have $f_a^c = f_b^c f_a^b$. We endow $(\mathbb{R}, \leq)$ with this categorical structure. When the context is clear, for the poset $(\mathbb{R}, \leq)$, we will denote this category simply as $\mathbb{R}$.

Definition 4.1.1. Let $b \geq 0$. The *translation functor* of offset b is the functor T_b defined by $T_b(a) = a + b$ and $T(f_a^{a'}) = f_{a+b}^{a'+b}$.

We also define the natural transformation $\eta_b \colon \mathrm{id}_T \to T_b$ using $(\eta_b)_a = f_a^{a+b}$ for all $a \in T$.

Note that η_b is indeed a natural transformation. Let $f_a^{a'} \colon a \to a'$, we have the following commutative diagram that gives the naturality condition:

$$\begin{tikzcd}
a \arrow[r, "(\eta_b)_a = f_a^{a+b}"] \arrow[d, "f_a^{a'}"'] & a+b \arrow[d, "T_b(f_a^{a'}) = f_{a+b}^{a'+b}"] \\
a' \arrow[r, "(\eta_b)_{a'} = f_{a'}^{a'+b}"'] & a'+b.
\end{tikzcd}$$

Let $\mathcal{C}$ be a category and F, G be two functors $\mathbb{R} \to \mathcal{C}$. Let $\epsilon \in \mathbb{R}$ be greater than or equal to 0.

Definition 4.1.2. An *ϵ-interleaving* of the functors F and G consists in the data of 2 natural transformations $\Phi\colon F \to GT_\epsilon$ and $\Psi\colon G \to FT_\epsilon$ such that

$$(\Psi T_\epsilon)\Phi = F\eta_{2\epsilon} \text{ and } (\Phi T_\epsilon)\Psi = G\eta_{2\epsilon}.$$

If such an ϵ-interleaving exists, we say that F and G are *ϵ-interleaved.*

Here, we use the convention developed in Section B.1.1.

In more detail, by the naturality of Φ, we have for all $a \leq a'$ in $\mathbb{R}$ the following commutative diagram.

$$\begin{tikzcd}
F(a) \arrow[r, "\Phi_a"] \arrow[d, "F(f_a^{a'})"'] & GT_\epsilon(a) = G(a+\epsilon) \arrow[d, "GT_\epsilon(f_a^{a'}) = G(f_{a+\epsilon}^{a'+\epsilon})"] \\
F(a') \arrow[r, "\Phi'_a"'] & GT_\epsilon(a') = G(a'+\epsilon).
\end{tikzcd}$$

Therefore, $G(f_{a+\epsilon}^{a'+\epsilon}) \circ \Phi_a = \Phi_{a'} \circ F(f_a^{a'})$. Similarly, using the naturality of Ψ, we have $F(f_{a+\epsilon}^{a'+\epsilon}) \circ \Psi_a = \Psi_{a'} \circ G(f_a^{a'})$ for all $a \leq a'$.

The equality $(\Psi T_\epsilon)\Phi = F\eta_{2\epsilon}$ translates to, for all $a \in T$,

$$\Psi_{a+\epsilon}\Phi_a = (\Psi T_\epsilon)_a \Phi_a = F(\eta_{2\epsilon})_a = F(f_a^{a+2\epsilon}).$$

Similarly, we have, for all $a \in T$,

$$\Phi_{a+\epsilon}\Psi_a = G(f_a^{a+2\epsilon}).$$

Using interleaving, we can define an extended pseudo-metric on the functors between $\mathbb{R}$ and $\mathcal{C}$. For that, we need the following lemma, which will justify the use of the infimum in the definition of the extended pseudo metric. Let us first recall what an extended pseudo-metric is.

An *extended pseudo-metric* on a set X is a function $d\colon X \times X \to \mathbb{R}^+ \cup \{+\infty\}$ such that for all x, y, z, we have $d(x,x) = 0$, it is symmetric : $d(x,y) = d(y,x)$ and such that it respects the triangular inequality: $d(x,z) \leq d(x,y) + d(y,z)$.

The difference between an extended pseudo metric and an extended metric is that $d(x,y) = 0$ does not imply that $x = y$. However, the relation $x \sim y \Leftrightarrow d(x,y) = 0$ is an equivalence relationship on X. Indeed, it is clearly symmetric and reflexive. It is also transitive as if $x \sim y$ and $y \sim z$, by the triangle inequality, we have $d(x,z) \leq d(x,y) + d(y,z) = 0 + 0 = 0$ and $x \sim z$. One can therefore quotient out X, and the function defined on $X/\sim$ induced by d is an extended metric. The term "extended" means that $d(x,y)$ can take the value $+\infty$.

Lemma 4.1.3. If F and G are ϵ-interleaved, then they are ϵ'-interleaved for each $\epsilon' \geq \epsilon$.

Proof. Let $\Phi\colon F \to GT_\epsilon$ and $\Psi\colon G \to FT_\epsilon$ be the natural transformations defining the ϵ-interleaving. Denote by $\bar{\epsilon} = \epsilon' - \epsilon \geq 0$. We define $\eta_{\bar{\epsilon}}T_\epsilon\colon T_\epsilon \to T_{\bar{\epsilon}}T_\epsilon = T_{\epsilon'}$ and we have $G\eta_{\bar{\epsilon}}T_\epsilon\colon GT_\epsilon \to GT_{\epsilon'}$. We then define $\hat{\Phi} = (G\eta_{\bar{\epsilon}}T_\epsilon)\Phi$. Similarly, we define $\hat{\Psi} = (F\eta_{\bar{\epsilon}}T_\epsilon)\Psi$. Note that $(G\eta_{\bar{\epsilon}}T_\epsilon)_a = G(\eta_{\bar{\epsilon}})_{a+\epsilon} = G(f_{a+\epsilon}^{a+\epsilon+\bar{\epsilon}}) = G(f_{a+\epsilon}^{a+\epsilon'})$ and similarly, $(F\eta_{\bar{\epsilon}}T_\epsilon)_a = F(f_{a+\epsilon}^{a+\epsilon'})$.

We need to check that $(\hat{\Psi}T_{\epsilon'})\hat{\Phi} = F(\eta_{2\epsilon'})$. On each component, it gives

$$\begin{aligned}((\hat{\Psi}T_{\epsilon'})\hat{\Phi})_a =&\hat{\Psi}_{a+\epsilon'}\hat{\Phi}_a\\ =&((F\eta_{\bar{\epsilon}}T_\epsilon)\Psi)_{a+\epsilon'}((G\eta_{\bar{\epsilon}}T_\epsilon)\Phi)_a\\ =&(F\eta_{\bar{\epsilon}}T_\epsilon)_{a+\epsilon'}\Psi_{a+\epsilon'}(G\eta_{\bar{\epsilon}}T_\epsilon)_a\Phi_a\\ =&F(f_{a+\epsilon'+\epsilon}^{a+2\epsilon'})\Psi_{a+\epsilon'}G(f_{a+\epsilon}^{a+\epsilon'})\Phi_a.\end{aligned}$$

The condition is therefore equivalent to showing that the following diagram commutes.

$$\begin{tikzcd}
F(a) \arrow[rr, "F(f_a^{a+2\epsilon})"] \arrow[dr, "\Phi_a"'] & & F(a+2\epsilon) \arrow[rr, "F(f_{a+2\epsilon}^{a+\epsilon'+\epsilon})"] & & F(a+\epsilon'+\epsilon) \arrow[r, "F(f_{a+\epsilon'+\epsilon}^{a+2\epsilon'})"] & F(a+2\epsilon). \\
& G(a+\epsilon) \arrow[ur, "\Psi_{a+\epsilon}"'] \arrow[rr, "G(f_{a+\epsilon}^{a+\epsilon'+\epsilon})"'] & & G(a+\epsilon') \arrow[ur, "\Psi_{a+\epsilon'}"'] & &
\end{tikzcd}$$

The left triangle commutes as F and G are ϵ-interleaved and the rectangle commutes too by naturality of Ψ. ☐

Definition 4.1.4. We say that $d(F, G) \leq \epsilon$ if F and G are ϵ-interleaved. and we set

$$d(F, G) = \inf\{\epsilon \geq 0 : F \text{ and } G \text{ are } \epsilon\text{-interleaved}\}.$$

We keep the convention that $d(F, G) = \infty$ if for every $\epsilon \geq 0$, F and G are not ϵ-interleaved.

Remark 4.1.5. If $d(F, G) = 0$, it does not imply that F and G are naturally equivalent.

Indeed, take $F\colon \mathbb{R} \to \mathbf{Vect}, a \mapsto 0$ and $G\colon \mathbb{R} \to \mathbf{Vect}, a \mapsto \begin{cases}\mathbf{k} & \text{if } a = 0,\\ 0 & \text{otherwise.}\end{cases}$ with the 0 map between objects. These two functors are not naturally equivalent as $F(0) \not\simeq G(0)$. However, for each $\epsilon > 0$, defining Φ and Ψ by $\Phi_a = 0$ and $\Psi_a = 0$ for all $a \in \mathbb{R}$, we have that they are natural and we have $\Phi_{a+\epsilon}\Psi_a = 0 = G(f_a^{a+2\epsilon})$ and similarly with F. It therefore provides an ϵ-interleaving.

Note, however, that if F and G are 0-interleaved, then they must be naturally equivalent. It suffices to take as maps for the equivalence the maps Ψ and Φ.

We will use the lemma we have proven prior to the definition in order to prove the triangle inequality and show that it is indeed an extended pseudo-metric.

Proposition 4.1.6. The function d defined above is an extended pseudometric.

Proof. The conditions $d(F,F)=0$ and $d(F,G)=d(G,F)$ are direct by the definition of interleaving. It only remains to show the triangle inequality.

Let F,G and H be three functors, $a=d(F,G)$ and $b=d(G,H)$. Let also $\epsilon>0$. By the definition of the infimum, there are $\epsilon'\leq\epsilon$ and $\epsilon''\leq\epsilon$ such that F and G are $(a+\epsilon')$-interleaved and G and H are $(b+\epsilon'')$-interleaved. By the previous lemma, it implies that F and G are $(a+\epsilon)$-interleaved and G and H are $(b+\epsilon)$-interleaved.

Let $\Phi'\colon F\to GT_{a+\epsilon}$, $\Psi'\colon G\to FT_{a+\epsilon}$, $\Phi''\colon G\to HT_{b+\epsilon}$ and $\Psi''\colon H\to GT_{b+\epsilon}$ be the different natural transformations defining the interleaving. Define

$$\begin{aligned}\Phi&=(\Phi''T_{a+\epsilon})\Phi\colon F\to HT_{b+\epsilon}T_{a+\epsilon}=HT_{a+b+2\epsilon}\\ \Psi&=(\Psi'T_{b+\epsilon})\Psi''\colon H\to FT_{a+b+2\epsilon}.\end{aligned}$$

We will show that Φ and Ψ define an $(a+b+2\epsilon)$-interleaving between F and H. As they are defined as the composition of natural transformations, they are natural transformations. It remains to show that $(\Psi T_{a+b+2\epsilon})\Phi=F\eta_{2(a+b+2\epsilon)}$ and $(\Phi T_{a+b+2\epsilon})\Psi=G\eta_{2(a+b+2\epsilon)}$. We will only show the first equality as the second one is totally similar.

Take $x\in\mathbb{R}$, we have

$$\begin{aligned}((\Psi T_{a+b+2\epsilon})\Phi)_x=&(\Psi_{x+a+b+2\epsilon})\Phi_x\\ =&((\Psi'T_{b+\epsilon})\Psi'')_{x+a+b+2\epsilon}((\Phi''T_{a+\epsilon})\Phi')_x\\ =&\Psi'_{x+a+2b+3\epsilon}\Psi''_{x+a+b+2\epsilon}\Phi''_{x+a+\epsilon}\Phi'_x.\end{aligned}$$

Moreover, as Φ'' and Ψ'' define an $(b+\epsilon)$-interleaving, we have

$$\Psi''_{x+a+b+2\epsilon}\Phi''_{x+a+\epsilon}=((\Psi''T_{b+\epsilon})\Phi'')_{x+a+\epsilon}=(G\eta_{2b+2\epsilon})_{x+a+\epsilon}=G(f_{x+a+\epsilon}^{x+a+2b+3\epsilon}).$$

By the naturality of Ψ', we have the following commutative diagram.

$$\begin{tikzcd}
G(x+a+\epsilon) \arrow[r, "\Psi'_{x+a+\epsilon}"] \arrow[d, "G(f_{x+a+\epsilon}^{x+a+2b+3\epsilon})"'] & F(x+2a+2\epsilon) \arrow[d, "F(f_{x+2a+2\epsilon}^{x+2a+2b+4\epsilon})"] \\
G(x+a+2b+3\epsilon) \arrow[r, "\Psi'_{x+a+2b+3\epsilon}"'] & F(x+2a+2b+4\epsilon)
\end{tikzcd}$$

Therefore, we have

$$\begin{aligned}((\Psi T_{a+b+2\epsilon})\Phi)_x=&\Psi'_{x+a+2b+3\epsilon}\Psi''_{x+a+b+2\epsilon}\Phi''_{x+a+\epsilon}\Phi'_x\\ =&\Psi'_{x+a+2b+3\epsilon}G\left(f_{x+a+\epsilon}^{x+a+2b+3\epsilon}\right)\Phi'_x\\ =&F\left(f_{x+2a+2\epsilon}^{x+2a+2b+4\epsilon}\right)\Psi'_{x+a+\epsilon}\Phi'_x\\ =&F\left(f_{x+2a+2\epsilon}^{x+2a+2b+4\epsilon}\right)((\Psi'T_{a+\epsilon})\Phi')_x\\ =&F\left(f_{x+2a+2\epsilon}^{x+2a+2b+4\epsilon}\right)(F\eta_{2(a+\epsilon)})_x\\ =&F\left(f_{x+2a+2\epsilon}^{x+2a+2b+4\epsilon}\right)F\left(f_{x}^{x+2a+2\epsilon}\right)\\ =&F\left(f_{x}^{x+2a+2b+4\epsilon}\right).\end{aligned}$$

It means that F and H are $(a+b+2\epsilon)$-interleaved for all $\epsilon>0$. It implies by the definition of the infimum that $d(F,H)\leq a+b$. ☐

Even though d is not a distance, only an extended pseudo-metric, we will use an abuse of notation and still call it the interleaving distance.

> **Proposition 4.1.7.** Let $F, G\colon \mathbb{R} \to \mathcal{C}$ and $H\colon \mathcal{C} \to \mathcal{D}$ be two functors. If F and G are ϵ-interleaved, then so are HF and HG. In particular, $d(HF, HG) \leq d(F, G)$.

Proof. Let $\Phi\colon F \to GT_\epsilon$ and $\Psi\colon G \to FT_\epsilon$ be the natural transformation defining the ϵ-interleaving between F and G. Then the natural transformations $H\Phi$ and $H\Psi$ define an ϵ-interleaving between HF and HG. □

4.2 The Interpolation Lemma

The goal of this section is to prove that if we have two functors F, G that are ϵ-interleaved, then for all $\delta \leq \epsilon$, there is a functor H such that F, H are δ-interleaved and H, G are $(\epsilon - \delta)$-interleaved. For this section, we restrict ourselves to functors from $\mathbb{R}$ to an *abelian* category $\mathcal{A}$.

First, we will consider the plane $\mathbb{R}^2$ with the order $(a, b) \leq (a', b') \Leftrightarrow a \leq a'$ and $b \leq b'$. Note that we have not defined an interleaving between functors of $\mathbb{R}^2 \to \mathcal{A}$. However, for all $x \in \mathbb{R}$, we can define

$$\Delta_x = \{(p, q) | p - q = 2x\} \subset \mathbb{R}^2 .$$

As posets, we have $\mathbb{R} \simeq \Delta_x\colon t \mapsto (t - x, t + x)$. Therefore, if we have functors defined over Δ_x, we can define their interleaving by using the isomorphism between Δ_x and $\mathbb{R}$. We can also relate the fact of being interleaved with some condition of existence of functors on some parts of $\mathbb{R}^2$.

> **Proposition 4.2.1.** The functors $F, G\colon \mathbb{R} \to \mathcal{A}$ are $|y - x|$-interleaved if and only if there is a functor $H\colon \Delta_x \cup \Delta_y \to \mathcal{A}$ such that $H|_{\Delta_x} \simeq F$ and $H|_{\Delta_y} \simeq G$.

Proof. If $x = y$, then we have the result as $F \simeq G$ if and only if F, G are 0-interleaved. Without loss of generality, we will assume that $x < y$.

If F and G are $(y - x)$-interleaved, then there are natural transformations $\Phi\colon F \to GT_{y-x}$ and $\Psi\colon G \to FT_{y-x}$ such that

$$\Psi_{a+y-x}\Phi_a = F\left(f_a^{a+2(y-x)}\right)$$

and

$$\Phi_{a+y-x}\Psi_a = G\left(f_a^{a+2(y-x)}\right) .$$

We define H as $H(a - x, a + x) = F(a)$, $H(a - y, a + y) = G(a)$, $H\left(f_{(a-x,a+x)}^{(b-x,b+x)}\right) = F\left(f_a^b\right)$, $H\left(f_{(a-y,a+y)}^{(b-y,b+y)}\right) = G\left(f_a^b\right)$, and, if $a + y \leq b + x$,

$$H\left(f_{(a-x,a+x)}^{(b-y,b+y)}\right) = G\left(f_{a+y-x}^b\right)\Phi_a$$

and

$$H\left(f_{(a-y,a+y)}^{(b-x,b+x)}\right) = F\left(f_{a+y-x}^b\right)\Psi_a.$$

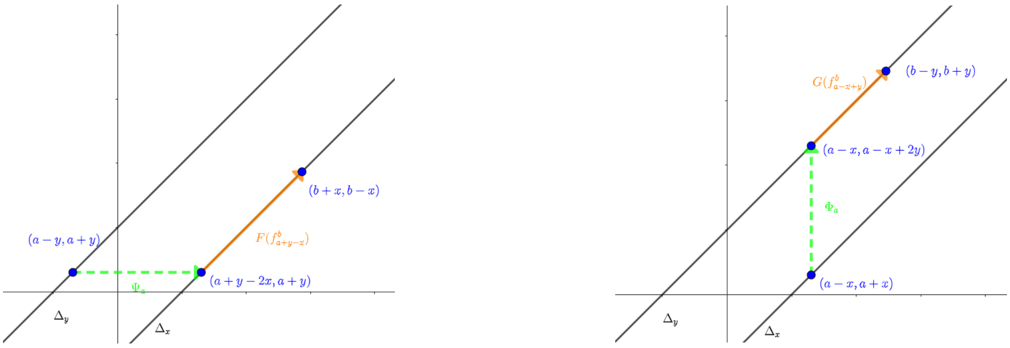

Figure 4.1: Definition of $H\left(f^{(b-y,b+y)}_{(a-x,a+x)}\right)$ and $H\left(f^{(b-x,b+x)}_{(a-y,a+y)}\right)$.

The following figure gives a visual representation of the definition of those maps.

From the definition, we directly have that $H|_{\Delta_x} \simeq F$ and $H|_{\Delta_y} \simeq G$. We also directly have that $H(\mathrm{id}_{(a,b)}) = \mathrm{id}_{H(a,b)}$. It only remains to show that $H(f \circ g) = H(f) \circ H(g)$. We will consider different cases. If the endpoints of f and g are all on the same diagonal, the equality follows from the definition of F and G. Let us compute now the result if we have one change of diagonal: First, assume that the change of diagonal happens in the first function. It gives the following equality.

$$\begin{aligned} H\left(f^{(c-y,c+y)}_{(b-y,b+y)}\right) H\left(f^{(b-y,b+y)}_{(a-x,a+x)}\right) =& G\left(f^c_b\right) G\left(f^b_{a+y-x}\right) \Phi_a \\ =& G\left(f^c_{a+y-x}\right) \Phi_a = H\left(f^{(c-y,c+y)}_{(a-x,a+x)}\right). \end{aligned}$$

Now, assume that the change of diagonal happens in the second function. It gives the following.

$$\begin{aligned} H\left(f^{(c-y,c+y)}_{(b-x,b+x)}\right) H\left(f^{(b-x,b+x)}_{(a-x,a+x)}\right) =& G\left(f^c_{b-x+y}\right) \Phi_b F\left(f^b_a\right) \\ =& G\left(f^c_{b-x+y}\right) G\left(f^{b-x+y}_{a+y-x}\right) \Phi_a \\ =& G\left(f^c_{a+y-x}\right) \Phi_a = H\left(f^{(c-y,c+y)}_{(a-x,a+x)}\right). \end{aligned}$$

The second equality of the second case comes from the naturality of Φ. The case going from Δ_y to Δ_x is completely symmetric to the one above. It only remains to show the case where there are 2 changes of diagonal. I will show the case where the first function goes from Δ_x to Δ_y and the second function goes from Δ_y to Δ_x. The other case is completely symmetric.

$$\begin{aligned} H\left(f^{(c-x,c+x)}_{(b-y,b+y)}\right) H\left(f^{(b-y,b+y)}_{(a-x,a+x)}\right) =& F\left(f^c_{b-x+y}\right) \Psi_b G\left(f^b_{a-x+y}\right) \Phi_a \\ =& F\left(f^c_{b-x+y}\right) F\left(f^{b-x+y}_{a-2x+2y}\right) \Psi_{a-x+y} \Phi_a \\ =& F\left(f^c_{b-x+y}\right) F\left(f^{b-x+y}_{a-2x+2y}\right) F\left(f^{a-2x+2y}_a\right) \\ =& F\left(f^c_a\right) = H\left(f^{(c-x,c+x)}_{(a-x,a+x)}\right). \end{aligned}$$

We first use the naturality of Ψ before using the interleaving equality. Therefore, H is indeed a functor from $\Delta_x \cup \Delta_y$ to $\mathcal{A}$.

Let us now assume that there exists a functor $H\colon \Delta_x \cup \Delta_y \to \mathcal{A}$ such that $H|_{\Delta_x} \simeq F$ and $H|_{\Delta_y} \simeq G$. Let $\alpha\colon H|_{\Delta_x} \to F$ and $\beta\colon H|_{\Delta_y} \to G$ be the equivalence of functors. Let us construct Φ and Ψ defining an $(y-x)$-interleaving (we suppose again $x < y$). Let $\Phi_a = \beta_{a-x+y} H\left(f^{(a-x,a-x+2y)}_{(a-x,a+x)}\right) \alpha_a^{-1}$ and $\Psi_a = \alpha_{a-x+y} H\left(f^{(a-2x+y,a+y)}_{(a-y,a+y)}\right) \beta_a^{-1}$.

We then have

$$\begin{aligned}
\Phi_b F(f^b_a) =& \beta_{b-x+y} H\left(f^{(b-x,b-x+2y)}_{(b-x,b+x)}\right) \alpha_b^{-1} F(f^b_a)\\
=& \beta_{b-x+y} H\left(f^{(b-x,b-x+2y)}_{(b-x,b+x)}\right) H\left(f^{(b-x,b+x)}_{(a-x,a+x)}\right) \alpha_a^{-1}\\
=& \beta_{b-x+y} H\left(f^{(b-x,b-x+2y)}_{(a-x,a+x)}\right) \alpha_a^{-1}\\
=& \beta_{b-x+y} H\left(f^{(b-x,b-x+2y)}_{(a-x,a-x+2y)}\right) H\left(f^{(a-x,a-x+2y)}_{(a-x,a+x)}\right) \alpha_a^{-1}\\
=& G\left(f^{b-x+y}_{a-x+y}\right) \beta_{a-x+y} H\left(f^{(a-x,a-x+2y)}_{(a-x,a+x)}\right) \alpha_a^{-1}\\
=& G\left(f^{b-x+y}_{a-x+y}\right) \Phi_a.
\end{aligned}$$

It gives the naturality of Φ; the one for Ψ is completely similar. For the equation giving interleaving, it comes from

$$\begin{aligned}
\Psi_{a-x+y}\Phi_a =& \alpha_{a-2x+2y} H\left(f^{(a-3x+2y,a-x+2y)}_{(a-x,a-x+2y)}\right) \beta^{-1}_{a-x+y} \beta_{a-x+y} H\left(f^{(a-x,a-x+2y)}_{(a-x,a+x)}\right) \alpha_a^{-1}\\
=& \alpha_{a-2x+2y} H\left(f^{(a-3x+2y,a-x+2y)}_{(a-x,a-x+2y)}\right) H\left(f^{(a-x,a-x+2y)}_{(a-x,a+x)}\right) \alpha_a^{-1}\\
=& \alpha_{a-2x+2y} H\left(f^{(a-3x+2y,a-x+2y)}_{(a-x,a+x)}\right) \alpha_a^{-1}\\
=& F\left(f^{a-2x+2y}_a\right) \alpha_a \alpha_a^{-1} = F\left(f^{a-2x+2y}_a\right)
\end{aligned}$$

Therefore, we have an interleaving. ☐

Note that in this proof, we did not use the fact that $\mathcal{A}$ was abelian; the result is actually true for any functors from $\mathbb{R}$ to any category $\mathcal{C}$.

Using this new perspective on the interleaving, we can prove the interpolation lemma by extending the functor H to a strip $\Delta_{[x,y]} = \{(a,b) : 2x \leq b-a \leq 2y\}$. Then, for every $x \leq z \leq y$, we will have that $H|_{\Delta_z}$ is $(z-x)$-interleaved with F and $(y-z)$-interleaved with G.

Theorem 4.2.2. Every functor $H\colon \Delta_x \cup \Delta_y \to \mathcal{A}$ with $\mathcal{A}$ an abelian category can be extended to a functor $\overline{H}\colon \Delta_{[x,y]} \to \mathcal{A}$. In other words, we have the following commutative diagram.

$$\begin{tikzcd}
\Delta_{[x,y]} \arrow[rd, dashed, "\overline{H}"] & \\
\Delta_x \cup \Delta_y \arrow[u, hook, "i"] \arrow[r, "H"'] & \mathcal{A}
\end{tikzcd}$$

Proof. We will assume that $x = -1$ and $y = 1$ to ease the notation. We can extend the result to arbitrary x and y by rescaling and translation. Let us denote by $F = H|_{\Delta_{-1}}$ and $G = H|_{\Delta_1}$. We will denote by

$$\Phi_t\colon F(t) = H(t+1, t-1) \to G(t+2) = H(t+1, t+3)$$

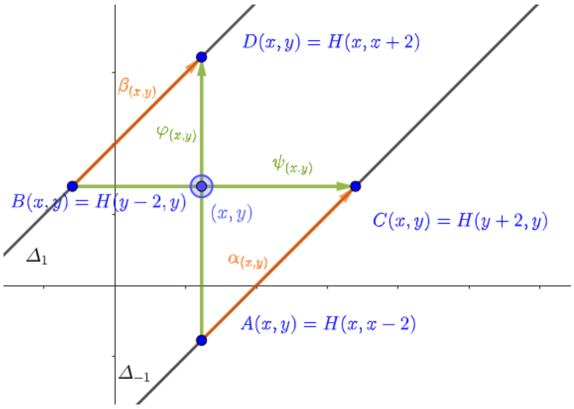

Figure 4.2: Representation of the four functors and four natural transformations.

and

$$\Psi_t \colon G(t) = H(t-1,t+1) \to F(t+2) = H(t+3,t+1).$$

We define four other functors over $\mathbb{R}^2$:

$$\begin{aligned}
A: &\quad A(x,y) := F(x-1) = H(x,x-2), & A\left(f_{(x,y)}^{(z,w)}\right) &:= F\left(f_{x-1}^{z-1}\right) = H\left(f_{(x,x-2)}^{(z,z-2)}\right)\\
B: &\quad B(x,y) := G(y-1) = H(y-2,y), & B\left(f_{(x,y)}^{(z,w)}\right) &:= G\left(f_{y-1}^{w-1}\right) = H\left(f_{(y-2,y)}^{(w-2,w)}\right)\\
C: &\quad C(x,y) := F(y+1) = H(y+2,y), & C\left(f_{(x,y)}^{(z,w)}\right) &:= F\left(f_{y+1}^{w+1}\right) = H\left(f_{(y+2,y)}^{(w+2,w)}\right)\\
D: &\quad D(x,y) := G(x+1) = H(x,x+2), & D\left(f_{(x,y)}^{(z,w)}\right) &:= G\left(f_{x+1}^{z+1}\right) = H\left(f_{(x,x+2)}^{(z,z+2)}\right).
\end{aligned}$$

We also define four natural transformations between these functors:

$$\begin{aligned}
\alpha \colon A \to C, &\quad \alpha_{(x,y)} = H\left(f_{(x,x-2)}^{(y+2,y)}\right) = F\left(f_{x-1}^{y+1}\right)\\
\varphi \colon A \to D, &\quad \varphi_{(x,y)} = H\left(f_{(x,x-2)}^{(x,x+2)}\right) = \Phi_{x-1}\\
\psi \colon B \to C, &\quad \psi_{(x,y)} = H\left(f_{(y-2,y)}^{(y+2,y)}\right) = \Psi_{q-1}\\
\beta \colon B \to D, &\quad \beta(x,y) = H\left(f_{(y-2,y)}^{(x,x+2)}\right) = G\left(f_{y-1}^{x+1}\right).
\end{aligned}$$

The morphisms φ and ψ are defined for all $(x,y) \in \mathbb{R}^2$ whereas α requires $x-1 \leq y+1$ and β requires $y-1 \leq x+1$. In total, they are all well defined when $-2 \leq x-y \leq 2$, i.e. on the strip $\Delta_{[-1,1]}$, we restrict all the functors and natural transformations on that strip. The figure gives a visual representation of the different functors and morphisms. Note that α, β, φ and ψ are all natural transformations.

We define

$$\Omega = \begin{pmatrix} \alpha & \psi \\ \varphi & \beta \end{pmatrix} : A \oplus B \to C \oplus D.$$

Note that Ω is well defined as $\mathcal{A}$ is abelian, therefore $\mathbf{Fun}(\Delta_{[-1,1]}, \mathcal{A})$ is also an abelian category, as discussed in Example B.3.6. Therefore, the direct sum is well defined. We will show that $\overline{H} = \mathrm{im}(\Omega)$ is an extension of H on $\Delta_{[-1,1]}$. Once again, it is well-defined as $\mathbf{Fun}(\Delta_{[-1,1]}, \mathcal{A})$ is abelian (recall the category-theoretic definition of the image of a morphism in Section B.3). $\overline{H}$ is a functor from $\Delta_{[-1,1]}$ to $\mathcal{A}$, it only remains to show that its restriction to $\Delta_{-1} \cup \Delta_1$ is H.

Let us first show that $\overline{H}|_{\Delta_{-1}} \simeq F$. We associate $\Delta_{-1} = \{(t+1, t-1)\}$ to $\mathbb{R}$ through the isomorphism $t \mapsto (t+1, t-1)$. We have $(A \oplus B)(t) = F(t) \oplus G(t-2)$ and $(C \oplus D)(t) = F(t) \oplus G(t+2)$. Moreover, on Δ_{-1}, we have

$$\begin{aligned}
\Omega_t &= \begin{pmatrix} \alpha_{(t+1,t-1)} & \psi_{(t+1,t-1)} \\ \varphi_{(t+1,t-1)} & \beta_{(t+1,t-1)} \end{pmatrix} \\
&= \begin{pmatrix} F\left(f_t^t\right) & H\left(f_{(t-3,t-1)}^{(t+1,t-1)}\right) \\ H\left(f_{(t+1,t-1)}^{(t+1,t+3)}\right) & G\left(f_{t-2}^{t+1}\right) \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{id}_t & \Psi_{t-2} \\ \Phi_t & G\left(f_{t-2}^{t+1}\right) \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{id}_t \\ \Phi_t \end{pmatrix} \begin{pmatrix} \mathrm{id}_t & \Psi_{t-2} \end{pmatrix}.
\end{aligned}$$

The last equality follows from $G\left(f_{t-2}^{t+1}\right) = H\left(f_{(t-3,t-1)}^{(t+1,t+3)}\right) = H\left(f_{(t+1,t-1)}^{(t+1,t+3)}\right) H\left(f_{(t-3,t-1)}^{(t+1,t-1)}\right)$
Denote $\Omega_{1,t} = \begin{pmatrix} \mathrm{id}_t & \Psi_{t-2} \end{pmatrix}$ and $\Omega_{2,t} = \begin{pmatrix} \mathrm{id}_t \\ \Phi_t \end{pmatrix}$.

Using this, we will show that the universal property of the image is satisfied.

$$\begin{tikzcd}
F(t) \oplus G(t-2) \arrow[rr, "\Omega_t"] \arrow[dr, "\Omega_{1,t}"] \arrow[ddr, "e"'] & & F(t) \oplus G(t+2) \\
& F(t) \arrow[ur, tail, "\Omega_{2,t}"'] \arrow[d, dashed, tail, "\exists! f"] & \\
& I \arrow[uur, tail, "m"'] &
\end{tikzcd}$$

We directly have that $\Omega_{2,t}$ is a mono as if $g \colon A \to F(t)$ is such that $\Omega_{2,t} g = 0$, it means that

$$\begin{pmatrix} g \\ \Phi_t g \end{pmatrix} = 0 \Rightarrow g = 0.$$

Let us now construct a map $f \colon F(t) \to I$ completing the diagram and let us show that this map is unique. First, we can write $e = \begin{pmatrix} e_1 & e_2 \end{pmatrix}$ and $m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$. We want f such that $f\Omega_{1,t} = e$ and $mf = \Omega_{2,t}$. In matrix form, it gives

$$\begin{pmatrix} f & f\Psi_{t-2} \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \end{pmatrix}$$

and

$$\begin{pmatrix} m_1 f \\ m_2 f \end{pmatrix} = \begin{pmatrix} \mathrm{id}_t \\ \Phi_t \end{pmatrix}.$$

The first component of the first equation directly gives that $f = e_1$ is the only possibility. Let us show that it works. We have $me = \Omega_t$, it therefore gives

$$\begin{pmatrix} m_1 e_1 & m_1 e_2 \\ m_2 e_1 & m_2 e_2 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_t & \Psi_{t-2} \\ \Phi_t & G\left(f_{t-2}^{t+1}\right) \end{pmatrix}.$$

In particular, it proves that $mf = \Omega_{2,t}$ as we have $\begin{pmatrix} m_1 e_1 \\ m_2 e_2 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_t \\ \Phi_t \end{pmatrix}$. It remains to show that $f\Omega_{1,t} = e$. In order to do that, we will use the fact that m is a monomorphism. Indeed, we have

$$me = \Omega_t = \Omega_{2,t}\Omega_{1,t} = mf\Omega_{1,t}.$$

As m is monomorphic, we have the required $e = f\Omega_{1,t}$. Therefore, $F(t) \simeq \mathrm{im}(\Omega_{t+1,t-1})$ for all $t \in \mathbb{R}$, therefore $F \simeq \mathrm{im}(\Omega)|_{\Delta_{-1}}$.

Similarly, we have $G \simeq \mathrm{im}(\Omega)|_{\Delta_1}$: on Δ_1, we have

$$\Omega_t = \begin{pmatrix} H\left(f_{(t-1,t+1)}^{(t+3,t+1)}\right) \\ \mathrm{id}_{(t-1,t+1)} \end{pmatrix} \begin{pmatrix} H\left(f_{(t-1,t-3)}^{(t-1,t+1)}\right) & \mathrm{id}_{(t-1,t+1)} \end{pmatrix}.$$

We denote by $\Omega_{3,t} = \begin{pmatrix} \Phi_{t-2} & \mathrm{id}_{(t-1,t+1)} \end{pmatrix}$ and $\Omega_{4,t} = \begin{pmatrix} \Psi_t \\ \mathrm{id}_{(t-1,t+1)} \end{pmatrix}$. Using this decomposition, we have that $G(t)$ describes the image of $\Omega_{t-1,t+1}$ for all $t \in \mathbb{R}$ in a similar manner as above.

It only remains to show that the maps from elements of the first diagonal to the second are the same as the ones in H. To prove that, it suffices to show that for all $t \in \mathbb{R}$, the vertical maps are given by Φ: $\overline{H}\left(f_{t+1,t-1}^{t+1,t+3}\right) \simeq \Phi_t$ and the horizontal maps are given by Ψ: $\overline{H}\left(f_{t-1,t+1}^{t+3,t+1}\right) \simeq \Psi_t$. We will show it for the vertical maps; the case of horizontal maps is similar.

For every $t \in \mathbb{R}$, the following diagram commutes.

$$\begin{tikzcd}
F(t) \oplus G(t-2) \arrow[r, "\Omega_{1,t}"] \arrow[d, "\mathrm{id}_{F(t)} \oplus G\left(f_{t-2}^{t+2}\right)"'] & F(t) \arrow[r, "\Omega_{2,t}"] \arrow[d, "\Phi_t"'] & F(t) \oplus G(t+2) \arrow[d, "F\left(f_t^{t+4}\right) \oplus \mathrm{id}_{G(t+2)}"] \\
F(t) \oplus G(t+2) \arrow[r, "\Omega_{3,t+2}"'] & G(t+2) \arrow[r, "\Omega_{4,t+2}"'] & F(t+4) \oplus G(t+2).
\end{tikzcd}$$

Indeed, for the left square, we have

$$\begin{aligned}
\Phi_t\Omega_{1,t} =& \Phi_t \begin{pmatrix} \mathrm{id}_t & \Psi_{t-2} \end{pmatrix} \\
=& \begin{pmatrix} \Phi_t & \Phi_t\Psi_{t-2} \end{pmatrix} \\
=& \begin{pmatrix} \Phi_t & G\left(f_{t-2}^{t+2}\right) \end{pmatrix} \\
=& \begin{pmatrix} \Phi_t & \mathrm{id}_{G(t+2)} \end{pmatrix} \begin{pmatrix} \mathrm{id}_t & 0 \\ 0 & G\left(f_{t-2}^{t+2}\right) \end{pmatrix} \\
=& \Omega_{3,t+2}\left(\mathrm{id}_{F(t)} \oplus G\left(f_{t-2}^{t+2}\right)\right).
\end{aligned}$$

For the right square, we have

$$\begin{aligned}
\Omega_{4,t+2}\Phi_t =& \begin{pmatrix} \Psi_{t+2}\Phi_t \\ \mathrm{id}_{G(t+2)}\Phi_t \end{pmatrix} = \begin{pmatrix} F\left(f_t^{t+4}\right) \\ \mathrm{id}_{G(t+2)}\Phi_t \end{pmatrix} \\
=& \begin{pmatrix} F\left(f_t^{t+4}\right) & 0 \\ 0 & \mathrm{id}_{G(t+2)} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{F(t)} \\ \Phi_t \end{pmatrix}.
\end{aligned}$$

As a result, for each t, Φ_t is the induced map $\overline{H}_{(t+1,t-1)} \to \overline{H}_{(t+1,t+3)}$ as it is the only map making the universal diagram defining the 2 images commute. It is indeed unique as the relation $f\Omega_{1,t} = \Omega_{3,t+2}\left(\mathrm{id}_{F(t)} \oplus G\left(f_{t-2}^{t+2}\right)\right)$ implies that $f = \Phi_t$.

$$\begin{tikzcd}
F(t) \oplus G(t-2) \arrow[rr, "\Omega_{(t+1,t-1)}"] \arrow[dr, "\Omega_{1,t}"] \arrow[ddd, "\mathrm{id}_{F(t)} \oplus G\left(f_{t-2}^{t+2}\right)"'] & & F(t) \oplus G(t+2) \arrow[ddd, "F\left(f_t^{t+4}\right) \oplus \mathrm{id}_{G(t+2)}"] \\
& F(t) = \mathrm{im}(\Omega_{(t+1,t-1)}) \arrow[ur, "\Omega_{2,t}"'] \arrow[d, "\Phi_t"'] & \\
& G(t+2) = \mathrm{im}(\Omega_{(t+1,t+2)}) \arrow[dr, "\Omega_{4,t+2}"] & \\
F(t) \oplus G(t+2) \arrow[ur, "\Omega_{3,t+2}"'] \arrow[rr, "\Omega_{(t+1,t+2)}"'] & & F(t+4) \oplus G(t+2).
\end{tikzcd}$$

Similarly, the horizontal maps Ψ_t come from the commutativity of the following diagram.

$$\begin{tikzcd}
F(t-2) \oplus G(t) \arrow[r, "\Omega_{3,t}"] \arrow[d, "F\left(f_{t-2}^{t+2}\right) \oplus \mathrm{id}_{G(t)}"'] & G(t) \arrow[r, "\Omega_{4,t}"] \arrow[d, "\Psi_t"'] & F(t+2) \oplus G(t) \arrow[d, "\mathrm{id}_{F(t+2)} \oplus G\left(f_t^{t+4}\right)"] \\
F(t+2) \oplus G(t) \arrow[r, "\Omega_{1,t+2}"'] & F(t+2) \arrow[r, "\Omega_{1,t+2}"'] & F(t+2) \oplus G(t+4).
\end{tikzcd}$$

□

4.3 Computation of some Interleaving Distances

We will look at some ways to compute the interleaving distance for persistence modules. For that, we restrict further to functors from $\mathbb{R}$ to **Vect**, the category of vector spaces. Note that **Vect** is an abelian category. In the following, we use the notation χ_I to denote the interval module of support I instead of the previous notation $\mathbb{I}^I$ as well as the notation of the functor F rather than $\mathbb{V}$ to denote persistence modules. We use these notations to emphasise the fact that we work with functors. We will first compute the interleaving distance between any two interval modules and prove the equalities put forward in [13].

Lemma 4.3.1. Assume I and I' are finite intervals. Then $d(\chi_I, \chi_{I'}) \leq \max(h, h')$ where $h = \frac{|I|}{2}$ is half the length of I and h' is half the length of I'.

Proof. Let $\epsilon > \max(h, h')$. We then have that $\chi_I \eta_{2\epsilon} = 0 = \chi_{I'}\eta_{2\epsilon}$ as for all real number a, $(\chi_I \eta_{2\epsilon})_a = \chi_I(f_a^{a+2\epsilon}) = 0$. Indeed, either a or $a + 2\epsilon$ is not in I as their difference is larger than the length of the interval. Therefore, $\chi_I(f_a^{a+2\epsilon}) = 0$. Similarly for $\chi_{I'}\eta_{2\epsilon}$.

Therefore, $\Phi = 0\colon \chi_I \to \chi_{I'}T_\epsilon$ and $\Psi = 0\colon \chi_{I'} \to \chi_I T_\epsilon$ defines an ϵ-interleaving. □

Lemma 4.3.2. Assume I and I' are finite intervals. Let m be the midpoint of I. If $m \notin I'$, then $d(\chi_I, \chi_{I'}) \geq h$ where, as above, h is the half distance of I.

Proof. Let $\epsilon < h$, we will show that there cannot be any ϵ-interleaving. Assume there are some (Φ, Ψ) defining an ϵ-interleaving. As $\epsilon < h$, we have $[m-\epsilon, m+\epsilon] \subset I$, in particular, $(\chi_I \eta_{2\epsilon})_{m-\epsilon} = \mathrm{id}_{\mathbf{k}}$. It implies that $((\Psi T_\epsilon)\Phi)_{m-\epsilon} = (\chi_I \eta_{2\epsilon})_{m-\epsilon} = \mathrm{id}_{\mathbf{k}}$. However, the codomain of $\Phi_{m-\epsilon}$ is $\chi_{I'}(m) = 0$ as $m \notin I'$. Therefore, $\Phi_{m-\epsilon}$ must be the zero map and thus $((\Psi T_\epsilon)\Phi)_{m-\epsilon} = 0$, a contradiction. □

Using these two lemmas, we can now compute the interleaving distance between two interval modules.

Proposition 4.3.3. Let I and I' be finite intervals. Then,

1. if $I = \emptyset = I'$, we have $d(\chi_I, \chi_{I'}) = 0$;
2. if $I' = \emptyset$ and I has endpoints a, b, then $d(\chi_I, \chi_{I'}) = \frac{b-a}{2}$;
3. if I,I' have endpoints a, b and a', b' respectively,

$$d(\chi_I, \chi_{I'}) = \min\left(\max(|a-a'|, |b-b'|), \max\left(\frac{b-a}{2}, \frac{b'-a'}{2}\right)\right).$$

Proof. 1. It is direct as $I = I'$.

2. By Lemma 4.3.1, we have $d(\chi_I, \chi_{I'}) \leq \max(0, \frac{b-a}{2}) = \frac{b-a}{2}$. In the meantime, as the midpoint $m = \frac{a+b}{2} \notin \emptyset$, by Lemma 4.3.2, $d(\chi_I, \chi_{I'}) \geq \frac{b-a}{2}$. We therefore have the equality.

3. We will treat it case by case.

-Assume first that $m \notin I'$ and $m' \notin I$. Then we have either $\frac{a+b}{2} \leq a'$ and $b \leq \frac{a'+b'}{2}$, or $\frac{a+b}{2} \geq b'$ and $\frac{a'+b'}{2} \leq a$. Note that we can't have $\frac{a+b}{2} < a'$ and $\frac{a'+b'}{2} < a$ as it implies that $a < \frac{a+b}{2} < a'$ and $a' < \frac{a'+b'}{2} < a$, which is impossible. Similarly, we can't have $\frac{a+b}{2} > b'$ and $b < \frac{a'+b'}{2}$. In the former case, we have $\frac{b-a}{2} = \frac{a+b}{2} - a \leq a' - a$ and we have $\frac{b'-a'}{2} = b' - \frac{b'+a'}{2} \leq b' - b$. In the latter case, we have $b - b' \geq \frac{b-a}{2}$ and $\frac{b'-a'}{2} \leq a - a'$. In both cases, we have

$$\max(|a-a'|, |b-b'|) \geq \max\left(\frac{b-a}{2}, \frac{b'-a'}{2}\right).$$

Therefore, we need to show that $d(\chi_I, \chi_{I'}) = \max(\frac{b-a}{2}, \frac{b'-a'}{2})$. By Lemma 4.3.1, we have $d(\chi_I, \chi_{I'}) \leq \max(\frac{b-a}{2}, \frac{b'-a'}{2})$. By Lemma 4.3.2, as both $m \notin I'$ and $m' \notin I$, we have $d(\chi_I, \chi_{I'}) \geq \max(\frac{b-a}{2}, \frac{b'-a'}{2})$. Therefore, we have the equality.

-Suppose now that $m \notin I'$ and $m' \in I$ (the case $m \in I'$ and $m' \notin I$ is completely symmetric). It means that $a \leq \frac{a'+b'}{2} \leq b$ and $\frac{a+b}{2} \leq a'$ or $\frac{a+b}{2} \geq b'$. Suppose first that $\frac{a+b}{2} \leq a'$. As above, we have that $\frac{b-a}{2} \leq a' - a$. We furthermore have $b - a = a + b - 2a \geq 2b' - 2a \geq 2b' - a' - b' = b' - a'$. It means that

$$\min\left(\max(|a-a'|, |b-b'|), \max(\frac{b-a}{2}, \frac{b'-a'}{2})\right) = \frac{b-a}{2}.$$

In the meantime, we have by Lemma 4.3.1 $d(\chi_I, \chi_{I'}) \leq \max(\frac{b-a}{2}, \frac{b'-a'}{2}) = \frac{b-a}{2}$. By Lemma 4.3.2, as $m \notin I'$, $d(\chi_I, \chi_{I'}) \geq \frac{b-a}{2}$ and we have the equality. Suppose now that $\frac{a+b}{2} \geq b'$, we then have $b - b' \geq \frac{b-a}{2}$ and $b' - a' = b' + a' - 2a' \leq 2b - 2a' \leq 2b - a - b = b - a$. Therefore, we have the same equalities as above: $\min\big(\max(|a-a'|, |b-b'|), \max(\frac{b-a}{2}, \frac{b'-a'}{2})\big) = \frac{b-a}{2}$, which allows us to conclude.

- Let us now assume that $m \in I'$ and $m' \in I$. It means that we have

$$\begin{cases} 2a \leq & a' + b' & \leq 2b, \\ 2a' \leq & a + b & \leq 2b'. \end{cases}$$

We then have the following.

$$\begin{array}{rcll} 2a' - 2a \leq & a + b - 2a & = b - a & \Rightarrow & \frac{b-a}{2} & \geq a' - a, \\ 2b' - 2b \leq & 2b' - a' - b' & = b' - a' & \Rightarrow & \frac{b'-a'}{2} & \geq b' - b, \\ 2b - 2b' \leq & 2b - a - b & = b - a & \Rightarrow & \frac{b-a}{2} & \geq b - b', \\ 2a - 2a' \leq & a' + b' - 2a' & = b' - a' & \Rightarrow & \frac{b'-a'}{2} & \geq a - a'. \end{array}$$

It implies that

$$\max(|a-a'|, |b-b'|) \leq \max(\frac{b-a}{2}, \frac{b'-a'}{2}).$$

Let us take $0 \leq \epsilon < |a - a'|$ and show that there is no ϵ-interleaving between the 2 functors. Without loss of generality, suppose $a < a'$. Then, there is $x \in \mathbb{R}$ such that $a < x < x + \epsilon < a' < m$. Moreover, we have $x + 2\epsilon < m + \epsilon < m + \frac{b-a}{2} \leq b$. Therefore, we have $(\chi_I \eta_{2\epsilon})_x = \chi_I(f_x^{x+2\epsilon}) = \mathrm{id}_{\mathbf{k}}$. Suppose there is an ϵ-interleaving defined by (Φ, Ψ), it implies that $\left((\Psi T_\epsilon)\Phi\right)_x = \mathrm{id}_{\mathbf{k}}$. However, the codomain of Φ_x is $\chi_{I'}(x + \epsilon) = 0$ as $x + \epsilon \notin I'$. Therefore, $\left((\Psi T_\epsilon)\Phi\right)_x = 0$. Similarly, there is no ϵ-interleaving for $\epsilon < |b - b'|$. It means that $d(\chi_I, \chi_{I'}) \geq \max(|a-a'|, |b-b'|)$.

Let us show the converse inequality. Let $\epsilon > \max(|a-a'|, |b-b'|)$ and let us construct an ϵ-interleaving between the functors. Define

$$(\Phi)_x = \begin{cases} \mathrm{id}_{\mathbf{k}} & \text{if } x \in I, x + \epsilon \in I', \\ 0 & \text{else.} \end{cases}$$

and

$$(\Psi)_x = \begin{cases} \mathrm{id}_{\mathbf{k}} & \text{if } x \in I', x + \epsilon \in I, \\ 0 & \text{else.} \end{cases}$$

Let us show that it forms an ϵ-interleaving. Let $x \leq y \in \mathbb{R}$.

First of all, Φ is a natural transformation. We have $\chi_{I'}(f_{x+\epsilon}^{y+\epsilon})\Phi_x = \mathrm{id}_{\mathbf{k}}$ if and only if $x \in I$ and $x + \epsilon, y + \epsilon \in I'$. On the other hand, $\Phi_y \chi_I(f_x^y) = \mathrm{id}_{\mathbf{k}}$ if and only if $x, y \in I$ and $y + \epsilon \in I'$. Let us show that it is equivalent. If $x + \epsilon, y + \epsilon \in I'$, then $a' \leq x + \epsilon \leq y + \epsilon \leq b$ then $a' - \epsilon \leq x \leq b' - \epsilon < b$ as $b' - b < \epsilon$. For the converse, if $x, y \in I$ and $y + \epsilon \in I'$, then $a \leq x \leq y \leq b$, which gives $a' < a + \epsilon \leq x + \epsilon \leq y + \epsilon$ and $x + \epsilon \in I'$. Similarly, Ψ is a natural transformation.

We only need to check the equality of the interleaving. Let $x \in \mathbb{R}$.

$$\chi_I(f_x^{x+2\epsilon}) = \begin{cases} \mathrm{id}_{\mathbf{k}} & \text{if } x, x + 2\epsilon \in I, \\ 0 & \text{else.} \end{cases}$$

and

$$\Psi_{x+\epsilon}\Phi_x = \begin{cases} \mathrm{id}_{\mathbf{k}} & \text{if } x \in I, x+\epsilon \in I' \text{ and } x+2\epsilon \in I, \\ 0 & \text{else.} \end{cases}$$

It is equivalent as we have $a \leq x \Rightarrow a' < a+\epsilon \leq x+\epsilon$ and $x+2\epsilon \leq b \Rightarrow x+\epsilon \leq b-\epsilon < b'$. Therefore, $x+\epsilon \in I'$. It forms an interleaving and we have $d(\chi_I, \chi_{I'}) \leq \max(|a-a'|, |b-b'|)$ and therefore the equality. □

Let us now consider the distance between interval functors with non-bounded intervals.

Proposition 4.3.4. Let I, I' be two intervals, one of which is non-bounded.

1. If $I, I' = \mathbb{R}$, then $d(\chi_I, \chi_{I'}) = 0$.
2. If $\inf(I) = \inf(I') = -\infty$ and I, I' have a finite supremum, namely b and b' respectively, then $d(\chi_I, \chi_{I'}) = |b-b'|$.
3. If $\sup(I) = \sup(I') = +\infty$ and I, I' have a finite infimum, namely a and a' respectively, then $d(\chi_I, \chi_{I'}) = |a-a'|$.
4. In all the other cases, we have $d(\chi_I, \chi_{I'}) = \infty$.

Proof. 1. We have $I = I'$, therefore $\chi_I = \chi_{I'}$ and the distance must be null.

2. Without loss of generality, we will assume that $I \subset I'$, which implies in particular $b \leq b'$. Let us first show that $d(\chi_I, \chi_{I'}) \geq |b-b'|$. Let $\epsilon < |b-b'|$. We can find an $x \in \mathbb{R}$ such that $b < x \leq x+\epsilon < b'$. Suppose we have an ϵ-interleaving (Φ, Ψ). Then $(\chi_{I'}\eta_{2\epsilon})_{x-\epsilon} = \mathrm{id}_{\mathbf{k}}$. However, the codomain of $\Psi_{x-\epsilon} = \chi_I(x) = 0$, which implies that $\Phi_x\Psi_{x-\epsilon} = 0 \neq (\chi_{I'}\eta_{2\epsilon})_{x-\epsilon}$.

Let us now show that $d(\chi_I, \chi_{I'}) \leq |b-b'|$. Let $\epsilon > |b-b'|$. Then, we define

$$(\Phi)_x = \begin{cases} \mathrm{id}_{\mathbf{k}} & \text{if } x \in I, x+\epsilon \in I', \\ 0 & \text{else.} \end{cases}$$

and

$$(\Psi)_x = \begin{cases} \mathrm{id}_{\mathbf{k}} & \text{if } x \in I', x+\epsilon \in I, \\ 0 & \text{else.} \end{cases}$$

Remark that the condition on Ψ is equivalent to $x+\epsilon \in I$ as $b \leq b'$. Let us show that it forms an ϵ-interleaving. It is a natural transformation: let $x \leq y$. We have $\chi_{I'}(f_{x+\epsilon}^{y+\epsilon})\Phi_x = \mathrm{id}_{\mathbf{k}}$ if and only if $x \in I$ and $x+\epsilon, y+\epsilon \in I'$, which is equivalent to $x \in I$ and $y+\epsilon \in I'$. On the other hand, $\Phi_y\chi_I(f_x^y) = \mathrm{id}_{\mathbf{k}}$ if and only if $x, y \in I$ and $y+\epsilon \in I'$, which is also equivalent to $x \in I$ and $y+\epsilon \in I'$ as we have $y \leq b'-\epsilon < b$. Therefore, the 2 functions are equal. The equalities of the interleaving are also true as $\Psi_{x+\epsilon}\Phi_x = \mathrm{id}_{\mathbf{k}}$ if and only if $x+\epsilon \in I'$ and $x+2\epsilon \in I$, which is equivalent to $x+2\epsilon \in I$. On the other hand, $\chi_I(f_x^{x+2\epsilon}) = \mathrm{id}_{\mathbf{k}}$ if and only if $x+2\epsilon \in I$. The other equality comes from the fact that $\Phi_{x+\epsilon}\Psi_x = \mathrm{id}_{\mathbf{k}} \Leftrightarrow x \in I', x+\epsilon \in I, x+2\epsilon \in I'$ and $\chi_{I'}(f_x^{x+2\epsilon}) = \mathrm{id}_{\mathbf{k}} \Leftrightarrow x+2\epsilon \in I'$. Both conditions are equivalent as $x+\epsilon = x+2\epsilon-\epsilon \leq b'-\epsilon < b$.

3. The result is completely symmetric to the one in 2.

4. In all the remaining cases, we have either that the infimum of an interval is finite, whereas for the other it is infinite, or the supremum of an interval is finite, whereas for the other interval, it is infinite. We will treat the case with the supremum; the case with the infimum is completely symmetric. Let us therefore suppose that $\sup(I) = b$ and $\sup(I') = +\infty$. Let us show that for all $\epsilon \geq 0$, there is no ϵ-interleaving. Suppose there is one (Φ, Ψ). There is some $x \in \mathbb{R}$ such that $x \in I'$ and $x \notin I$. We then have $\Phi_{x+\epsilon}\Psi_x = 0$ but $\chi_{I'}(f_x^{x+2\epsilon}) = \mathrm{id}_{\mathbf{k}}$, a contradiction. □

Remark 4.3.5. In the light of the two previous propositions, we have that for all $\alpha \leq \beta \in \mathbb{R} \cup \{-\infty, +\infty\}$, when the intervals are defined, the intervals

$$\begin{matrix} (\alpha, \beta) & (\alpha, \beta] \\ [\alpha, \beta) & [\alpha, \beta] \end{matrix}$$

all have an interleaving distance of 0.

The next result will allow us to put an upper bound on the interleaving distance between two persistence modules.

Proposition 4.3.6. Let F_1, F_2, G_1 and $G_2 \colon \mathbb{R} \to \mathcal{A}$ be functors with $\mathcal{A}$ an abelian category such that F_1 and G_1 are ϵ-interleaved and F_2, G_2 are also ϵ-interleaved. Then, the functors $F_1 \oplus F_2$ and $G_1 \oplus G_2$ are ϵ-interleaved.

In particular, we have that

$$d\left(\bigoplus_{j \in J} F_j, \bigoplus_{j \in J} G_j\right) \leq \sup_{j \in J} d(F_j, G_j).$$

Proof. Suppose F_j and G_j are ϵ-interleaved for all $j \in J$. Therefore, there is $\Phi^j \colon F_j \to G_j T_\epsilon$ and $\Psi^j \colon G_j \to F_j T_\epsilon$ defining an interleaving. Then, $\Phi = \bigoplus_{j \in J} \Phi^j$ and $\Psi = \bigoplus_{j \in J} \Psi^j$ defines an ϵ-interleaving between $\bigoplus_{j \in J} F_j$ and $\bigoplus_{j \in J} G_j$ as all the required equalities are checked on the components. □

4.4 Bottleneck Distance

Recall that in section 3.3, we defined the notion of barcode and persistence diagram of a decomposable persistence module. They are formally multisets. We will now define a distance on multisets to later compare that distance on barcodes with the interleaving distance of the persistence modules defining those barcodes. We use the definition of [13].

Definition 4.4.1. Given two multisets A and B, we define the multiset A_B as the disjoint union of A with the multiset containing the empty set $\emptyset$ the cardinality of B times. A *stable bijection* or *partial matching* between two multisets A and B is a bijection $f \colon A_B \to B_A$. We write $f \colon A \rightleftharpoons B$.

Another way to put it is to say that a *partial matching* is a bijection between a subset A' of A and a subset B' of B by restricting f to points of A such that their image is not the empty set. Such a couple $(a, f(a))$ is called a matching and elements in $A \backslash A'$ and $B \backslash B'$ are said to be unmatched.

EXAMPLE 4.4.2. Taking $A = \{1,1,2,3\}$ and $B = \{a,b,b\}$, we can construct 2 multisets. We get $A_B = \{1,1,2,3,\emptyset,\emptyset,\emptyset\}$ and $B_A = \{a,b,b,\emptyset,\emptyset,\emptyset,\emptyset\}$. The map $f\colon (1,1,2,3,\emptyset,\emptyset,\emptyset) \mapsto (\emptyset,b,\emptyset,a,\emptyset,b,\emptyset)$ is a partial matching.

DEFINITION 4.4.3. Let A and A' be two barcodes of persistence modules. The *bottleneck distance* is given by

$$d_B(A,A') = \inf_{f\colon A \rightleftharpoons A'} \sup_{I \in A_{A'}} d(\chi_I, \chi_{f(I)})$$

where the distance on the right-hand side is the interleaving distance.

Recall that in the case of Morse functions, we defined the bottleneck distance between two multisets of points of $\mathbb{R}^n$ of same cardinality X, Y in Definition 1.2.8 as

$$W(X,Y) = \inf_{\varphi\colon X \xrightarrow{\sim} Y} \sup_{x \in X} ||x - \varphi(x)||_\infty.$$

We will show in the following that the definitions are equivalent if we consider the barcode or persistence diagram of a persistence module.

Let F, G be two decomposable persistence modules. Let $X = \mathrm{dgm}(F), Y = \mathrm{dgm}(G)$ be their respective persistence diagram and $A = B(F), B = B(G)$ their respective barcode (i.e. $F = \bigoplus_{I \in A} \chi_I$ and $G = \bigoplus_{I \in B} \chi_I$). As discussed in remark 4.3.5, the fact that the endpoints of the intervals are in the interval or not relevant to the computation of the interleaving distance. I will therefore suppose that all the intervals of the barcodes are open. It does not change the distance but removes cases.

Let us first show that $W(X,Y) \geq d_B(A,B)$

Let $\varphi\colon X \to Y$ be a bijection between the persistence diagram and let $(a,b) \in X$, denote by $(c,d) = \varphi(a,b) \in Y$. We will construct a stable matching f between A and B such that $\sup_{I \in A_B} d(\chi_I, \chi_{f(I)}) \leq \sup_{x \in X} ||x - \varphi(x)||_\infty$. For each pairs of points $(a,b) \in X$, $(c,d) = \varphi(a,b) \in Y$, we have 4 possibilities.

-$(a,b) \notin \Delta$ and $(c,d) \notin \Delta$. Then we have $I = (a,b) \in A$ and $J = (c,d) \in B$. Then, we define $f(I) = J$. Moreover, we have

$$||(a,b) - \varphi(a,b)||_\infty = \max(|a-c|, |b-d|) \geq d(\chi_I, \chi_J).$$

Where we take the convention that $\infty - (-\infty) = 0$ and $+\infty - (+\infty) = 0$. The inequality comes from Propositions 4.3.3 and 4.3.4.

-$(a,b) \notin \Delta$ and $(c,d) \in \Delta$. Then we can suppose $(c,d) = (x,x)$. Define $f(I) = \emptyset$. If a or b is infinite, then $||(a,b) - \varphi(a,b)||_\infty = \infty$ and we have $||(a,b) - \varphi(a,b)||_\infty \geq d(\chi_I, \chi_\emptyset)$. If both a, b are finite, then

$$||(a,b) - \varphi(a,b)||_\infty = \max(|a-x|, |b-x|) \geq \frac{b-a}{2} = d(\chi_I, \chi_\emptyset).$$

-$(a,b) \in \Delta$ and $(c,d) \notin \Delta$, then we define $f(\emptyset) = J = (c,d)$ and as above, we have the inequality.

- $(a,b) \in \Delta$ and $(c,d) \in \Delta$, then $f(\emptyset) = \emptyset$ and $||(a,b) - \varphi(a,b)||_\infty \geq 0 = d(\chi_\emptyset, \chi_\emptyset)$.

In general, f defines a stable matching between A and B such that

$$\sup_{I \in A_B} d(\chi_I, \chi_{f(I)}) \leq \sup_{x \in X} ||x - f(x)||_\infty.$$

Indeed, for each $I, f(I)$, we have the associated $((a,b), \varphi(a,b))$ with greater distance.

Let us now show that $W(X,Y) \leq d_B(A,B)$.

Let $f\colon A \rightleftharpoons B$ be a stable matching. We will construct a bijection φ between X and Y such that $\sup_{I \in A_B} d(\chi_I, \chi_{f(I)}) \geq \sup_{x \in X} ||x - \varphi(x)||_\infty$.

Let $I = (a,b) \in A$ be finite and non-empty. Suppose first that $J = f(I) \neq \emptyset$. If both intervals are finite: $J = (c,d)$, then

$$d(\chi_I, \chi_J) = \min\left(\max(|a-c|, |b-d|), \max(\frac{b-a}{2}, \frac{d-c}{2})\right).$$

If $d(\chi_I, \chi_J) = \max(|a-c|, |b-d|)$, then define $\varphi(a,b) = (c,d)$, we have $d(\chi_I, \chi_J) = ||(a,b) - \varphi(a,b)||_\infty$. If $d(\chi_I, \chi_J) = \max(\frac{b-a}{2}, \frac{d-c}{2})$, then define $\varphi(a,b) = (\frac{a+b}{2}, \frac{a+b}{2})$ and $\varphi(\frac{c+d}{2}, \frac{c+d}{2}) = (c,d)$. We then have $d(\chi_I, \chi_J) \geq ||(a,b) - \varphi(a,b)||_\infty$ and $d(\chi_I, \chi_J) \geq ||(\frac{c+d}{2}, \frac{c+d}{2}) - (c,d)||_\infty$. If J is infinite, then the distance between the 2 intervals is infinite; in this case, any bijection between X and Y will satisfy the inequality of distances.

If $f(I) = \emptyset$, then define $\varphi(a,b) = (\frac{a+b}{2}, \frac{a+b}{2})$, we have the equality of distances.

If I is empty, such that $f(I) = J$ is non-empty, then if J is infinite, we have an infinite distance and any bijection between X and Y will satisfy the inequality of distances. If $J = (c,d)$ is finite, then define $\varphi(\frac{c+d}{2}, \frac{c+d}{2}) = (c,d)$. We have $d(\chi_I, \chi_J) = \frac{c+d}{2} = ||(\frac{c+d}{2}, \frac{c+d}{2}) - (c,d)||_\infty$.

If I is infinite, then if $f(I)$ is finite or empty, the distance between χ_I and χ_J is infinite. Suppose then that J is infinite then we have $d(\chi_I, \chi_J) = \max(|a-c|, |b-d|)$ if we take the convention that $\infty - (-\infty) = 0$ and $+\infty - (+\infty) = 0$. Then we can define $\varphi(a,b) = \varphi(c,d)$ and we have the equality of distances.

We have defined φ on all the points outside of the diagonal and all the points whose image is outside of the diagonal. Define φ to be the identity on the rest of the diagonal. It produces a bijection between X and Y such that $\sup_{I \in A_B} d(\chi_I, \chi_{f(I)}) \geq \sup_{x \in X} ||x - \varphi(x)||_\infty$.

In particular, it shows that barcodes with d_B and persistence diagrams with W are isometric.

Another way to compute this distance is through ϵ-matching.

Definition 4.4.4. A partial matching $f\colon A \rightleftharpoons B$ between two multisets in the extended half plane $H = \{(x,y) \in \overline{\mathbb{R}}^2 | x < y\}$ is an ϵ-matching if

- for all $(\alpha, f(\alpha))$ with $f(\alpha) \neq \emptyset$, we have $d_\infty(\alpha, f(\alpha)) \leq \epsilon$
- if $\alpha \in X$ is such that $f(\alpha) = \emptyset$ (i.e. is unmatched), then $d_\infty(\alpha, \Delta) \leq \epsilon$
- if $\beta \in B$ is unmatched, then $d_\infty(\beta, \Delta) \leq \epsilon$.

It is direct to see that for decomposable persistence modules F, G with $\mathrm{dgm}(F)\backslash\Delta = A$ and $\mathrm{dgm}(G)\backslash\Delta = B$, we have

$$d_p(A, B) = \inf(\epsilon \leq 0: \text{ there is an } \epsilon\text{-matching}\} = d_B(B(F), B(G))$$

as $\sup_{I \in B(F)_{B(G)}} d(\chi_I, \chi_{f(I)}) \leq \epsilon$ is equivalent to the condition that that f is an ϵ-matching between A and B.

4.5 Isometry Theorem

Now that we have defined an extended (pseudo-)metric on the functors defining persistence modules and the persistence diagram, we will try to relate them. It turns out that they help define an isometry between 2 subsets. In this section, we only consider decomposable persistent, but it turns out that the result can be extended to other persistence modules (see, for example, [18])

Proposition 4.5.1. Let $F, G \colon \mathbb{R} \to \mathbf{Vect}$ be two decomposable persistence modules. Then

$$d(F, G) \leq d_B(\mathrm{dgm}(F), \mathrm{dgm}(G))$$

Proof. We will show the result for the barcode, as the persistence diagram and barcode are isometric, the result will follow. Let C be the barcode of F and D the barcode of G. Suppose $d_B(C, D) < \epsilon$, then there is a stable matching $f \colon C \rightleftharpoons D$ such that $d(\chi_I, \chi_{f(I)}) < \epsilon$ for all $I \in C_D$. As F and G are decomposable and as f is a stable matching, we can write

$$F = \bigoplus_{I \in C_D} \chi_I$$

and

$$G = \bigoplus_{I \in C_D} \chi_{f(I)}.$$

The only difference compared to before is that we added some characteristic functors of the empty sets. However, note that it does not change the functor to add $\chi_\emptyset$ in the decomposition. Using Proposition 4.3.6, as for each I, we have $d(\chi_I, \chi_{f(I)}) < \epsilon$, it gives $d(F, G) < \epsilon$. As $d_B(C, D) < \epsilon$ is arbitrary, we have the result. ☐

4.5.1 Stability Theorem

Let us now look at the converse inequality. Unfortunately, the proof is not as straightforward as for the previous inequality. This bound was first proven in [21], then in [18] some restrictions were removed. These two proofs used a geometric flavour; one main ingredient was the interpolation lemma (Theorem 4.2.2). The other main ingredient used by Chazal and al. was the notion of rectangle measure to generalise the notion of persistence diagram, although very interesting and offering a useful extension of the persistence diagram, I have decided to use a more straightforward proof due to [7]. The result presented will be for persistence modules that are pointwise finite-dimensional, i.e. functors from $\mathbb{R}$ to **vect**, the category of finite-dimensional vector spaces. Using corollary 3.2.30, all these modules are decomposable. Therefore, our definition of persistence diagram and barcode makes sense. On the other hand, not all decomposable modules are pointwise finite-dimensional, as shown in Theorem 3.2.12.

The main tool that we will use to prove our result is Hall's[1] marriage theorem, which we will state without proof.

THEOREM 4.5.2 (HALL'S MARRIAGE THEOREM). If S is a family of finite (not necessarily disjoint) sets, then the two conditions are equivalent.

1. For all $S' \subset S$, $|S'| \leq |\bigcup_{s \in S'} s|$.
2. There is a set T and a *bijection* $\sigma\colon S \to T$ such that $\sigma(s) \in s$ for all $s \in S$.

The name of this theorem comes from the interpretation that we can imagine a (possibly infinite) set of boys, each acquainted with a finite number of girls (the girls a boy fancies represent a set of S). It is possible to marry each boy to a girl (condition 2) if and only if every set of n boys is collectively acquainted with at least n girls (condition 1).

We will use several lemmas to show that condition 1 is fulfilled by some subset of the power set of $B(G)$.

In this section, we will say that two intervals are *of the same type* if $I \backslash J$ and $J \backslash I$ are both bounded. In particular, there are 4 types of intervals: finite intervals, intervals of the type $(-\infty, b\rangle$, intervals of the type $\langle a, +\infty)$ and $(-\infty, +\infty)$ with $a, b \in \mathbb{R}$. Here, the notation $\langle \cdot, \cdot \rangle$ allows us to write the endpoints of the interval without considering whether it is closed or open.

In this section, we will also suppose that F and G are pointwise finite-dimensional and ϵ-interleaved. As previously, we will denote by $\Phi\colon F \to GT_\epsilon$ and $\Psi\colon G \to FT_\epsilon$ morphisms defining an ϵ-interleaving. We will denote by $B(F)$ and $B(G)$ the barcode of F and G respectively.

Taking $I \in B(F)$ and $J \in B(G)$, we also denote by $\Phi_{I,J} = (\pi_J T_\epsilon)\Phi i_I\colon \chi_I \to \chi_J T_\epsilon$ where i_I and π_J are the canonical injection and projection. Similarly, we denote by $\Psi_{J,I} = (\pi_J T_\epsilon)\Psi i_I$. In other words, $\Phi_{I,J}$ is the following composition of functions:

$$\chi_I \xrightarrow{i_I} \bigoplus_{I \in B(F)} \chi_I \simeq F \xrightarrow{\Phi} \bigoplus_{J \in B(G)} \chi_J \simeq G \xrightarrow{\pi_J T_\epsilon} \chi_J T_\epsilon.$$

Moreover, for $I \in B(F)$, we have

$$\begin{aligned}(\pi_{I'} T_{2\epsilon})(F\eta_{2\epsilon}) i_I &= (\pi_{I'} T_{2\epsilon})(\Psi T_\epsilon)(\Phi i_I) \\ &= (\pi_{I'} T_{2\epsilon}) \left((\sum_{J \in B(G)} \Psi i_J \pi_J) T_\epsilon \right) (\Phi i_I) \\ &= \sum_{J \in B(G)} (\pi_{I'} T_{2\epsilon})((\Psi T_\epsilon)(i_J T_\epsilon)(\pi_J T_\epsilon))(\Phi i_I)\end{aligned}$$

---

[1] Philip Hall (1904–1982) was an English mathematician. He worked on group theory, especially on finite groups and solvable groups. Not to be confused with Marshall Hall Jr. (1910-1990), who was an American mathematician who made contributions to group theory and combinatorics.

$$\begin{aligned}
&= \sum_{J\in B(G)} \Big((\pi_{I'}T_{2\epsilon}(\Psi T_\epsilon)(i_J T_\epsilon)\Big)\,((\pi_J T_\epsilon)(\Phi i_I))\\
&= \sum_{J\in B(G)} (\Psi_{J,I'}T_\epsilon)\Phi_{I,J}. \qquad (4.1)
\end{aligned}$$

Furthermore, as $F\eta_{2\epsilon}$ is zero between different components of F, for $I \neq I' \in B(F)$, we have

$$0 = (\pi_{I'}T_{2\epsilon})F\eta_{2\epsilon}e_I = \sum_{J\in B(G)} (\Psi_{J,I'}T_\epsilon)\Phi_{I,J}. \tag{4.2}$$

This first lemma gives some conditions on having non-zero morphisms between interval modules.

Lemma 4.5.3. Let $\alpha\colon \chi_I \to \chi_J$ be a morphism between interval modules. Then for all $a, b \in I \cap J$, we have $\alpha_a = \alpha_b$ seen as $\mathbf{k}$-endomorphisms. Moreover, if α is non-zero, we have

$$\inf(J) \leq \inf(I) \text{ and } \sup(J) \leq \sup(I).$$

Proof. If α is the zero map, then the result is obvious. If α is non-zero, let $s \in I$ be a real number such that $\alpha_s\colon \chi_I(s) \to \chi_J(s)$ is a non-zero map. It implies that $\chi_J(s) \neq 0$ and therefore $s \in I \cap J$.

Let us take $r \in I$ such that $r \leq s$. It gives the following commutative diagram

$$\begin{tikzcd}
\chi_I(r) = \mathbf{k} \arrow[r, "\chi_I(f_r^s)=\mathrm{id}_{\mathbf{k}}"] \arrow[d, "\alpha_r"'] & \chi_I(s) = \mathbf{k} \arrow[d, "\alpha_s"] \\
\chi_J(r) \arrow[r, "\chi_J(f_r^s)"'] & \chi_J(s) = \mathbf{k}.
\end{tikzcd}$$

If $r \notin J$, we have $\alpha_s \,\mathrm{id}_{\mathbf{k}} = \chi_J(f_r^s)\alpha_r = 0$, which is not possible. Therefore, $r \in J$ and then $\chi_J(f_r^s) = \mathrm{id}_{\mathbf{k}}$, which then implies $\alpha_r = \alpha_s$.

Similarly, if we take $t \in J$ such that $t \geq s$, we have the following commutative diagram

$$\begin{tikzcd}
\mathbf{k} \arrow[r, "\chi_I(f_s^t)"] \arrow[d, "\alpha_s"'] & \chi_I(t) \arrow[d, "\alpha_t"] \\
\mathbf{k} \arrow[r, "\mathrm{id}_{\mathbf{k}}"'] & \mathbf{k}.
\end{tikzcd}$$

It also implies that $t \in I$ and $\alpha_s = \alpha_t$. Therefore, we have the first result. Furthermore, the condition $r \in I$ with $r \leq s$ implies $r \in J$ gives that $\inf(I) \geq \inf(J)$. Meanwhile, the condition $t \in J, t \geq s$ implies $t \in I$ gives that $\sup(I) \geq \sup(J)$. Or in other words, $\alpha_p \neq 0$ for all $p \in (\inf(I), \sup(J))$, which implies $(\inf(I), \sup(J)) \subset I \cap J$. □

It is well known that the $\mathbf{k}$-endomorphisms are just multiplication by a constant, which is equal to the image of the unit. We can then define the following function that we will use in a later lemma to give a lower bound to the number of elements of some sets:

$$w\colon (B(F)\times B(G))\sqcup (B(G)\times B(F))\to \mathbf{k},$$
$$(I,J)\mapsto \begin{cases} (\Phi_{I,J})_t(1) & \text{if } (I,J)\in B(F)\times B(G) \text{ and } t\in I\cap J\neq\emptyset,\\ (\Psi_{I,J})_t(1) & \text{if } (I,J)\in B(G)\times B(F) \text{ and } t\in I\cap J\neq\emptyset,\\ 0 & \text{else.}\end{cases}$$

We will now show that we can assume that all the intervals of $B(F)$ and $B(G)$ are of same type. It will help to reduce the number of cases. To do that, let us first prove a small lemma.

Lemma 4.5.4. Let I and K be intervals of the same type. Let J be an interval of different type. Then, $\beta\alpha = 0$ for all $\alpha\colon \chi_I\to\chi_J$ and $\beta\colon \chi_J\to\chi_K$.

Proof. By contradiction, suppose $\beta\alpha\neq 0$. It implies by the previous lemma that $\inf(K)\leq\inf(J)\leq\inf(I)$ and $\sup(K)\leq\sup(J)\leq\sup(I)$. This gives that J is of the same type as K and I, a contradiction. □

Let us denote by φ the function $F\to GT_\epsilon$ such that for all $I\in B(F)$ and $J\in B(G)$, we have

$$\varphi_{I,J} = \begin{cases}\Phi_{I,J} & \text{if } I,J \text{ are of same type,}\\ 0 & \text{else.}\end{cases}$$

Similarly, we denote $\psi\colon G\to FT_\epsilon$ such that

$$\psi_{J,I} = \begin{cases}\Psi_{J,I} & \text{if } I,J \text{ are of same type,}\\ 0 & \text{else.}\end{cases}$$

For all $I,I'\in B(F)$ and $J\in B(G)$, we have the following equality:

$$\sum_{J\in B(G)} (\Psi_{J,I'}T_\epsilon)\Phi_{I,J} = \sum_{J\in B(G)} (\psi_{J,I'}T_\epsilon)\varphi_{I,J}$$

Indeed, if I and I' are of different types, then the right-hand side is null by definition of φ and ψ as J will be of a different type from I or I'. The left hand side is also zero as Φ and Ψ are ϵ-interleaving, therefore, using Equations 4.1 and 4.2, we get $\sum_{J\in B(G)}(\Psi_{J,I'}T_\epsilon)\Phi_{I,J} = (\pi_{I'}T_{2\epsilon})(F\eta_{2\epsilon})i_I = 0$. If I and I' are of the same type, we have the result by the previous lemma.

Therefore, we have that $\psi(T_\epsilon)\varphi = \Psi(T_\epsilon)\Phi$ and, repeating the argument, we also get $\varphi(T_\epsilon)\psi = \Phi(T_\epsilon)\Psi$. It means that φ and ψ are ϵ-interleaving morphisms between the components of F and G of the same type. We have thus reduced our problem to the case where the intervals of $B(F)$ and $B(G)$ are of the same type.

Let us now define a preorder on the intervals. To do that, we first define the function α taking intervals and outputting real numbers defined as

$$\begin{aligned}\alpha(\langle a,b\rangle) &\mapsto a+b;\\ \alpha((-\infty,b\rangle) &\mapsto b;\\ \alpha(\langle a,+\infty) &\mapsto a;\\ \alpha((-\infty,+\infty)) &\mapsto 0.\end{aligned}$$

Thanks to this function, we can define the relation $I \leq_\alpha J$ if

$$\alpha(I) < \alpha(J)$$

or if all these conditions are fulfilled:

$$\begin{cases} \alpha(I) = \alpha(J), \\ I \text{ is closed to the left or } J \text{ is open to the left}, \\ I \text{ is open to the right or } J \text{ is closed to the right}. \end{cases}$$

Example 4.5.5. We have

$$(-\infty, +\infty) \leq_\alpha [1,2) \leq_\alpha (1,2) \leq_\alpha (1,2] \leq_\alpha (-\infty, 3] \leq_\alpha (-\infty, 3).$$

Note that it is not antisymmetric as $[1,2) \leq_\alpha [0,3)$ and $[0,3) \leq_\alpha [1,2)$. It is also not complete as $(1,2) \not\leq_\alpha [1,2]$ and $[1,2] \not\leq_\alpha (1,2)$.

However, it is transitive and reflexive. Therefore, it defines a preorder on the intervals. We will denote by $I <_\alpha J$ the condition $I \leq_\alpha J$ and $J \not\leq_\alpha I$.

We will now relate the closeness of intervals with the preorder defined above.

Lemma 4.5.6. Let I, J, K be intervals of the same type with $I \not>_\alpha K$ (i.e. $I \leq_\alpha K$ or $(I \not\leq_\alpha K$ and $I \not\geq_\alpha K)$). Suppose furthermore that there are non-zero functions $\beta\colon \chi_I \to \chi_J T_\epsilon$ and $\gamma\colon \chi_J \to \chi_K T_\epsilon$. Then χ_J is ϵ-interleaved with χ_I or with χ_J.

Proof. As the map β is not zero, using Lemma 4.5.3, we have

$$\inf(J) - \epsilon = \inf(\operatorname{supp}(\chi_J T_\epsilon)) \leq \inf(I)$$

where $\operatorname{supp}(F)$ is the set of $t \in \mathbb{R}$ such that $F(t) \neq 0$. Similarly, as β and γ are non-zero maps, we get

$$\inf(K) \leq \inf(J) + \epsilon, \quad \sup(J) \leq \sup(I) + \epsilon, \quad \sup(K) \leq \sup(J) + \epsilon.$$

Suppose χ_I and χ_J are not ϵ-interleaved. It means that $d(I,J) \geq \epsilon$. If I, J are finite, using Proposition 4.3.3, we have $\max(|\inf(I) - \inf(J)|, |\sup(I) - \sup(J)|) \geq \epsilon$. If I and J are non-bounded on the left, then $|\sup(I) - \sup(J)| \geq \epsilon$ and if I and J are non-bounded on the right, then $|\inf(I) - \inf(J)| \geq \epsilon$, using Proposition 4.3.4. Using the conditions $\inf(J) \leq \inf(I) + \epsilon$ and $\sup(J) \leq \sup(I) + \epsilon$, we get that

$$\inf(I) \geq \inf(J) + \epsilon \text{ and } \inf(J) \neq -\infty$$

or

$$\sup(I) \geq \sup(J) + \epsilon \text{ and } \sup(J) \neq +\infty.$$

Let us suppose the first condition. If I, J, K are with infinite right endpoint, then $\alpha(J) = \inf(J) \leq \inf(I) - \epsilon < \inf(I) = \alpha(I)$. Therefore, $J <_\alpha I$.

Suppose I, J, K are finite intervals. we have

$$\alpha(J) = \inf(J) + \sup(J) \leq (\inf(I) - \epsilon) + (\sup(I) + \epsilon) = \alpha(I)$$

where he have the equality if and only if $J = \langle \inf(I) - \epsilon, \sup(I) + \epsilon \rangle$. In that case, since there are no non-zero functions $f\colon \chi_J \to \chi_I T_\epsilon$, it implies that J is closed to the left and I is open to the left. On the other hand, as we have $\beta\colon \chi_I \to \chi_J T_\epsilon$ non-zero, we can't have I open to the right and J closed to the right. All in all, it shows that $I >_\alpha J$.

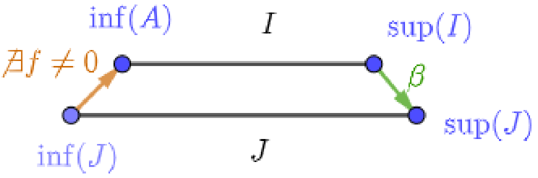

Figure 4.3: Maps between the endpoints of I and J.

Similarly, if χ_J and χ_K are not ϵ-interleaved, then $J >_\alpha K$. Therefore, if χ_J is not ϵ-interleaved to both χ_I and χ_J, it would mean that $I >_\alpha K$, a contradiction. ☐

Definition 4.5.7. A persistence module F is *ϵ-significant* it there is a real number $t \in \mathbb{R}$ such that $F(f_t^{t+\epsilon}) \neq 0$. Otherwise, it is *$\epsilon$-trivial*.

Example 4.5.8. An interval module χ_I is ϵ-significant if and only if the length of I is strictly bigger than ϵ or if it is exactly equal to ϵ and I is closed.

Lemma 4.5.9. Let I, J, K be intervals of the same type with I and K 2ϵ-significant and $\alpha(I) \leq \alpha(K)$. Suppose further that there are non-zero morphisms $\beta\colon \chi_I \to \chi_J T_\epsilon$ and $\gamma\colon \chi_J \to \chi_K T_\epsilon$. Then $(\gamma T_\epsilon)\beta \neq 0$.

Proof. Using Lemma 4.5.3, we have $\inf(K) - 2\epsilon \leq \inf(I) - \epsilon \leq \inf(I)$ and $\sup(K) - 2\epsilon \leq \sup(J) \leq \sup(I)$. It therefore gives $(\inf(I), \sup(K) - 2\epsilon) \subset I \cap (J - \epsilon) \cap (K - 2\epsilon)$. Suppose $(\gamma T_\epsilon)\beta = 0$. Then, $(J - \epsilon) \cap (K - 2\epsilon) = \emptyset$, which implies $\inf(I) \geq \sup(K) - 2\epsilon$ and I, J, K finite intervals. We then get

$$\alpha(I) = \inf(I) + \sup(I) \geq 2\sup(I) + 2\epsilon \geq 2\sup(K) - 2\epsilon \geq \inf(K) + \sup(K) = \alpha(K)$$

using the fact that I and K are 2ϵ-significant. However, as $\alpha(I) \leq \alpha(K)$, it implies that $\alpha(I) = \alpha(K)$. In the chain of inequalities above, we have the equality if $\inf(I) - 2\epsilon = \sup(I)$, $\inf(K) - 2\epsilon = \sup(K)$ and $\inf(I) = \sup(K) - 2\epsilon = \inf(K)$. Therefore, we have $I = \langle \inf(I), \inf(I) + 2\epsilon \rangle$ and $K = \langle \inf(I), \inf(I) + 2\epsilon \rangle$ (note that at this point, they might not have the same left and right openness. However, as K and I are both 2ϵ-significant, they must be closed. Therefore, we get that $I = K = [\inf(I), \inf(I) + 2\epsilon]$. But then $((\gamma T_\epsilon)\beta)_{\inf(I)} \neq 0$, a contradiction. ☐

We define the function

$$\begin{aligned}\mu\colon B(F) &\to \mathcal{P}(B(G))\\ I &\mapsto \{J \in B(G) : \chi_I, \chi_J \text{ are } \epsilon\text{-interleaved}\}.\end{aligned}$$

If χ_I is 2ϵ-significant, let $t \in \mathbb{R}$ be such that $t, t+2\epsilon \in I$. It implies that $t+\epsilon \in J$ for all $J \in \mu(I)$. Furthermore, as G is a point-wise finite-dimensional persistence module, we have that $\dim(G(t+\epsilon)) < \infty$, which implies that $|\mu(I)| < \infty$ for all 2ϵ-significant intervals modules I. For $A \subset B(F)$, we also write $\mu(A) = \bigcup_{I\in A} \mu(I)$.

Lemma 4.5.10. Let A be a countable subset of $B(F)$ containing no 2ϵ-trivial elements then

$$|A| \le |\mu(A)|.$$

Proof. Assume first that A is finitpie. As $\le_\alpha$ is a preorder, we can order[2] $A = \{I_1, I_2, \cdots, I_r\}$ such that $I_i \not>_\alpha I_j$ for all $i < j$. It suffices to take a minimal element I_1 (which exists as the set is finite) and to determine the ith element, take a minimal element of $A \backslash \{I_1, \cdots I_{i-1}\}$. In particular, note that $\alpha(I_i) \le \alpha(I_j)$ for all $i \le j$. We denote the elements of $\mu(A) = \{J_1, J_2, \cdots, J_s\}$.

By Lemma 4.5.6, if $(\Psi_{J,I'}T_\epsilon)\Phi_{I,J} \neq 0$ and $I \not>_\alpha I'$, then J is ϵ-interleaved to I or I'. Therefore, for $i < i'$, we have

$$0 = \sum_{J\in B(G)} (\Psi_{J,I'}T_\epsilon)\Phi_{I,J} = \sum_{J\in\mu(A)} (\Psi_{J,I'}T_\epsilon)\Phi_{I,J} \tag{4.3}$$

where the first equality comes from Equation 4.2 and the second uses the fact that if J is not ϵ-interleaved with both I and J, we have $(\Psi_{J,I'}T_\epsilon)\Phi_{I,J} = 0$.

Equation 4.1 and Lemma 4.5.6 also gives

$$\chi_I\eta_{2\epsilon} = (\pi_I T_{2\epsilon})F\eta_{2\epsilon} = \sum_{J\in B(G)} (\Psi_{J,I}T_\epsilon)\Phi_{I,J} = \sum_{J\in\mu(A)} (\Psi_{J,I}T_\epsilon)\Phi_{I,J} \tag{4.4}$$

Therefore, we get the following matrix equality.

$$\begin{pmatrix}\Psi_{J_1,I_1}T_\epsilon & \cdots & \Psi_{J_s,I_1}T_\epsilon\\ \vdots & & \vdots\\ \Psi_{J_1,I_r}T_\epsilon & \cdots & \Psi_{J_s,I_r}T_\epsilon\end{pmatrix}\begin{pmatrix}\Phi_{I_1,J_1} & \cdots & \Phi_{I_r,J_1}\\ \vdots & & \vdots\\ \Phi_{I_1,J_s} & \cdots & \Phi_{I_r,J_s}\end{pmatrix} = \begin{pmatrix}\chi_{I_1}\eta_{2\epsilon} & & *\\ & \ddots & \\ 0 & & \chi_{I_r}\eta_{2\epsilon}\end{pmatrix}$$

The equality with the diagonal elements comes from Equation 4.4, and the one for elements below the diagonal is given by Equation 4.3. Using Lemma 4.5.3, we know that a morphism between interval modules can be identified with a **k**-endomorphism using the map w. In general, we will denote by $w(\beta) = c$ for a morphism $\beta\colon \chi_I \to \chi_J$ if there is $t \in \mathbb{R}$ such that β_t correspond to the multiplication by c and 0 otherwise. By Lemma 4.5.9, we have that the composition $(\Psi_{J,I'}T_\epsilon)\Phi_{I,J}$ is non zero if $\alpha(I) \le \alpha(I')$, therefore, for all $i \le j$,we have

$$w((\Psi_{J,I_j}T_\epsilon)\Phi_{I_i,J}) = w((\Psi_{J,I_j}T_\epsilon))w(\Phi_{I_i,J}) = w(J, I_i)w(I_j, J)$$

---

[2] Note that opposite to what is said in [7], as the order is not total (as mentioned in Example 4.5.5), we cannot order A such that $I_i \le_\alpha I_j$ if $i \le j$.

Therefore, it gives

$$\begin{aligned}
1 &= w(\chi_{I_i}\eta_{2\epsilon}) \\
&= w(\sum_{J\in\mu(A)} (\Psi_{J,I_i}T_\epsilon)\Phi_{I_i,J}) \\
&= \sum_{J\in\mu(A)} w((\Psi_{J,I_i}T_\epsilon)\Phi_{I_i,J}) \\
&= \sum_{J\in\mu(A)} w(\Psi_{J,I_i}T_\epsilon)w(\Phi_{I_i,J}) \\
&= \sum_{J\in\mu(A)} w(J,I_i)w(I_i,J)
\end{aligned}$$

and for $i < j$,

$$0 = \sum_{J\in\mu(A)} w(J,I_j)w(I_i,J).$$

From a matrix perspective, it gives

$$\begin{pmatrix} w(J_1,I_1) & \cdots & w(J_s,I_1) \\ \vdots & & \vdots \\ w(J_1,I_r)T_\epsilon & \cdots & w(J_s,I_r) \end{pmatrix} \begin{pmatrix} w(I_1,J_1) & \cdots & w(I_r,J_1) \\ \vdots & & \vdots \\ w(I_1,J_s) & \cdots & w(I_r,J_s) \end{pmatrix} = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} := M.$$

It gives

$$|A| = r = \operatorname{rank}(M) \leq s = |\mu(A)|.$$

Let us now assume that A is countable infinite. Let us denote by $A = \{I_1, I_2, \cdots\}$ the elements of A and $A_n = \{I_1, \cdots, I_n\}$. We have $\mu(A_n) \subset \mu(A)$ and by the first part, $n = |A_n| \leq |\mu(A_n)| \leq |\mu(A)|$. Therefore, $|\mu(A)|$ is infinite. □

Now that we have this lemma, we can prove the result we've been working on following the proof of [7].

Theorem 4.5.11. Let F, G be ϵ-interleaved point-wise finite-dimensional persistence modules, then there is an ϵ-matching between $B(F)$ and $B(G)$

In particular, we have

$$d(F,G) \geq d_B(\operatorname{dgm}(F), \operatorname{dgm}(G))$$

Proof. Let $S = \{\mu(I) | I \in B(F) : I \text{ is } 2\epsilon\text{-significant}\}$. The previous lemma tells us that the first condition of Hall's marriage theorem (Theorem 4.5.2) is fulfilled. Therefore, there is a matching $\gamma\colon B(F) \rightleftharpoons B(G)$ that matches each 2ϵ-significant $I \in B(F)$ with an element of $\mu(I)$. By symmetry, there is also a matching $\tau\colon B(G) \rightleftharpoons B(F)$ that matches each 2ϵ-significant $J \in B(G)$ with an element of $\mu(J)$. However, note that γ and τ might not be ϵ-matching. Let us construct σ that is.

1. If $I \in B(F)$ is 2ϵ-significant, of the form $(\tau\gamma)^i(I')$ for some $i \geq 0$, $I' \in B(F)$, 2ϵ-significant such that $I' \notin \operatorname{im}(\tau)$. Then we define $\sigma(I) = \gamma(I)$. In particular, if I is 2ϵ-significant and if it is not in the image of τ, then $I = (\tau\gamma)^0(I)$ and $\sigma(I) = \gamma(I)$.

2. Otherwise, if $I \in \operatorname{im}(\tau)$ (note that I might not be 2ϵ-significant). We define $\sigma(I) = \tau^{-1}(I)$.

Let us first show that σ is well defined: we need to show that I can't be written in the above form in more than one way: let $i, j \geq 0$ and I, J such that $I, J \notin \operatorname{im} \tau$ and $(\tau\gamma)^i(I) = (\tau\gamma)^j(J)$. Without loss of generality, suppose $i \leq j$. As γ and τ are injective when defined, they are left invertible, we therefore have $I = (\tau\gamma)^{j-i}(J)$. If $j > i$, it means that $I \in \operatorname{im}(\tau)$, which is impossible. If $j = i$, then $I = J$. We therefore have the uniqueness and σ is well defined on a subset of $B(F)$ that includes the 2ϵ-significant intervals.

Let us now show that σ is injective. Suppose $\sigma(I) = \sigma(J)$ for $I \neq J$. As τ and γ are injective, we can't have $\sigma(I) = \tau^{-1}(I)$ and $\sigma(J) = \tau^{-1}(J)$ simultaneously. Similarly, we can't have $\sigma(I) = \gamma(I)$ and $\sigma(J) = \gamma(J)$ at the same time. Therefore, we either have $\gamma(I) = \sigma(I) = \sigma(J) = \tau^{-1}(J)$ or $\gamma(J) = \tau^{-1}(I)$. Suppose the former. It means that $I = (\tau\gamma)^i(I')$ for I' a 2ϵ-significant interval not in the image of τ. As $\gamma(I) = \tau^{-1}(J)$, we have $J = (\tau\gamma)(I) = (\tau\gamma)^{i+1}(I')$ which then implies that $\sigma(J) = \gamma(J) \neq \gamma(I) = \sigma(I) = \sigma(J)$, where the inequality comes from the injectiveness of γ.

It only remains to show that σ is an ϵ-matching. Let us show that all the 2ϵ-significant intervals of $B(G)$ are in the image of σ. If J is 2ϵ-significant, it is in the domain of τ. Therefore, we either have $\tau(J)$ is of the form $(\tau\gamma)^i(I)$ for $i \geq 0$ and $I \in B(F)$ a 2ϵ-significant interval not in the image of τ. We then have

$$J = \tau^{-1}\tau(J) = \tau^{-1}(\tau\gamma)^i(I) = \gamma(\tau\gamma)^{i-1}(I) = \sigma((\tau\gamma)^{i-1}(I))$$

or we directly have $J = \sigma(\tau(J))$. □

4.5.2 Isometries

Using the 2 inequalities proved earlier, we can prove the isometry between the barcodes and the persistence modules.

Theorem 4.5.12. Let $\mathcal{B}$ denote the set of element-wise finite barcodes (each interval has a finite multiplicity). Then the mapping

$$\lambda\colon (\mathcal{B}, d_B) \xrightarrow{\sim} (\mathbf{vect}^{\mathbb{R}}, d), \{I_k\}_{k\in K} \mapsto \bigoplus_{k\in K} \chi_{I_k}$$

is an isometry (i.e. it is bijective and preserves the pseudo-metric).

Proof. The fact that it is an injection comes from the uniqueness of the decomposition (Theorem 3.1.3). The function λ is also surjective by Theorem 3.2.30. The fact that it is an isometry follows directly from Theorem 4.5.1 and Theorem 4.5.11. □

This isometry allows us to study in more detail the homology and topology of certain manifolds. In particular, it justifies the interests of persistent homology in the study of the topology of manifolds. Let X be a topological space. Assume $f, g\colon X \to \mathbb{R}$ are **not necessarily continuous** maps. We define the functor $F\colon \mathbb{R} \to \mathbf{Top}$ by $F(a) = f^{-1}(-\infty, a]$ and the morphism $F(f_a^b)$ being the inclusion

$f^{-1}(-\infty, a] \hookrightarrow f^{-1}(-\infty, b]$. Similarly, we define the functor G using the function g. Let us also take $H\colon \mathbf{Top} \to \mathcal{C}$ to be any functor.

We then have the following, which is of prime importance in topological data analysis.

Theorem 4.5.13 (Stability Theorem). Using the above notations, we have

$$d(HF, HG) \leq ||f - g||_\infty.$$

Proof. Let $\epsilon = ||f - g||_\infty$. Let us show that F and G are ϵ-interleaved. Let $a \in \mathbb{R}$. We have

$$F(a) = f^{-1}(-\infty, a] \subset g^{-1}(-\infty, a + \epsilon] = G(a + \epsilon)$$

where the inclusion comes from the fact that $\epsilon = ||f - g||_\infty$. Similarly, $G(a) \subset F(a + 2\epsilon)$. We can therefore define an ϵ-interleaving between F and G by taking Φ_a as the inclusion of $F(a)$ into $G(a + \epsilon)$ and Ψ_a as the inclusion of $G(a)$ into $F(a + \epsilon)$. Those inclusion are natural and we have $\Psi_{a+\epsilon}\Phi_a$ is the inclusion of $F(a)$ into $F(a+2\epsilon)$, which is exactly $F(f_a^{a+2\epsilon})$ (and similarly for $\Phi_{a+\epsilon}\Psi_a$). By Proposition 4.1.7, we know that HF and HG are also ϵ-interleaved. Therefore, we have the inequality. □

Taking H to be a functor from $\mathbb{R}$ to **vect**, we can further compare the barcodes and persistence diagrams with the infinite norm between functions.

In particular, using the previous theorem with H being the singular homology with coefficients in $\mathbf{k}$ and the isometry theorem justifies the use of the barcode and persistence diagram to study the topology of some manifold under some small perturbations (see the end of Section 1.2.3).

5 | Conclusion

Persistent homology is, in my opinion, a fascinating subject as it provides an extremely diverse field of research. One can see them as an abstract object and study them from an algebraic perspective, or even category-theoretic perspective. One can also see them as a tool and study their applications in other fields, or focus on the computer-theoretic point of view and focus on the implementation of robust and efficient algorithms for computing this homology.

The results presented in this master thesis provide an overview of and justification for the interest of persistent homology in data analysis. In particular, we showed that from every filtration of a complex or topological space over one parameter, we can construct the persistent homology of that filtration. We then showed that given a persistent homology, we can associate a persistence module that carries the same information.

We then studied such persistence modules from an algebraic point of view. We showed that if those modules were indexed by a finite set or if they were finite-dimensional at each index, then they admitted a decomposition into interval modules, and this decomposition was essentially unique. Using the uniqueness of the decomposition, we defined the persistence diagram and barcode as a way to synthesise the information carried by the persistence diagram.

Afterwards, we defined two (extended) pseudo-metrics on the set of persistence modules and persistence diagrams and showed that these "metric" defined an isometry between the set of point-wise finite-dimensional persistence modules and point-wise finite-dimensional persistence diagrams. This isometry allowed us to show the stability theorem.

5.1 Further Work

Of course, much more can be said on persistent homology, persistence modules and topological data analysis. In this section, I will give some further research that has been carried out in this field. Several directions of research other than the following ones exist, and research is still carried out.

Change of Indexing Set

One of the first generalisations mathematicians started working on is the case of multiparameter persistence modules and, in general, persistence modules indexed by other sets than $\mathbb{R}$ or completely ordered sets. This led to the notion of *zigzag* persistence modules as a first, easier generalisation.

Zigzag persistence modules can be defined as being quiver representations of a quiver of type A_n, where, in contrast to the persistence modules, not all the arrows of the quiver are necessarily in the same direction. This configuration allows us to study several subsets $X_1, \cdots X_n$ that do not necessarily have an order relation between them. We can for example construct the union sequence

$$X_1 \to X_1 \cup X_2 \leftarrow X_2 \to X_2 \cup X_3 \leftarrow \cdots \leftarrow X_n.$$

Using Gabriel's theorem (Theorem 2.4.22) and Theorem 3.2.12, we have that a zigzag module admits a decomposition if the indexing set is finite. It is also shown in [8] that zigzag modules also respect some kind of algebraic stability condition. Other results and introduction concerning zigzag modules can be found in [14, 44].

This also raises the question of considering other posets as indexing sets P. Gabriel's theorem directly tells us that if P has a Hasse diagram that is not of simply laced Dynkin type, then it has an infinite number of different indecomposable representations and the persistence modules indexed by P cannot be described and classified by discrete invariants (like the persistence diagram). In particular, the study of persistence modules in higher dimensions is much harder than in one dimension. In [15], some partial classification using a generalisation of persistence diagram is presented. Some discussion about the metrics that can be used and the stability theorem is developed in [39, 11].

Persistence Modules as Sheaves

In [37, 23], persistence modules were studied from a sheaf-theoretic point of view, either as a sheaf on a set with a γ-topology or with an Alexandrov topology. This point of view is suited for a higher-dimensional study of persistence. The one-dimensional case was treated in [5]. Some more work was also done in [6].

Homological Algebra on Persistence Modules

Considering persistence modules from a graded module or sheaf perspective allows for the definition of 2 types of internal Hom-functor and tensor product. Deriving those functors gives two different Ext and Tor functors, which provide some information about the category of persistence modules. In [42, 12], the homological algebraic study of persistence modules is developed. Some further study was also conducted in [16].

Persistent homology with coefficient in a ring R

The majority of the research done on persistent homology is by considering the homology with coefficients in a field, as it allows us to work with persistence modules seen either as a graded module over $\mathbf{k}[t]$, which is a PID and therefore admits some decomposition. However, it erases some information like the torsion groups when computing the homology over $\mathbb{Z}$ for instance. In [48], the notion of persistence modules is generalised to include homologies with coefficients in a ring. In [40], an algorithm determining whether such a module splits is presented.

Computation and optimisation of Persistent Homology and Persistence Diagram

In this master thesis, I only presented the Vietoris-Rips complex and a basic algorithm algorithm computing the persistence diagram. However, several other complexes, such as the Delaunay complex or the witness complex, can be used. Furthermore, the algorithm presented is not time efficient, and several other algorithms have been developed in the setting of this master thesis as well as in generalisation to compute the persistent homology and persistence diagram. See, for example, [46].

A | Some Algebraic Geometry

As its name suggests, algebraic geometry is at the intersection of algebra and geometry. It allows the generalisation of geometric concepts and provides an efficient framework to abstract geometric notions. The starting point is to consider the zero set of a set of polynomials as our main object of interest.

This chapter is loosely based on [31, 54].

A.1 Some Basic Results

This section is used to recall the most basic facts of algebraic geometry. Most results are stated without proofs, one can find those in [31].

Let $\mathbf{k}$ be an algebraically closed field. Let $\mathbb{A}^n = \{(a_1, \cdots, a_n) : a_i \in \mathbf{k}\}$. The reason why we use $\mathbb{A}^n$ instead of $\mathbf{k}^n$ is to emphasize that we forget the field structure of $\mathbf{k}$ when considering $\mathbb{A}^n$.

Definition A.1.1. For a subset $S \subset \mathbf{k}[x_1, \cdots, x_n]$, we denote by

$$V(S) = \{x \in \mathbb{A}^n : f(x) = 0 \ \forall f \in S\}$$

the affine *zero locus* of S. Subsets of $\mathbb{A}^n$ that can be written as $V(S)$ are called *affine varieties*.

Dually, given a subset $X \subset \mathbb{A}^n$, we denote by

$$I(X) = \{f \in \mathbf{k}[x_1, \cdots, x_n] : f(x) = 0 \ \forall x \in X\}$$

the *ideal* of X.

A primordial result of algebraic geometry is the following, which allows us to link varieties (geometric objects) with ideals (algebraic objects).

Theorem A.1.2 (Hilbert's Nullstellensatz).

1. for any affine variety $X \subset \mathbb{A}^n$, we have $V(I(X)) = X$.
2. For any ideal $J \trianglelefteq \mathbf{k}[x_1, \cdots, x_n]$, we have $I(V(J)) = \sqrt{J}$.

Recall that if J is an ideal in a commutative ring R, then the *radical ideal* $\sqrt{J}$ is the ideal $\{x \in R : \exists n \in \mathbb{N} : x^n \in J\}$.

Definition A.1.3. If X is an affine variety, we denote by $A(X) = \mathbf{k}[T]/I(X)$ its *coordinate ring*.

On an affine variety X, we define the *Zariski topology* for which the closed subsets are exactly the subsets of the form $V(S)$ for some $S \subset A(X)$.

If $f \in A(X)$, we denote by $D(f) = X \backslash V(f) = \{x \in X : f(x) \neq 0\}$ the *distinguished open subset* of f in X.

It is direct to show, using Hilbert's Nullstellensatz, that the Zariski topology is indeed a topology on X. It can also be shown that the set $D(f)$ for $f \in A(X)$ forms a basis of topology.

We say that a topological space X is *reducible* if it can be written as $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subsetneq X$. Otherwise, the set is called *irreducible*.

Definition A.1.4. Let X be a non-empty topological space.

The *dimension* $\dim X \in \mathbb{N} \cup \{\infty\}$ is the supremum over all $n \in \mathbb{N}$ such that there is a chain

$$\emptyset \neq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subset X$$

with $Y_0, \cdots, Y_n$ closed irreducible subsets of X.

If $Y \subset X$ is non-empty, the *codimension* $\operatorname{codim}_X Y$ of Y in X is the supremum over all $n \in \mathbb{N}$ such that there is a chain

$$Y \subset Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subset X$$

with $Y_0, \cdots, Y_n$ closed irreducible subsets of X.

To give a more precise definition of a variety and morphism of varieties, we need the notion of sheaves and ringed spaces. More results on sheaves can be found in [38].

Definition A.1.5. A *presheaf* (of rings) $\mathcal{F}$ on a topological space X consists of

- For every open subset $U \subset X$, a ring $\mathcal{F}(U)$.
- For every inclusion of open subsets $U \subset V$, a ring homomorphism $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ called the *restriction map*.

such that

1. $\mathcal{F}(\emptyset) = 0$.
2. $\rho_{U,U}$ is the identity map on $\mathcal{F}(U)$ for all open subsets of X.
3. For any inclusion $U \subset V \subset W$ of open subsets of X, we have $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,V}$.

The elements of $\mathcal{F}(U)$ are called the *sections* of $\mathcal{F}$ over U and the restriction maps are also denoted by $\rho_{V,U}(\varphi) = \varphi|_U$ for all $\varphi \in \mathcal{F}(V)$.

Definition A.1.6. A presheaf is called a *sheaf* if it also satisfies the following glueing properties:

1. For any open subset $U \subset X$ and any covering $U = \bigcup_{i \in I} U_i$, we have that any section $s \in \mathcal{F}(U)$ satisfying $s|_{U_i} = 0$ for all $i \in I$ is such that $s = 0$.
2. For any open subset $U \subset X$, any covering $U = \bigcup_{i \in I} U_i$, and any family $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all pairs $i, j \in I$, there is some $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Example A.1.7. 1. Let X, Y be two topological spaces. For all open subsets of X, let $\mathcal{F}(U)$ be the set of continuous functions on U to Y. Let $\rho_{V,U}$ be the restriction from U to V function. It is clear to see that it forms a sheaf on X.

2. On the other hand, let U be an open subset of $\mathbb{R}$, denote by $\mathcal{F}(U)$ the set of bounded functions from U to $\mathbb{R}$ and $\rho_{V,U}$ the restriction of such functions. Then $\mathcal{F}$ forms a presheaf on $\mathbb{R}$ but not a sheaf as it does not respect the second property: taking $U_i = (i-1, i+1)$ a covering of $\mathbb{R}$ and s_i to be the identity on U_i, the only candidate for the function s is the identity on $\mathbb{R}$, which is not bounded.

Definition A.1.8. Let X be an affine variety, let U be an open subset of X. A *regular function* on U is a map $\varphi\colon U \to \mathbf{k}$ such that for all $a \in U$, there are polynomial functions $f, g \in A(X)$ with $f(x) \neq 0$ and $\varphi(x) = \frac{g(x)}{f(x)}$ for all x in an open subset U_a of U containing a.

The set of all regular functions on U is denoted by $\mathcal{O}_X(U)$.

It is direct to see that $\mathcal{O}_X$ forms a sheaf on X, which allows us to make sense of the following definition.

Definition A.1.9. A *ringed space* is a topological space X together with a sheaf of rings on X.

An affine variety can therefore be seen as a ringed space with its sheaf of regular functions. We will only focus on the case when the sheaf is a sheaf of $\mathbf{k}$-valued functions.

Definition A.1.10. Let X, Y be two ringed spaces. Let $f\colon X \to Y$ be a map of sets. It induces for any map $\varphi\colon U \to \mathbf{k}$ (where $U \subset Y$ is open in Y) a composition $f^*\varphi = \varphi \circ f\colon f^{-1}(U) \to \mathbf{k}$ called the *pullback* of φ by f.

The map f is called a *morphism* of ringed spaces if it is continuous and if for all open subsets $U \subset Y$ and $\varphi \in \mathcal{O}_Y(U)$, we have $f^*\varphi \in \mathcal{O}_X(f^{-1}(U))$.

When f is a morphism, then we have that the pull-back by f yields a $\mathbf{k}$-algebra homomorphism $f^*\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$. More than that, the pull-back actually yields a bijection between morphisms and $\mathbf{k}$-algebra homomorphisms.

Proposition A.1.11. For any two affine varieties X and Y, there is a bijection

$$\begin{array}{ccc}\{\text{morphisms } X \to Y\} & \leftrightarrow & \{\mathbf{k}\text{-algebra homomorphisms } A(Y) \to A(X)\} \\ f & \mapsto & f^*.\end{array}$$

We also have the following bijection.

Lemma A.1.12. We have an isomorphism

$$\mathrm{Hom}_{\mathbf{k}-alg}(A(X), \mathbf{k}) \simeq X.$$

Proof. Take $\varphi\colon X \to \mathrm{Hom}_{\mathbf{k}-alg}(A(X), \mathbf{k})$, $x \mapsto (\beta\colon f \mapsto f(x))$. Recall that $A(X) = \mathbf{k}[t]/I(x)$. Thus, if $\alpha \in \mathrm{Hom}_{\mathbf{k}-alg}(A(X), \mathbf{k})$, it is uniquely determined by $\alpha(T) \in \mathbf{k}$ such that $f(\alpha(T)) = 0$ for all $f \in I(X)$: if $g \in A(X)$, then $g = h + l$ with $l \in I(X)$ and $h = h_0 + h_1 T + \cdots + h_n T^n \in \mathbf{k}[T]$, then $\alpha(g) = h(\alpha(T))$. Therefore, as $f(\alpha(T)) = 0$ for all $f \in I(X)$, it means that $\alpha(T) \in V(I(X)) = X$ using Hilbert's Nullstellensatz. Now take $\psi\colon \mathrm{Hom}_{\mathbf{k}-alg}(A(X), \mathbf{k}) \to X$, $\alpha \mapsto \alpha(T)$. We have

$$\varphi \circ \psi(\alpha) = \varphi(\alpha(T)) = (\beta\colon f \mapsto f(\alpha(T))) = \alpha$$

as $\beta(T) = \alpha(T)$. We also have $\psi \circ \varphi(x) = \psi(\beta\colon f \mapsto f(x)) = T(x) = x$. We therefore have the result. ☐

We can now define what varieties are.

Definition A.1.13. A *prevariety* is a ringed space X that has a finite open cover by affine varieties.

A prevariety is a *variety* if the diagonal $\Delta_X = \{(x, x) : x \in X\}$ is closed in $X \times X$.

Example A.1.14. 1. Open subsets of varieties are again varieties. Let $U \subset X$ be an open set. Then taking the structure sheaf $\mathcal{O}_U = \mathcal{O}_X|_U$ on U, it forms a ringed space. Moreover, as X is covered by affine varieties, U can be covered by open subsets of affine varieties, which themselves can be covered by affine varieties (for example, the distinguished open subsets). Therefore, U is a prevariety. Considering the inclusion morphism $i\colon U \times U \to X \times X$, we have $\Delta_U = i^{-1}(\Delta_X)$, which is closed as i is continuous and Δ_X closed.

2. Closed subsets of varieties are again varieties. Let $F \subset X$ be a closed subset of X. Define $\mathcal{O}_F(U) = \{\varphi\colon U \to k : \forall a \in U, \exists V$ open neighbourhood of a in X, there is $\psi \in \mathcal{O}_X(V) : \phi = \psi$ on $U \cap V\}$. It is a sheaf on F. Moreover, for any affine open subset $U \subset X$, we can consider $U \cap F$ as an open subset of the ringed space F or as an affine subvariety of U and those two interpretations are isomorphic. Therefore, F is a prevariety, and the same reasoning as for the open subset gives that it is a variety.

A.2 Chevalley's Lemma

The goal of this section is to prove that the image of a morphism between two varieties contains a non-empty open subset of its closure. We use this result in the proof of Gabriel's theorem. To prove this result, we will first prove 2 results on the extension of some morphisms. Let us also first recall some definitions.

Definition A.2.1. A commutative ring R is said to be *reduced* if 0 is the only nilpotent element. In other words, it means that if $x \in R$ is such that $x^2 = 0$, then $x = 0$.

Such a ring is an *integral domain* if for all $x, y \in R$, $xy = 0$ implies that $x = 0$ or $y = 0$.

Let B be a reduced ring, A a subring of B such that B is of finite type over A, i.e. $B = A[b_1, \cdots, b_n]$ for some $b_1, \cdots, b_n \in B$. Assume that $n = 1$ and $B = A[b]$, then $B = A[T]/I$ where $I = \{f \in A[T] : f(b) = 0\}$. Denote by

$$\mathcal{J}(I) = \{0\} \cup \{a : a \text{ is the leading coefficient of } f \text{ for } f \in I\}.$$

It is direct to see that $\mathcal{J}(I)$ is an ideal of A.

Lemma A.2.2. With the notation used above, let $\phi\colon A \to \mathbf{k}$ be a ring homomorphism with $\mathbf{k}$ an algebraically closed field such that $\phi(\mathcal{J}(I)) \neq \{0\}$. Then ϕ can be extended into a ring homomorphism $B \to \mathbf{k}$.

Proof. As $\phi(\mathcal{J}(I)) \neq \{0\}$, there is some $f \in I$, $f = f_0 + f_1T + \cdots + f_mT^m$ such that $\phi(f_m) \neq 0$. Assume that m is minimal. We can extend $\phi\colon A \to \mathbf{k}$ to $\phi\colon A[T] \to \mathbf{k}[T]$ by linearity. If we assume that $\phi(I)$ does not contain any non-zero constant, then $\langle\phi(I)\rangle \subsetneq \mathbf{k}[T]$. Let $z \in \mathbf{k}$ be such that $f(z) = 0$ for all $f \in \langle\phi(I)\rangle$. Note that since $\mathbf{k}$ is algebraically closed, such a z exists as if $f, g \in \langle\phi(I)\rangle$ have no common root, then they must be co-prime, so $\langle\phi(I)\rangle = \mathbf{k}[T]$. In general, if there is no $z \in \mathbf{k}$ such that $f(z) = 0$ for all $f \in \langle\phi I\rangle$, then the set $\langle\phi(I)\rangle$ spans $\mathbf{k}[T]$. Then, by setting $\phi(b) = z$, we define an extension of ϕ to B.

To complete the proof, it remains to show that $\phi(I)$ does not contain any non-zero constant. By contradiction, suppose that there exists some $g \in I$ with $g = g_0 + g_1T + \cdots + g_nT^n$ such that $\phi(g_0) \neq 0$ and $\phi(g_i) \neq 0$ for $i > 0$. Suppose furthermore that n is minimal. By the division algorithm, there are some $d \geq 0$, $q, r \in A[T]$ such that $f_m^d g = qf + r$ and $\deg(r) < m$. Applying ϕ to the equality, one gets

$$\phi(f_m)^d\phi(g_0) = \phi(f_m^d g) = \phi(q)\phi(f) + \phi(r).$$

The left side is a non-zero constant, whereas $\phi(f)$ is a polynomial of degree $m > 0$. It implies that $\phi(q) = 0$ and $\phi(r) \in \mathbf{k}\backslash\{0\}$. It implies that $n \leq \deg r < m$. Let us now proceed by induction on m.

- For $m = 1$, as $n < m$, g can't exist as it would mean that $g = a$ is a constant and therefore $a \in \mathcal{J}(I)$ and by our hypothesis, $\phi(a) \neq 0$.

-Induction: For a polynomial $h = h_0 + \cdots + h_sT^s \in A[T]$ with $h_s \neq 0$, let us denote by $\tilde{h} = T^s h(\frac{1}{T}) = h_s + \cdots + h_0T^s$. We write $\tilde{I} = \langle\tilde{h} : h \in I\rangle \subset A[T]$. Let us also denote $\mathcal{J} = \tilde{I} \cap A$, it is an ideal. Moreover, if $a \in \mathcal{J} = \tilde{I} \cap A$, it means that there is some $s \in \mathbb{N}$ such that $aT^s \in I$. As I is an ideal of A, it implies that $a^{s-1}aT^s = (aT)^s \in I$. Moreover, as $B = A[T]/I$ is reduced, it implies that $aT \in I$. Therefore we have the equality

$$\mathcal{J} = \tilde{I} \cap A = \{a \in A : aT \in I\}.$$

If $\phi(\mathcal{J}) \neq \{0\}$ then we fall back to the case $m = 1$. Therefore $\phi(\mathcal{J}) = \{0\}$. Denote $\tilde{A} = A/\mathcal{J}$, $\tilde{B} = A[T]/\tilde{I} = \tilde{A}[T]/\tilde{I} = \tilde{A}[\tilde{b}]$ for some $\tilde{b} \in B$. We have that $\tilde{B}$ is reduced: let $f \in A[T]$ be such that $f^s \in \tilde{I}$, then $f^s = \sum_{k=1}^{r} \tilde{h_k}$ for $r \geq 1$ and $h_k \in I$ for all k. Then for some $u \in \mathbb{N}$, one get $T^u \tilde{f^s} = T^{u+s} f^s(\frac{1}{T}) = \sum_{k=1} r T^{u+s} \tilde{h_r}(\frac{1}{T})$. The right-hand side is in I. It means that $T^u \tilde{f^s} \in I$ for some $u \geq 0$. Therefore $(T\tilde{f})^{u+s} \in I$ and as B is reduced, $T\tilde{f} \in I$, which implies that $f \in \tilde{I}$. Moreover, $\tilde{g} = g_n + \cdots + g_0 T^n \in \tilde{I}$ and $\phi(g_0) \neq 0$. Therefore, as $\phi\colon A \to \mathbf{k}$ defines a morphism $\tilde{\phi}\colon \tilde{A} \to \mathbf{k}$ and as $n < m$, we have by induction that $\tilde{\phi}$ extends to $\tilde{B} \to \mathbf{k}$. However, as $\phi(g_i) = 0$ for $i > 0$, one gets $0 = \tilde{\phi}(\tilde{g}) = \phi(g_0)\phi(T^n)$. Hence, $0 = \phi(T^n) = \phi(T)^n$ and $\phi(T) = 0$. It implies that $\tilde{\phi}(\tilde{b}) = 0$, which further implies that we have $\tilde{f} \in \tilde{I}$ with $\tilde{\phi}(\tilde{f}) = \phi(f_m) \neq 0$. It yields a contradiction.

□

We will use this lemma in the next lemma.

> **Lemma A.2.3.** Let B be an integral domain. Let A be a subring of B such that B is of finite type over A. Then given $b \neq 0$ in B, there is some $a \in A \backslash \{0\}$ such that any ring homomorphism $\phi\colon A \to \mathbf{k}$ with $\phi(a) \neq 0$ can be extended to $\phi\colon B \to \mathbf{k}$ with $\phi(b) \neq 0$.

Proof. As B is of finite type over A, we have $B = A[b_1, \cdots, b_n]$. By a direct induction, we can suppose that $n = 1$: $B = A[b_1] \simeq A[T]/I$ with as above $I = \{f \in A[T] : f(b_1) = 0\}$.

If $I = \{0\}$, then $B \simeq A[T]$ therefore taking $b \in B$, we can write $h = a_0 + a_1 T + \cdots + a_m T^m$ the polynomial representing b with $a_m \neq 0$. Then taking $a = a_m$, we can extend ϕ to $\phi\colon A[T] \to \mathbf{k}$ and then send T to $x \in \mathbf{k}$ such that x is not a root of $\phi(a_0) + \phi(a_1)T + \cdots + \phi(a_m)T^m$. Note that such an x exists as $\mathbf{k}$ is algebraically closed, therefore infinite, whereas the polynomial has a finite number of roots.

Let us treat the case when $I \neq \{0\}$. Let $f \in I \backslash \{0\}$ of minimal degree. Let a_1 be its leading coefficient. By the division algorithm, we have that $g \in I$ if and only if there is $d \geq 0$ such that $f | a_1^d g$. Let $h \in A[T]$ be a polynomial representing b, i.e. a polynomial such that $h(b_1) = b$. As $b \neq 0$, $h \notin I$. We have that f is irreducible over $\mathrm{Frac}(A)$ its field of fraction: if $f = f_1 f_2$ for some $f_1, f_2 \in \mathrm{Frac}(A)[T]$, then $0 = f(b_1) = f_1(b_1) f_2(b_1)$. As B is an integral domain, either $f_1(b_1) = 0$ or $f_2(b_1) = 0$, without loss of generality, suppose the former. It implies that $a' f_1 \in I$ with $a' \in A \backslash \{0\}$ being the product of the denominator of the coefficients of f_1 (to insure that $a' f_1 \in A[T]$). By minimality of the degree of f, it implies that $\deg f = \deg f_1$ and f is irreducible. Hence, f and h are co-prime over $\mathrm{Frac}(A)$ and therefore there are $u, v \in A[T]$ and $a_2 \in A \backslash \{0\}$ such that $uf + vh = a_2$. Then, let us take $a = a_1 a_2$. If $\phi(a) \neq 0$, then $\phi(a_1) \neq 0$, so $\phi(\mathcal{J}(I)) \neq \{0\}$ and we can apply the previous lemma. Let us still denote by $\phi\colon B \to \mathbf{k}$ the extension of $\phi\colon A \to \mathbf{k}$. We have

$$\phi(v(b_1))\phi(b) = \phi(v(b_1))\phi(h(b_1)) = \phi(v(b_1)h(b_1)) + \phi(u(b_1)f(b_1)) = \phi(a_2) \neq 0.$$

Hence, $\phi(b) \neq 0$ and we have the result. $\square$

We can now prove Chevalley's lemma.

Theorem A.2.4 (Chevalley's Lemma). Let $\psi\colon X \to Y$ be a morphism of varieties. Then the image of the morphism, $\psi(X)$, contains a non-empty open subset of $\overline{\psi(X)}$.

Proof. Let us first assume that X and Y are affine and X is irreducible. We can assume that $Y = \overline{\psi(X)}$. Recall that $X \simeq \mathrm{Hom}_{\mathbf{k}-alg}(k(X), \mathbf{k})$ (Lemma A.1.12). Moreover, as ψ is a morphism of varieties, we have that $\psi^*\colon \mathbf{k}[Y] \to \mathbf{k}[X], f \mapsto f \circ \psi$ is a morphism of $\mathbf{k}$-algebras. As $\psi(X)$ is dense in Y, this morphism is a monomorphism. We can then associate $\mathbf{k}[Y]$ to a subset $\psi^*\mathbf{k}[Y]$ of $\mathbf{k}[X]$. As X is irreducible, $\mathbf{k}[X]$ is an integral domain. Hence, taking $b = 1 \in \mathbf{k}[X]$, by the preceding lemma (with $A = \mathbf{k}[Y], B = \mathbf{k}[X]$), there is $a \in \mathbf{k}[Y]$ such that for all $f\colon \mathbf{k}[Y] \to \mathbf{k}$ with $f(a) \neq 0$, there is $\overline{f}\colon \mathbf{k}[X] \to \mathbf{k}$ such that $\overline{f} \circ \psi = f$ and $\overline{f}(1) \neq 0$. Using the isomorphism $X \simeq \mathrm{Hom}_{\mathbf{k}-alg}(k(X), \mathbf{k})$, it is equivalent to say that there is $a \in \mathbf{k}[Y]$ such that for all $y \in Y$ with $a(y) \neq 0$, there is some $x \in X$ with $\psi(x) = y$ and such that $1(x) \neq 0$. In particular, it means that $D(a) \subset \psi(X)$.

If X is no longer irreducible, let $X_1, \cdots, X_r$ be its irreducible components. Then $\psi(X_i)$ contains a non empty subset of $\overline{\psi(X_i)}$. As $\overline{\psi(X)} = \cup_i \overline{\psi(X_i)}$, the result is still correct. Assume X is no longer affine. By taking a cover of X by affine subsets $X_1, \cdots, X_r$, we have that $\psi(X_1)$ contains V_1, a non empty open subset of $\overline{\psi(X_1)}$, so $\psi(X) \supset \psi(X_1) \supset V_1$. Let us now assume that Y is no longer affine. We can still assume that $\overline{\psi(X)} = Y$. Let $W_1, \cdots, W_n$ be an affine cover of Y. We know that $\psi(X) \cap W_i$ is dense in W_i. Let $U_i = \psi^{-1}(W_i)$. We have $\psi(U_i) = \psi(X) \cap W_i$. Therefore, $\overline{\psi(U_i)} \cap W_i = \overline{\psi(X) \cap W_i} \cap W_i = W_i$ as $\psi(X) \cap W_i$ is dense in W_i. Therefore, if we take $V_i \subset \overline{\psi(U_i)} \cap W_i$ open and non empty, then V_i is open in Y and $V_i \subset \psi(U_i) \subset \psi(X)$. $\square$

A.3 Algebraic Groups

Similarly to the fact that in topology there is the notion of topological groups, which are topological spaces with a group structure that respects the continuity conditions, one can also define the notion of algebraic groups as being algebraic varieties with a group structure. The following definition, coming from [54], sets it formally.

Definition A.3.1. An *algebraic group* is an algebraic variety G which is also a group such that the maps defining the group structure $\mu\colon G \times G \to G, (x, y) \mapsto xy$ and $i\colon G \to G, x \mapsto x^{-1}$ are morphisms of algebraic varieties.

One can study those algebraic groups in themselves, for example, their subgroups, quotients, and morphisms between such groups. More results on those are given in [54]. We will only restrict ourselves to the bare minimum in what we use in this document. We will therefore mainly focus on algebraic group actions.

Definition A.3.2. A *G-space* is a variety X on which G, an algebraic group, acts. In other words, there is a morphism of varieties $\alpha\colon G\times X \to X,\ (g,x)\mapsto g\cdot x$ such that $g\cdot(h\cdot x)=(gh)\cdot x$ and $e\cdot x = x$.

Given a G-space and $x\in X$, the *orbit* is the set $G\cdot x=\{g\cdot x | g\in G\}$.

Proposition A.3.3. An orbit $G\cdot x$ is locally closed, i.e. it is open in its closure. Moreover, it is a variety.

Proof. Using Chevalley's lemma A.2.4 to the morphism $g\mapsto g\cdot x$ shows that $G\cdot x$ contains a non-empty open subset U of its closure $\overline{G\cdot x}$. We therefore have $U\subset G\cdot x\subset \overline{G\cdot x}$. Let us show that

$$\bigcup_{g\in G} g\cdot U = G\cdot x :$$

$\bigcup_{g\in G} g\cdot U \subset G\cdot x$: Let $y\in\bigcup_{g\in G} g\cdot U$, we therefore have $y = g\cdot u$ for some $g\in G$ and $u\in U$. As $u\in U\subset G\cdot x$, we have $u = g'\cdot x$. Therefore $y = g\cdot(g'\cdot x)=(gg')\cdot x\in G\cdot x$.

$G\cdot x\subset\bigcup_{g\in G} g\cdot U$: Let $y\in G\cdot x$, then $y=g\cdot x$ for some $g\in G$. As $U\subset G\cdot x$ is non empty, there is $u\in U$, $u=g'\cdot x$. Then $y=g\cdot x=(gg'^{-1})\cdot(g'\cdot x)=(gg'^{-1})\cdot u\in\bigcup_{g\in G} g\cdot U$.

We have that $g\cdot U$ is an open set as $g\cdot U\simeq U$ with $\phi\colon g\cdot U\to U, v\mapsto g^{-1}\cdot v$ and $\psi\colon U\to g\cdot U,\ u\mapsto g\cdot u$ being the inverse diffeomorphisms. Therefore, $\bigcup_{g\in G} g\cdot U$ is an open subset of $\overline{G\cdot x}$ and we have the first result.

The second one comes from the fact that, as stated in example A.1.14, a closed subset of a variety is a variety; therefore $\overline{G\cdot x}$ is a variety. Furthermore, an open subset of a variety is also a variety, giving the result. ☐

B | An Introduction to Category Theory

The goal of this chapter is to give a gentle introduction to category theory, with a view towards homological algebra. Only the main concepts and results will be presented, without proofs. For instance, I will not develop the concept of adjunction, Yoneda's lemma, the homotopy category, etc. The main sources are [41, 29] as well as the course of Homological Algebra taught by Prof. Vitoria dos Santos that I attended in Padua during the Spring semester of 2024.

B.1 First Definitions

The main idea of category theory is to reunite already available concepts in other fields of Mathematics to be able to apply them to different concepts, generalise them and develop new ones. The focus of study in category theory is the relationship between objects, not the objects in themselves.

Definition B.1.1. A *category* $\mathcal{C}$ consists of

- A class of objects, denoted by $\mathrm{Ob}(\mathcal{C})$,
- For each pair $X, Y \in \mathrm{Ob}(\mathcal{C})$ of objects in $\mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ whose elements are called *morphisms* and denoted by $f\colon X \to Y$,
- A composition law satisfying the following two properties:
    1. *Associativity*: for all $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$, for all $f\colon X \to Y, g\colon Y \to Z$ and $h\colon Z \to W$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.
    2. *Unit*: For all object X of $\mathcal{C}$, there is a morphism $1_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that, for all $f\colon X \to Y$ and $g\colon Y \to X$, we have $f \circ 1_X = f$ and $1_X \circ g = g$.

The elements of the Hom-sets are called morphisms but they are *formal* morphisms: unless we are in a specific category where we know the type of objects and morphisms between those objects, we will never apply those to elements of the object (we don't even know if the objects have elements in them!).

Some morphisms in a category have some special properties, we will present some of them.

Definition B.1.2. Let $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ be a morphism in the category $\mathcal{C}$. Then,

- f is a *monomorphism* if for all $h, g \colon Z \to X$, the equality $f \circ g = f \circ h$ implies $g = h$. Monomorphisms are (sometimes) denoted by $f \colon X \hookrightarrow Y$ or $f \colon X \rightarrowtail Y$.
- f is a *epimorphism* if for all $h, g \colon Z \to X$, the equality $g \circ f = h \circ f$ implies $g = h$. Epimorphisms are (sometimes) denoted by $f \colon X \twoheadrightarrow Y$.
- f is an *isomorphism* if there is a morphism $g \colon Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

Before going further, let us see some examples of categories.

Example B.1.3. -The category **Set** whose objects are sets and if X, Y are two sets, $\mathrm{Hom}_{\mathbf{Set}}(X, Y)$ is the set of all (set-theoretical) maps from X to Y. Monomorphisms in **Set** are injective functions, epimorphisms are surjections and isomorphisms are bijections.

-The category $\mathbf{Vect_k}$ ($\mathbf{k}$ a field) with objects being $\mathbf{k}$-vector spaces and morphisms being $\mathbf{k}$-linear maps between vector spaces. Monomorphisms are injective linear maps, epimorphisms are surjective linear maps and isomorphisms are, as defined in linear algebra, bijective linear maps.

-Given a poset $(T, \leq)$, we have the category $\mathcal{C}_{(T,\leq)}$ with as objects the elements of T and

$$\mathrm{Hom}_{\mathcal{C}_{(T,\leq)}}(a, a') = \begin{cases} \{f_a^{a'}\} & \text{if } a \leq a' \\ \emptyset & \text{otherwise.} \end{cases}$$

Remark B.1.4. Note that if a morphism is a monomorphism and an epimorphism, it does not imply that it is an isomorphism. For instance, in the category **Ring** with object rings and as morphisms ring homomorphisms, the inclusion $i \colon \mathbb{Z} \to \mathbb{Q}$ is a monomorphism and an epimorphism, but it is not an isomorphism.

A common construction of category theory is to “reverse” all the arrows. This gives another category.

Definition B.1.5. Let $\mathcal{C}$ be a category, the *opposite category* or *dual category* is the category $\mathcal{C}^{\mathrm{op}}$ defined by $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ and the composition is given by $g^* \circ f^* = (f \circ g)^*$, where the $\cdot^*$ represent the morphism considered in $\mathcal{C}^{\mathrm{op}}$.

Now that we have defined categories, let us define “functions” between categories. As often in Mathematics, we would like those “functions” to respect in some ways the structure of the categories, i.e. it must respect the composition and identity. We will therefore define that as follows.

Definition B.1.6. Let $\mathcal{C}, \mathcal{D}$ be categories. A (covariant) *functor* $F \colon \mathcal{C} \to \mathcal{D}$ consists of

- An assignment $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D}), X \mapsto F(X)$,

- For any pair of objects X, Y in $\mathcal{C}$, we have a function

  $$F_{X,Y}\colon \operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{D}}(F(X),F(Y))$$

  such that

  1. For all object X of $\mathcal{C}$, we have $F(1_X) = 1_{F(X)}$,
  2. For all maps $f\colon X \to Y, g\colon Y \to Z$, we have $F_{X,Z}(g \circ f) = F_{Y,Z}(g) \circ F_{X,Y}(f)$.

A *contravariant functor* F is a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$. It means if we have the morphism $f\colon X \to Y$ in $\mathcal{C}$, we will have the morphism $F(f)\colon F(Y) \to F(X)$ in $\mathcal{D}$ and $F(f \circ g) = F(g) \circ F(f)$.

In the following, we will drop the indices of $F_{X,Y}(f) = F(f)$.

The following examples are 2 functors widely used in the study of categories.

Example B.1.7. Let $\mathcal{C}$ be a category and $X \in \operatorname{Ob}(\mathcal{C})$, we can define the functor

$$h_x = \operatorname{Hom}_{\mathcal{C}}(X,-)\colon \mathcal{C} \to \mathbf{Set}$$

as, on the objects, $h_X(Y) = \operatorname{Hom}_{\mathcal{C}}(X,Y)$ and if $f\colon Y \to Z$ is a morphism, $h_X(f) = \operatorname{Hom}_{\mathcal{C}}(X,f)\colon \operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{C}}(X,Z)$ is defined as $h_X(f)(g) = f \circ g$. Note that we have $h_X(f \circ g)(h) = f \circ h \circ g = h_X(f) \circ h_X(g)(h)$. It is a covariant functor.

On the other hand, we can define the contravariant functor

$$h^X = \operatorname{Hom}_{\mathcal{C}}(-,X)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

as $h^X(Y) = \operatorname{Hom}_{\mathcal{C}}(Y,X)$ and if $f\colon Y \to Z$ in $\mathcal{C}$,

$$h^X(f)\colon \operatorname{Hom}_{\mathcal{C}}(Z,X) \to \operatorname{Hom}_{\mathcal{C}}(Y,X), g \mapsto g \circ f.$$

We could use the functors as a way to states that 2 categories are equivalent: by stating that $\mathcal{C}, \mathcal{D}$ are equivalent if there exists $F\colon \mathcal{C} \to \mathcal{D}$ and $F\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{D}}$. However, this definition is pretty restrictive, and some categories that are "similar enough" are not equivalent in the sense of that definition. We will therefore need to introduce a new concept.

Going one level further in the abstraction, we can define "functions" between functors. For that, we introduce the (naïve) concept of *diagrams* to represent a part of the category as a (multi-)graph with objects represented as vertices and morphisms between them as arrows. We say that a diagram *commutes* if for every two vertices and any two directed paths joining the first vertex to the second one, the resulting maps obtained by composing the maps represented by each arrow are the same.

Definition B.1.8. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* is a collection of maps $\alpha = (\alpha_X)_{X \in \operatorname{Ob}(\mathcal{C})}$ where α_X is a morphism $F(X) \to G(X)$ in $\mathcal{D}$ (i.e. $\alpha_X \in \operatorname{Hom}_{\mathcal{D}}(F(X), G(X))$) such that for all $f \in \operatorname{Hom}_{\mathcal{C}}(X,Y)$, we have $G(f) \circ \alpha_X = \alpha_Y \circ F(f)$. In other words, we want the following diagram in $\mathcal{D}$ to commute

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\alpha_Y"'] & G(Y).
\end{tikzcd}$$

A natural transformation $\alpha\colon F \to G$ for which all α_X are isomorphisms in $\mathcal{D}$ is called a *natural equivalence*. The two functors are said to be naturally equivalent, and we denote it by $F \simeq G$.

These two definitions allow us to abstract even further and create some "meta-categories". We only need to introduce a technical definition to make sure that the Hom-sets are indeed sets. In the following, a *small* category is a category whose class of objects form a set.

Example B.1.9. - The small categories with morphisms between them given by functors form a category.

- If $\mathcal{C}$ is small, let $\mathcal{D}$ be any category. Then we can define the functor category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ by taking as objects the functors form $\mathcal{C}$ to $\mathcal{D}$ and for all F, G functors, $\mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}(F, G)$ is the set of natural transformation from F to G. The composition is given by the composition of the morphisms on each object, i.e. for all objects $X \in \mathrm{Ob}(\mathcal{C})$, we have $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$.

The natural transformation also allows us to have a more suitable notion of equivalence of categories.

Definition B.1.10. We say that the categories $\mathcal{C}$ is *equivalent* to the category $\mathcal{D}$ if there is a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F \simeq 1_{\mathcal{C}}$ and $F \circ G \simeq 1_{\mathcal{D}}$ where the functor $1_{\mathcal{C}}$ is the identity: $1_{\mathcal{C}}(X) = X$ and $1_{\mathcal{C}}(f) = f$ for all object and morphism of $\mathcal{C}$.

B.1.1 Composition of Functors and Natural Transformations

Let us now define the composition between functors and natural transformations. Note that these definitions are pretty specific to this master thesis and not widely used (although the definitions are pretty canonical). They were used (without definition) in [13]. We use these compositions as a shorthand notation in Chapter 4.

Let $F, G\colon \mathcal{C} \to \mathcal{D}$, $J\colon \mathcal{B} \to \mathcal{C}$ and $H\colon \mathcal{D} \to \mathcal{E}$ be functors. Let $\alpha\colon F \to G$ be a natural transformation. We define the natural transformation $H\alpha\colon HF \to HG$ by, for all object $X \in \mathrm{Ob}\,\mathcal{C}$,

$$(H\alpha)_X = H \circ \alpha_X \colon HF(X) \to HG(X).$$

Let us check that it is indeed a natural transformation. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. In $\mathcal{E}$, we have the following diagram; let us show that it commutes.

$$\begin{tikzcd}
HF(X) \arrow[r, "H\circ\alpha_X"] \arrow[d, "HF(f)"'] & HG(X) \arrow[d, "HG(f)"] \\
HF(Y) \arrow[r, "H\circ\alpha_Y"'] & HG(Y)
\end{tikzcd}$$

As α is a natural transformation, we have $G(f)\alpha_X = \alpha_Y F(f)$ and therefore,

$$\begin{aligned} HG(f) \circ (H \circ \alpha_X) &= H(G(f) \circ \alpha_X) \\ &= H(\alpha_Y \circ F(f)) \\ &= H\alpha_Y \circ HF(f). \end{aligned}$$

We can also define the composition starting with the functor and composing it with a natural transformation. We define the natural transformation $\alpha J \colon FJ \to GJ$ by, for all objects A of $\mathcal{B}$,

$$(\alpha J)_A = \alpha_{J(A)}.$$

Once again, it is again a natural transformation as, for all $f \colon A \to B$ morphism of $\mathcal{B}$, we have the morphism $J(f) \colon J(A) \to J(B)$ in $\mathcal{C}$ and by naturality of α, we have $\alpha_{J(B)} FJ(f) = GJ(f)\alpha_{J(A)}$. In other words, the following diagram commutes

$$\begin{tikzcd} FJ(A) \arrow[r, "\alpha_{J(A)}"] \arrow[d, "FJ(f)"'] & GJ(A) \arrow[d, "GJ(A)"] \\ FJ(B) \arrow[r, "\alpha_{J(B)}"'] & GJ(B). \end{tikzcd}$$

B.2 Objects with Special Properties

Now that we have the basic definitions of categories, we can start to construct/define some specific objects of those categories. One of the most used methods to define objects is through *universal properties*. We define the objects and some of their morphisms through a commutative diagram and the existence and uniqueness of a specific arrow of the diagram. The uniqueness of the arrows implies that the object constructed, if it exists, is unique up to unique isomorphism. This (partially) justifies the abuse of notation and use of the definite article when referring to that object.

The best way to understand those universal properties is through some examples. We will define the (categorical) product and coproduct as well as the inductive and projective limit, but other objects defined in that way can be constructed (such as (co)equaliser, etc.).

B.2.1 Product and Coproduct

Let J be a set and for all $j \in J$, let X_j be an object of $\mathcal{C}$.

The *product* of $(X_j)_{j \in J}$ (if it exists) is the object P with, for all $j \in J$ morphisms $\pi_j \colon P \to X_j$ such that for any object Z of $\mathcal{C}$ with morphisms $f_j \colon Z \to X_j$, there is a unique morphism $f \colon Z \to P$ such that $f_j = \pi_j \circ f$ for all $j \in J$. In other words, we have the following universal diagram

$$\begin{tikzcd}
 & Z \arrow[ddl, dashed, "f_j"'] \arrow[ddr, dashed, "f_k"] \arrow[d, dotted, "\exists ! f"] & \\
 & P \arrow[dl, "\pi_j"] \arrow[dr, "\pi_k"'] & \\
X_j & & X_k.
\end{tikzcd}$$

The *coproduct* is pretty similar, just reversing the arrows. It is the object C with, for all $j \in J$ morphisms $i_j \colon X_j \to C$ such that for any object Z of $\mathcal{C}$ with morphisms $f_j \colon X_j \to Z$, there is a unique morphism $f \colon C \to Z$ such that $f_j = f \circ i_j$ for all $j \in J$. In other words, we have the following universal diagram.

$$\begin{tikzcd}
X_j \arrow[dr, "i_j"] \arrow[ddr, dashed, "f_j"'] & & X_k \arrow[dl, "i_k"'] \arrow[ddl, dashed, "f_k"] \\
 & C \arrow[d, dotted, "\exists ! f"] & \\
 & Z. &
\end{tikzcd}$$

We denote the product of the X_j by $(\prod_J X_j, \pi_j)$ and the coproduct by $(\coprod_J X_j, i_j)$.

B.2.2 Pushouts and Pullbacks

Let $f : A \to B$ and $g : A \to C$ be two morphisms of $\mathcal{C}$. The *pushout* of (A, f, g) is the object D with morphisms $u_1 : B \to D$ and $u_2 : C \to D$ such that $u_1 f = u_2 g$ and such for every other object D' and pair of morphisms $v_1 : B \to D$, $v_2 : C \to D$ such that $v_1 f = v_2 g$, then there is a unique morphism $\varphi : D \to D'$ such that $v_1 = \varphi u_1$ and $v_2 = \varphi u_2$. In other words, we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "u_2"] \arrow[ddr, dashed, bend left, "v_2"] & \\
B \arrow[r, "u_1"'] \arrow[drr, dashed, bend right, "v_1"'] & D \arrow[dr, dotted, "\exists ! \varphi"] & \\
 & & D'.
\end{tikzcd}$$

To denote that D is a pushout, we use an extra arrowhead at the bottom right corner of the square, and we say that the square is a *pushout square*.

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near end] & C \arrow[d, "u_2"] \\
B \arrow[r, "u_1"'] & D.
\end{tikzcd}$$

If we reverse the arrows, we get a *pullback square*:

$$\begin{tikzcd}
D \arrow[dr, dotted, "\exists!\varphi"] \arrow[drr, dashed, bend left, "v_2"] \arrow[ddr, dashed, bend right, "v_1"'] & & \\
& D \arrow[r, "u_2"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "g"] \\
& B \arrow[r, "f"'] & A.
\end{tikzcd}$$

B.2.3 Limits

Instead of taking J as a set, we can take $\mathcal{J}$ as a category and $F\colon \mathcal{J} \to \mathcal{C}$ a functor. We can then define the limit of the functor F as being the object L with the morphisms $\varphi_j\colon L \to F(j)$ for all $j \in \operatorname{Ob}\mathcal{J}$ such that for all morphism $f_{kj}\colon j \to k$ in $\mathcal{J}$, we have $F(f_{kj})\varphi_j = \varphi_k$. Moreover, we want this object to be the "biggest" for which it works, more formally, for any other object X with $\psi_j\colon X \to F(j)$ such that $F(f_{kj})\psi_j = \psi_k$, there is a unique morphism $\psi\colon X \to L$ such that $\psi_j = \varphi_j\psi$. In other words, the following diagram commutes.

$$\begin{tikzcd}
& & F(k) \\
X \arrow[urr, dashed, "\psi_k"] \arrow[r, dotted, "\exists!\psi"] \arrow[drr, dashed, "\psi_j"'] & L \arrow[ur, "\varphi_k"] \arrow[dr, "\varphi_j"'] & \\
& & F(j). \arrow[uu, "F(f_{ij})"']
\end{tikzcd}$$

If it exists, we denote this object by $\varprojlim_{i\in I} F(i)$.

If we "reverse" all the arrows, we can construct the colimit or direct limit. To be more precise, it is the object C with the morphisms $\rho_j\colon F(j) \to C$ for all $j \in \operatorname{Ob}(\mathcal{J})$ such that for all morphism $f_{kj}\colon j \to k$ in $\mathcal{J}$, we have $\rho_k F(f_{kj}) = \rho_j$. Moreover, for any other object X with $\psi_j\colon F(j) \to X$ such that $\psi_k F(f_{kj}) = \psi_j$, there is a unique morphism $\psi\colon C \to X$ such that $\psi_j = \varphi_j\psi$. In other words, the following diagram commutes.

$$\begin{tikzcd}
F(k) \arrow[dr, "\rho_k"'] \arrow[drr, dashed, "\psi_k"] & & \\
& C \arrow[r, dotted, "\exists!\psi"] & X. \\
F(j) \arrow[uu, "F(f_{kj})"] \arrow[ur, "\rho_j"] \arrow[urr, dashed, "\psi_j"'] & &
\end{tikzcd}$$

If it exists, we denote this object by $\varinjlim_{j\in\mathcal{J}} F(j)$.

We can see that if $\mathcal{J}$ is a category with only arrows being the identity, then the product of the $F(j)$ is the projective limit while the coproduct is the inductive limit.

B.3 Abelian Category

Although generic categories are already some mathematical concepts really interesting to study, and we could say much more about them, we will add some more structure to them.

Definition B.3.1. A *preadditive* structure on a category $\mathcal{C}$ is a class of operations $+_{X,Y}$ on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ (we will drop the indices) such that $(\mathrm{Hom}_{\mathcal{C}}(X,Y),+_{X,Y})$ is an abelian group and such that the composition is bi-additive: $f\circ(g+h) = f\circ g + f\circ h$ and $(f+g)\circ h = f\circ h + g\circ h$ whenever the operations are defined. A category with such a preadditive structure is called a *preadditive category*.

A functor F between two preadditive categories is said to be *additive* if

$$F(f+g) = F(f) + F(g).$$

Suppose that the category $\mathcal{C}$ has a *zero object*[1] Z, then the composition $X \to Z \to Y$ provides a unique morphism that is the neutral element in $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ for the abelian group structure. We denote this morphism by 0 and call it the *zero morphism*.

This allows us to define the kernel and cokernel of a morphism. Similarly to the previous section, we use universal properties to define such an object. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. The *kernel* of f is a pair $(K,\epsilon\colon K \to X)$ such that $f\circ\epsilon = 0$ and for all $(K',\epsilon'\colon K' \to X)$ such that $f\circ\epsilon' = 0$, there is a unique morphism $g\colon K' \to K$ such that $\epsilon\circ g = \epsilon'$.

$$\begin{tikzcd}
K \arrow[r, "\epsilon"] \arrow[rr, bend left, "0"] & X \arrow[r, "f"] & Y. \\
& K' \arrow[u, "\epsilon'"] \arrow[ul, dotted, "\exists! g"] \arrow[ur, "0"'] &
\end{tikzcd}$$

On the other hand, the *cokernel* of f is a pair $(C,\rho\colon Y \to C)$ such that $\rho\circ f = 0$ and for all $\rho'\colon Y \to C'$ such that $\rho'\circ f = 0$, then there is a unique $g\colon C \to C'$ such that $g\rho = \rho'$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "0"] \arrow[dr, "0"'] & Y \arrow[r, "\rho"] \arrow[d, "\rho'"] & C. \arrow[dl, dotted, "\exists! g"] \\
& C' &
\end{tikzcd}$$

In linear algebra, we know that a linear map is injective if and only if its kernel is zero. We can generalise this statement in categories.

Proposition B.3.2. Let $\mathcal{C}$ be a preadditive category with zero object and kernels. Then, a morphism $f\colon X \to Y$ is a monomorphism if and only if $\ker f = 0$.

Definition B.3.3. A category is *additive* if it is preadditive, has a zero object and admits finite products.

In that case, the finite product $X\prod Y$ will be isomorphic to the finite coproduct $X\coprod Y$ and we will denote them by $X\oplus Y$. The object $X\oplus Y$ will have 4 canonical maps associated to it (coming from the universal property of the product and coproduct): $i_X\colon X \to X\oplus Y$, $i_Y\colon Y \to X\oplus Y$, $\pi_X\colon X\oplus Y \to X$ and $\pi_Y\colon X\oplus Y \to Y$.

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[1] An object $Z \in \mathrm{Ob}(\mathcal{C})$ such that for any other object X of $\mathcal{C}$, we have $\mathrm{Hom}_{\mathcal{C}}(X,Z) = \{*\}$ and $\mathrm{Hom}_{\mathcal{C}}(Z,X) = \{*\}$. i.e. there is one and only one morphism coming from this object to any other object and there is one unique morphism coming from any object to the zero object.

Moreover, if $X = X_1 \oplus \cdots X_n$ and $Y = Y_1 \oplus \cdots \oplus Y_m$, then a morphism $f\colon X \to Y$ is uniquely determined by the $f_{kj} = \pi_{Y_k} \circ f \circ i_{X_j}$. We can represent it as

$$f = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mn} \end{pmatrix}$$

and the composition of functions is given by matrix multiplication.

Definition B.3.4. A category is *abelian* if it is additive, it has kernels and cokernels, every monomorphism is a kernel and every epimorphism is a cokernel.

The last condition is equivalent to asking that for any morphism $f\colon X \to Y$ with kernel (K, ϵ) and cokernel (C, ρ), the (canonical) map $\overline{f}\colon \operatorname{coker}(\epsilon) \to \ker(\rho)$ is an isomorphism.

$$\begin{tikzcd}
K \arrow[r, "\epsilon"] & X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[r, "\rho"] & C. \\
 & \operatorname{coker}(\epsilon) \arrow[r, "\overline{f}"'] & \ker(\rho) \arrow[u, "\psi"'] &
\end{tikzcd}$$

Example B.3.5. If R is a ring, the category R-**Mod** of left modules over the ring R with linear maps is abelian. The abelian structure on the morphisms is given by point-wise addition. The zero object is the module 0, the finite products are given by the module products, similarly for the kernels, if $f\colon X \to Y$, we have

$$\ker f = \{x \in X : f(x) = 0\}$$

and

$$\operatorname{coker} f = Y/\operatorname{im}(f).$$

The fact that $\overline{f}$ is isomorphic comes from the first isomorphism theorem.

In particular, as $\mathbf{k}$-vector spaces can be seen as $\mathbf{k}$-modules with $\mathbf{k}$ a field, we have that the category $\mathbf{Vect_k}$ is abelian.

Example B.3.6. If $\mathcal{A}$ is an abelian category, then so is $\mathbf{Fun}(\mathcal{C}, \mathcal{A})$ for any category $\mathcal{C}$. The idea is that we define all the constructions pointwise. If $\alpha, \beta\colon F \to G$ are two natural morphisms, we define the sum $\alpha + \beta$ by

$$(\alpha + \beta)_X = \alpha_X + \beta_X\colon F(X) \to G(X),$$

as α_X and β_X are morphisms in $\mathcal{A}$ with the same source and target, their sum is well defined and it forms an abelian group. The zero object of this category is the functor sending every object to the zero object of the category $\mathcal{A}$ and every morphism to the zero morphism. We also define $F \prod G$ as the functor $(F \prod G)(X) = F(X) \prod G(X)$ and if $f\colon X \to Y$, $(F \prod G)(f)$ is the unique morphism making the following diagram commute,

$$\begin{tikzcd}
 & & F(Y) \\
F(X)\prod G(X) \arrow[urr, "F(f)\circ\pi_{F(X)}"] \arrow[drr, "G(f)\circ\pi_{G(X)}"'] \arrow[r, dotted] & F(Y)\prod G(Y) \arrow[ur, "\pi_{F(Y)}"'] \arrow[dr, "\pi_{G(Y)}"] & \\
 & & G(Y).
\end{tikzcd}$$

If $\alpha\colon F\to G$ is a natural morphism, the functor $\ker\alpha$ is defined by $(\ker\alpha)(X)=\ker(\alpha_X)$ and the map $(\ker\alpha)(f)$ is the unique map making the diagram commute. We construct it using the universal property of the kernel of α_Y:

$$\begin{tikzcd}
\ker\alpha_X \arrow[r, "\epsilon_X"] \arrow[d, dashed] & F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
\ker\alpha_Y \arrow[r, "\epsilon_Y"'] & F(Y) \arrow[r, "\alpha_Y"'] & G(Y).
\end{tikzcd}$$

The construction of $\text{coker}(\alpha)$ is symmetric. It still remains to show that these constructions respect the relevant universal properties; it can be done through diagram chasing.

If α is a monomorphism, then $\ker\alpha=0$, therefore $\ker\alpha_X=0$ for all X. It means that α_X is a monomorphism for each X. As $\mathcal{A}$ is abelian, it implies that α_X is a kernel for each X. Let us say that $\alpha_X=\ker\beta_X$ for each X. It only remains to show that $(\beta_X)_{X\in\text{Ob}(\mathcal{A})}$ defines a commutative diagram. It is direct through a simple diagram chasing. The case of epimorphisms is totally symmetric.

If $f\colon X\to Y$ is a morphism, we can define the *image* $\text{im}(f)$ of f through the following universal property: it is the object I and the monomorphism $m\colon I\to Y$ such that there is a morphism $e\colon X\to I$ with $f=me$ and if there is another object I', a morphism $e'\colon X\to I'$, and monomorphism $m'\colon I'\to Y$ such that $f=m'e'$ then there is a unique morphism $\varphi\colon I\to I'$ such that $m=m'\varphi$. In other words, we have the following commutative diagram.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "e"] \arrow[ddr, dashed, "e'"'] & & Y \\
 & I \arrow[ur, tail, "m"'] \arrow[d, dotted, "\exists!\varphi"] & \\
 & I' \arrow[uur, dashed, tail, "m'"'] &
\end{tikzcd}$$

For abelian categories, we have $\text{im}(f)=\ker(\text{coker}(f))$.

B.4 Exact Sequence

One of the main reasons for the study of abelian categories is that they provide a suitable framework to define exact sequences. These types of sequences are fundamental in homological algebra.

Proposition B.4.1. Let $\mathcal{A}$ be an abelian category, $f\colon X \to Y$ and $g\colon Y \to Z$ two morphisms of $\mathcal{A}$ such that $g \circ f = 0$. Denote by (C, ρ) the cokernel of f and $(\ker g, \alpha)$ the kernel of g. There is a unique morphism $\Theta_{f,g}\colon \ker(\rho) \to \ker(g)$ making the following diagram commute. Moreover, $\Theta_{f,g}$ is a monomorphism.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\overline{f}\varphi"'] & Y \\
\ker(\rho) \arrow[r, "\Theta_{f,g}"'] & \ker(g). \arrow[u, "\alpha"']
\end{tikzcd}$$

Using the morphism we have just constructed, we can now define exact sequences.

Definition B.4.2. If $\mathcal{A}$ is an abelian category, we say that the sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is *exact* if $g \circ f = 0$ and if the morphism $\Theta_{f,g}$ is an isomorphism.

A sequence of the form $X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_n$ is *exact* if $X_i \xrightarrow{f_i} X_{i+1} \xrightarrow{f_{i+1}} X_{i+2}$ is exact for all $1 \leq i \leq n-2$.

An exact sequence of the form $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is called a *short exact sequence*, also denoted as SES.

Example B.4.3. One of the most direct examples of a short exact sequence is obtained by using the biproduct. The sequence

$$0 \to X \xrightarrow{i_X} X \oplus Y \xrightarrow{\pi_Y} Y \to 0,$$

where i_X is the canonical injection from the coproduct and π_Y is the canonical projection from the product, is exact.

The above example is fundamental in the study of exact sequences as it has the following properties.

Proposition B.4.4. Let $\mathcal{A}$ be an abelian category. For a short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$, the following are equivalent.

1. There is a morphism $f'\colon Y \to X$ such that $f' \circ f = 1_X$ (f is said to be a *split monomorphism*);
2. There is a morphism $g'\colon Z \to Y$ such that $g \circ g' = 1_Z$ (g is said to be a *split epimorphism*);
3. There is an isomorphism $\gamma\colon Y \to X \oplus Z$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "1_X"'] & Y \arrow[r, "g"] \arrow[d, "\gamma"'] & Z \arrow[r] \arrow[d, "1_Z"'] & 0 \\
0 \arrow[r] & X \arrow[r, "i_X"'] & X \oplus Z \arrow[r, "\pi_Z"'] & Z \arrow[r] & 0.
\end{tikzcd}$$

If one of those conditions is fulfilled, the sequence is said to be *split exact*.

Definition B.4.5. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and F an additive functor between them.

- The functor F is *left exact* if for every $0 \to A \xrightarrow{f} B \xrightarrow{g} Z$ exact sequence of $\mathcal{A}$, the sequence $0 \to F(X) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is exact in $\mathcal{B}$.
- The functor F is *right exact* if for every $A \xrightarrow{f} B \xrightarrow{g} Z \to 0$ exact sequence of $\mathcal{A}$, the sequence $F(X) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ is exact in $\mathcal{B}$.
- The functor F is *exact* if any SES in $\mathcal{A}$ is sent by F to a SES in $\mathcal{B}$.

It can be shown that F is exact if and only if it is both left and right exact.

Example B.4.6. Let $\mathcal{A}$ be an abelian category and $X \in \operatorname{Ob}\mathcal{A}$. Then the functor $h_X = \operatorname{Hom}_{\mathcal{A}}(X,-)$ is left exact. Similarly, the functor $h^X = \operatorname{Hom}_{\mathcal{A}}(-,X)$ is left exact in the sense that if $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact, then

$$0 \to \operatorname{Hom}_{\mathcal{A}}(C,X) \xrightarrow{\operatorname{Hom}_{\mathcal{A}}(g,X)} \operatorname{Hom}_{\mathcal{A}}(B,X) \xrightarrow{\operatorname{Hom}_{\mathcal{A}}(f,X)} \operatorname{Hom}_{\mathcal{A}}(A,X)$$

is exact.

We will now present some classical results related to exact sequences.

Proposition B.4.7 (5-Lemma). Let $\mathcal{A}$ be an abelian category. Consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] & A_3 \arrow[r, "f_3"] \arrow[d, "h_3"'] & A_4 \arrow[r, "f_4"] \arrow[d, "h_4"'] & A_5 \arrow[d, "h_5"'] \\
B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r, "g_3"'] & B_4 \arrow[r, "g_4"'] & B_5.
\end{tikzcd}$$

Then,

1. If h_2, h_4 are monomorphisms and h_1 is an epimorphism, then h_3 is a monomorphism;
2. If h_2, h_4 are epimorphisms and h_5 is a monomorphism, then h_3 is an epimorphism;
3. If h_2, h_4 are isomorphism, h_1 is an epimorphism and h_5 is a monomorphism, then h_3 is an isomorphism.

We have the following well-known[2] lemma.

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[2] Its proof was even included in the 1980 movie "It's my Turn"! See for instance `https://www.youtube.com/watch?v=etbcKWEKnvg&ab_channel=philsproof`.

Proposition B.4.8 (Snake's Lemma). Let $\mathcal{A}$ be an abelian category. Consider the following diagram with exact rows

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d, "f"'] & B \arrow[r, "p"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "i"'] & B' \arrow[r] & C'.
\end{tikzcd}$$

Then there is a map $\partial\colon\ \ker h \to \operatorname{coker} f$ making the sequence

$$\ker f \to \ker g \to \ker h \overset{\partial}{\to} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h$$

exact.

The name of the result comes from the shape that the morphism ∂ takes when tracing the diagram.

$$\begin{tikzcd}
 & \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] \arrow[dddll, rounded corners, "\partial"] & \\
 & A \arrow[r] \arrow[d, "f"] & B \arrow[r, "p"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i"] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h. &
\end{tikzcd}$$

B.5 Basic Facts on Homological Algebra

Homological algebra aims to study homology, i.e. the quotients “$\ker d/\operatorname{im} d$” of complexes from a purely theoretical point of view. Different types of homologies can be constructed: Simplicial, de Rahm, Hochschild, Dolbeaux, etc., depending on the choice of complex. The goal of homological algebra is to provide a general framework to study them. These quotients provide some relevant information on the complexes and, more generally, on the category the complexes are constructed on. This small section will only briefly present the basics of the subject. Much more can be done, such as defining the Tor functor, spectral sequences, linking the homology with topology, etc. In addition to the previously mentioned course I followed, I also used [56] as a source for this section.

B.5.1 Projective and Injective Objects

We will start by defining a crucial class of objects in an abelian category.

Definition B.5.1. Let $\mathcal{A}$ be an abelian category. An object X in $\mathcal{A}$ is *projective* if the functor $h_X = \mathrm{Hom}_{\mathcal{A}}(X, -)$ is exact.

Dually, an object X of $\mathcal{A}$ is *injective* if the functor $\mathrm{Hom}_{\mathcal{A}}(-, X)$ is exact.

This exactness condition on the functor h_X is equivalent to several other conditions.

Proposition B.5.2. Let $\mathcal{A}$ be abelian and $X \in \mathrm{Ob}(\mathcal{A})$. The following are equivalent.

1. X is projective;
2. For all $f\colon A \twoheadrightarrow B$ epimorphism, for all $g\colon X \to B$, there is a morphism $\overline{g}\colon X \to A$ such that $f\overline{g} = g$ (note that $\overline{g}$ is not necessarily unique):
$$\begin{tikzcd} & X \arrow[d, "g"] \arrow[dl, dashed, "\exists\ \overline{g}"'] \\ A \arrow[r, two heads, "f"'] & B; \end{tikzcd}$$
3. Any SES $0 \to A \xrightarrow{f} B \xrightarrow{g} X \to 0$ splits;
4. Any epimorphism $g\colon B \twoheadrightarrow X$ is a split epimorphism.

Proposition B.5.3. Let $\mathcal{A}$ be an abelian category. Consider $(X_i)_{i\in I}$ a family of objects in $\mathcal{A}$. Then $\coprod_{i\in I} X_i$ is projective if and only if each X_i is projective.

In particular, taking $I = \{1, \cdots, n\}$ to be a finite set, we have that $\bigoplus_{i=1}^{n} X_i$ is projective if and only if X_i is projective for all $1 \leq i \leq n$.

The result on the injective objects can be obtained similarly by passing through the opposite category: X is projective in $\mathcal{A}$ if and only if X is injective in $\mathcal{A}^{\mathrm{op}}$.

B.5.2 Cochain Complex

Definition B.5.4. Let $\mathcal{A}$ be an abelian category. A (cochain) *complex* in $\mathcal{A}$ is a sequence
$$X^\bullet = \cdots \to X^{-1} \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^0} X^1 \to \cdots$$
with X^i objects of $\mathcal{A}$ for $i \in \mathbb{Z}$ and $d_X^i\colon X^i \to X^{i+1}$ morphisms such that $d_X^i \circ d_X^{i-1} = 0$ for all $i \in \mathbb{Z}$.

A morphism of complexes $f^\bullet\colon X^\bullet \to Y^\bullet$ is a family of morphisms $f^i\colon X^i \to Y^i$ such that for all $i \in \mathbb{Z}$, we have $d_Y^i f^i = f^{i+1} d_x^i$. Composition of morphisms is done component-wise: $(gf)^i = g^i f^i$.

This defines the category of complexes denoted by $\mathcal{C}(\mathcal{A})$.

We can define the *chain complex* similarly, with objects X_i and maps $d_i^X\colon X_i \to X_{i-1}$, the only difference is that the map d_i^X decreases the degree of the object instead of increasing it. We use the convention of numbering the elements of a chain complex with indices and the ones of a cochain complex with superscripts.

Proposition B.5.5. If $\mathcal{A}$ is abelian, so is $\mathcal{C}(\mathcal{A})$.

We can now define the cohomology of a cochain.

Definition B.5.6. Given an abelian category $\mathcal{A}$, we have define these functors:

- The n-*cocycle* $Z^n \colon \mathcal{C}(\mathcal{A}) \to \mathcal{A}, X^\bullet \mapsto \ker(d^n)$.
- The n-*coboundary* $B^n \colon \mathcal{C}(\mathcal{A}) \to \mathcal{A}, X^\bullet \mapsto \operatorname{im}(d^{n-1})$.
- The n-*cohomology* $H^n \colon \mathcal{C}(\mathcal{A}) \to \mathcal{A}, X^\bullet \mapsto \operatorname{coker}(\operatorname{im} d^{n-1} \to \ker d^n)$.

Proposition B.5.7 (Long exact cohomological sequence). If

$$0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$$

is a SES in $\mathcal{C}(\mathcal{A})$, then there is a long exact sequence in $\mathcal{A}$ of the form

$$\begin{tikzcd}
\cdots \arrow[r] & H^{-1}(X) \arrow[r] & H^{-1}(Y) \arrow[r] & H^{-1}(Z) \arrow[dll, out=0, in=180] & \\
& H^0(X) \arrow[r] & H^0(Y) \arrow[r] & H^0(Z) \arrow[dll, out=0, in=180] & \\
& H^1(X) \arrow[r] & H^1(Y)) \arrow[r] & H^1(Z) \arrow[ddll, dashed, out=0, in=180] & \\
& & & & \\
& H^n(X) \arrow[r] & H^n(Y) \arrow[r] & H^n(Z) \arrow[r] & \cdots .
\end{tikzcd}$$

B.5.3 The Ext Functor

Now that we have defined projective objects and cochain complexes, we can combine those notions to define the Ext functor. It is an example of a derived functor. In order to do that, we first need to define the projective and injective resolutions.

Definition B.5.8. Let $\mathcal{A}$ be an abelian category. This category is said to *have enough projectives* if for every object $X \in \mathcal{A}$ there is a projective object $P \in \mathcal{A}$ and an epimorphism $P \twoheadrightarrow X$.

It has *enough injectives* if for every object $X \in \operatorname{Ob}(\mathcal{A})$ there is an injective object $I \in \operatorname{Ob}(\mathcal{A})$ and a monomorphism $X \hookrightarrow I$.

This technical definition allows us to make sense of the following definition.

Definition B.5.9. Let $\mathcal{A}$ be an abelian category with enough projectives. For an object $X \in \mathcal{A}$, we can construct a *projective resolution* of X as a complex $P^\bullet$ with $P^i = 0$ for $i > 0$, P^i projective for $i \leq 0$ and such that the augmented complex (i.e. the complex $P^\bullet$ with $P^1 = X$ and the map d^0 given by the epimorphism $P \twoheadrightarrow X$)

$$\cdots \to P^{-3} \to P^{-2} \to P^{-1} \to P^0 \to X \to 0 \to 0 \to \cdots$$

is exact.

Dually, if $\mathcal{A}$ has enough injective, an *injective resolution* of X is a complex $I^\bullet$ such that $I^i = 0$ for $i < 0$, I^i is injective for all $i \geq 0$ and such that the augmented complex

$$\cdots \to 0 \to 0 \to X \to I^0 \to I^1 \to I^2 \to I^3 \to \cdots$$

is exact.

Note that such a resolution is not unique, but it can be rendered unique by passing to the homotopy category. This is outside the scope of this work, therefore, we will not go into further details.

Definition B.5.10. Let $\mathcal{A}$ be an abelian category with enough projectives. The *projective dimension* of an object $A \in \mathcal{A}$ is

$$\text{pr.dim}(A) = \inf\{j \in \mathbb{N} : \exists \text{ a projective resolution } P \text{ of } A \;:\; P_k = 0 \;\forall k > j\}$$

if A has a finite projective resolution and $\text{pr.dim}(A) = \infty$ otherwise. We can also define the *global dimension* of $\mathcal{A}$ as

$$\text{gl.dim}(\mathcal{A}) = \sup\{\text{pr.dim}(A) : A \in \mathcal{A}\} \in \mathbb{N} \cup \{\infty\}.$$

Definition B.5.11. Let $\mathcal{A}$ be an abelian category with enough injectives and $X \in \text{Ob}(\mathcal{A})$. Then, we define the functor $\text{Ext}^n_{\mathcal{A}}(X, -) : \mathcal{A} \to \mathbf{Ab}$ on each object Y of $\mathcal{A}$ by first constructing an injective resolution

$$\cdots \to 0 \to I^0 \to I^1 \to I^2 \to I^3 \to \cdots$$

of Y. Then we apply the functor $\text{Hom}_{\mathcal{A}}(X, -)$ to this sequence, it gives

$$\cdots \to 0 \to \text{Hom}_{\mathcal{A}}(X, I^0) \to \text{Hom}_{\mathcal{A}}(X, I^1) \to \text{Hom}_{\mathcal{A}}(X, I^2) \to \text{Hom}_{\mathcal{A}}(X, I^3) \to \cdots .$$

After which, we apply the functor H^n to this complex.

Note that this definition is pretty sketchy: we should use the notion of homotopy category to properly define the image of the morphisms in $\mathcal{A}$ by this functor. However, although fascinating, it is rather technical and outside the scope of the (very) limited use we will do of this functor in this master thesis. We will only use the following results.

Proposition B.5.12 (Long exact Ext sequence). If

$$0 \to X \to Y \to Z \to 0$$

is a SES in $\mathcal{A}$ and $M \in \text{Ob}(\mathcal{A})$, then there is a long exact sequence in $\mathcal{A}$ of the form

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^0_{\mathcal{A}}(M,X) \arrow[r] & \mathrm{Ext}^0_{\mathcal{A}}(M,Y) \arrow[r] & \mathrm{Ext}^0_{\mathcal{A}}(M,Z) \arrow[dll] \\
 & \mathrm{Ext}^1_{\mathcal{A}}(M,X) \arrow[r] & \mathrm{Ext}^1_{\mathcal{A}}(M,Y) \arrow[r] & \mathrm{Ext}^1_{\mathcal{A}}(M,Z) \arrow[dll] \\
 & \mathrm{Ext}^2_{\mathcal{A}}(M,X) \arrow[r] & \mathrm{Ext}^2_{\mathcal{A}}(M,Y) \arrow[r] & \mathrm{Ext}^2_{\mathcal{A}}(M,Z) \arrow[ddll, dashed] \\
 & & & \\
 & \mathrm{Ext}^n_{\mathcal{A}}(M,X) \arrow[r] & \mathrm{Ext}^n_{\mathcal{A}}(M,Y) \arrow[r] & \mathrm{Ext}^n_{\mathcal{A}}(M,Z) \arrow[r] & \cdots .
\end{tikzcd}$$

Proposition B.5.13. The functor $\mathrm{Ext}^1(P,-) \simeq 0$ if and only if P is projective. Moreover, in this case, we further have that $\mathrm{Ext}^n(P,-) \simeq 0$ for all $n \geq 1$.

Proposition B.5.14. If the category $\mathcal{A}$ is of global dimension less than n, then $\mathrm{Ext}^k(A,B) = 0$ for all $k > n$ and $A, B \in \mathcal{A}$.

Proposition B.5.15. The Ext functor commutes with direct products in the second variable and it takes direct sums in the first variable to products. In other words, for a set I and $X_i \in \mathcal{A}$ for all $i \in I$, we have

$$\mathrm{Ext}^1(\bigoplus_{i\in I} M_i, X) \simeq \prod_{i\in I} \mathrm{Ext}^1(M_i, X)$$

and

$$\mathrm{Ext}^1(X, \prod_{i\in I} M_i) \simeq \prod_{i\in I} \mathrm{Ext}^1(X, M_i).$$

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List of Symbols

Categories

$\mathcal{C}, \mathcal{D}$	Generic categories
$\mathcal{A}$	Abelian category
Set	Category of Sets
Ab	Category of abelian groups
Vect	Category of vector spaces
vect	Category of finite dimensional vector spaces
R-**Mod**	Category of finite-dimensional left modules over the ring R
Top	Category of topological spaces
$\mathbf{Fun}(\mathcal{C}, \mathcal{D})$	Category of functors from $\mathcal{C}$ to $\mathcal{D}$
$\mathscr{C}(\mathcal{A})$	Category of chain complexes on $\mathcal{A}$
$\mathrm{Rep}(Q)$	The category of finite dimensional representations of Q
$\widetilde{\mathrm{Rep}(Q)}$	The category of representations of Q
$\mathcal{C}^{\mathrm{op}}$	The dual category
$f : X \rightarrowtail Y$	f is a monomorphism
$f : X \hookrightarrow Y$	f is a monomorphism
$f : X \twoheadrightarrow Y$	f is an epimorphism
$\varprojlim_{i \in I} F(i)$	A projective limit of F
$\varinjlim_{i \in I} F(i)$	An inductive limit/colimit of F
$\mathrm{Ext}^n_{\mathcal{A}}(X, -)$	The Ext functor

Number Sets

$\mathbb{N}$	The set of natural numbers $\{0, 1, 2, 3, 4, \cdots\}$

$\mathbb{N}_0$	The set $\mathbb{N}\backslash\{0\} = \{1, 2, 3, 4, \cdots\}$
$\mathbb{Z}$	The set of integer numbers $\{\cdots, -2, -1, 0, 1, 2, 3, 4, \cdots\}$
$\mathbf{k}$	A field
$\emptyset$	The empty set
$\mathcal{P}(S)$	The power set of the set S

Quivers

Q	Quiver
Q_0	Set of vertices of the quiver Q
Q_1	Set of arrows of the quiver Q
ϵ_i	Lazy path starting at i
kQ	Path algebra
$S(i)$	Simple representation associated to the vertex i
$P(i)$	Projective representation associated to the vertex i
R_d	Space of representation of dimension vector d
q	Tits form
$V([i, j])$	Interval representation

Other Symbols

i_j	The canonical injection $i_j : E_j \to \bigoplus_{j\in J} E_j$
π_j	The canonical projection $\pi_j : \bigoplus_{j\in J} E_j \to E_j$
M^t	Sublevel set
$H_n(-)$	The n-homology with coefficient in a field $\mathbf{k}$ functor
$\mathbb{I}^J, \chi_J$	Interval module of support J
dgm	Persistence diagram of the persistence module $\mathbb{V}$
$B(\mathbb{V})$	Barcode of the persistence module $\mathbb{V}$

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