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Auteur : Bétermier, Alvaro
Promoteur(s) : Nicolay, Samuel
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Department of Mathematics

Nuclearity in locally convex spaces and its application to Gelfand–Shilov spaces

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Author :
Alvaro Bétermier

Supervisor :
Samuel Nicolay

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Introduction

This master's thesis explores the theory of nuclear spaces and a particular class of spaces of ultradifferentiable functions known as Gelfand–Shilov spaces. The theory of nuclearity was initially developed by Alexandre Grothendieck in his seminal work *Produits tensoriels et espaces nucléaires* [4], where he investigated the possible locally convex topologies that may be imposed on the tensor product of locally convex spaces. Given two locally convex spaces E and F, there are several ways to endow their tensor product $E \otimes F$ with a topology; among the most natural are the π-topology and the ε-topology. Grothendieck's original definition of nuclear spaces was based on these constructions. However, in line with contemporary conventions, this thesis adopts a different approach, defining nuclear spaces via the concept of nuclear operators and local Banach spaces. Connections with tensor products will be examined, though not immediately.

Beyond their relevance to tensor products, nuclear spaces—particularly nuclear Fréchet spaces—exhibit numerous remarkable properties, notably their connections to weakly and absolutely summable sequences and the Dynin–Mityagin theorem, which will be presented in detail.

The final part of this thesis is devoted to the study of a variant of the so-called Gelfand–Shilov spaces, introduced by Gelfand and Shilov in [3, 2]. These are spaces of ultradifferentiable functions characterized by simultaneous constraints on both the decay of the function at infinity and the growth of its derivatives. Formally, given a positive weight function w, a multi-indexed sequence $M = (M_\alpha)_{\alpha \in \mathbb{N}^d}$ of positive real numbers, and a parameter $q \in [1, +\infty]$, we define the Banach space

$$S^M_{w,q} := \left\{ f \in C^\infty(\mathbb{R}^n) : \|f\|_{S^M_{w,q}} := \sup_{\alpha \in \mathbb{N}^d} \left(M_\alpha \|f^{(\alpha)} w\|_q \right) < +\infty \right\}.$$

Here, w acts as a weight on f and its derivatives, while M governs the growth constraints. The Gelfand–Shilov spaces studied in this thesis are constructed as suitable projective limits of such Banach spaces. Compared to the classical Schwartz space $\mathscr{S}(\mathbb{R}^d)$, which also imposes decay and growth conditions, Gelfand–Shilov spaces feature more intricate constraints: for fixed w and M, the norm depends on the simultaneous control of the decay of f and all of its derivatives, making the role of M more transparent.

The first section of the thesis provides preliminary definitions and results that do not naturally fit into later sections but are essential for the development of the theory. These include standard notions such as the operator norm for continuous linear maps between Banach spaces, duality theory, and the completion of locally convex spaces. Classical theorems are occasionally stated without proof, accompanied by appropriate references. However, in the case of Banach space completions, a detailed construction is included to shed light on the underlying concepts.

The second section presents a detailed and self-contained introduction to nuclear spaces and some of their fundamental properties. A central result is a characterization of nuclear Fréchet spaces in terms of weakly and absolutely summable sequences. Much of this material is drawn from Pietsch's monograph *Nuclear Locally Convex Spaces* [11], but the results have been carefully selected and reorganized to provide clear and efficient proofs of the main theorems. A notable highlight is the Dynin–Mityagin theorem, which asserts that, under suitable conditions, if a Fréchet space E possesses a Schauder basis, then the norms of elements in E can be estimated solely in terms of their basis coefficients. More precisely, if the topology of a nuclear Fréchet space E is generated by a system of norms and $(e_j)_{j\in\mathbb{N}}$ is a Schauder basis with corresponding coefficient functionals $(c_j)_{j\in\mathbb{N}}$, i.e.,

$$x = \sum_j c_j(x)e_j, \quad \forall x \in E,$$

then for every continuous norm p on E, there exists another continuous norm q such that

$$p(x) \le p'(x) := \sum_j |c_j(x)|p(e_j) \le q(x).$$

In particular, the locally convex topology of E can be recovered from the family of norms p' constructed in this way. While the presentation in Meise and Vogt's book *Introduction to Functional Analysis* [8] omits the assumption that E possesses a fundamental system of norms, this assumption is needed in certain steps of the proof. The section concludes with a complete characterization of a class of Fréchet spaces known as Köthe spaces, which generalize classical ℓ^p spaces and serve as important examples in the theory of nuclearity.

The third section is a concise exposition on tensor products. It introduces the definition of the tensor product of locally convex spaces and presents the two canonical topologies mentioned earlier: the π- and ε-topologies. A central result is that these two topologies coincide when at least one of the spaces involved is nuclear—a fact that forms the basis of Grothendieck's original definition of nuclear spaces.

The final section of the thesis is devoted to the study of Gelfand–Shilov spaces. This part is primarily based on the doctoral thesis of Lenny Neyt, *Topological Properties and Asymptotic Behavior of Generalized Functions* [9], particularly the section titled "Topological properties of ultradifferentiable function spaces and their duals." The first half of the section introduces the Gelfand–Shilov spaces, explores some of their basic properties, and investigates the influence of the parameters involved. The second half presents a precise characterization of the nuclearity of these spaces under natural conditions on the defining parameters. The proof draws upon many concepts developed earlier in the thesis, including summable sequences, the Dynin–Mityagin theorem, and the nuclearity of Köthe spaces. Certain results that are not proved in Neyt's thesis are worked out in full here—for example, the completeness of the spaces $S^M_{w,q}$, which is not an immediate consequence of the definitions. The proof relies on a lemma adapted from Neyt's argument, modified to hold in arbitrary dimensions, as the original proof appears valid only in the one-dimensional setting.

Contents

Notations

- We always includes zero in the set $\mathbb{N}$ of all natural numbers.
- The field of scalars of the vectorial spaces considered will always be written $\mathbb{K}$ and can be either $\mathbb{R}$ or $\mathbb{C}$.
- All the locally convex spaces are supposed to be Hausdorff spaces and will often be refered as lcs.
- The topological dual of a locally convex space E is written E'.
- The set of all continuous seminorms on E will be denoted by $\operatorname{csn}(E)$, it is the largest fundamental system of seminorms of E.
- Given a locally convex space E and a seminorm $p \in \operatorname{csn}(E)$, we write $b_p(x, r)$ for the ball associated to p centered at $x \in E$ and of radius $r > 0$, $b_p(x) = b_p(x, 1), b_p = b_p(0, 1)$. Moreover, we write $b_p^\circ = \{a \in E' : |a| \leq p\}$, the polar of b_p.
- If possible, all families on a topological space will be denoted by bold letters except for families on the field of scalars that will be represented by Greek letters.
- Given two topological spaces X and Y, the space $C(X, Y)$ is the space of all continuous mapping from X to Y. If $Y = \mathbb{K}$, we write $C(X)$ instead of $C(X, \mathbb{K})$.
- Except in section 3, if f is a function on a set X and g a function on a space Y and if f and g takes values in $\mathbb{K}$, we write $f \otimes g : X \times Y : (x, y) \mapsto f(x)g(y)$.
- In the last section, the letter d will always stand for the dimension of some finite dimensional space and will therefore always be a non-zero natural number.
- The notation $K \Subset \mathbb{R}^d$ signifies that K is a compact subset of $\mathbb{R}^d$.

1 Preliminaries

This section contains some definitions and results that will be needed in the following sections.

1.1 Linear operators between normed spaces

Definition 1.1.1. Let (E, p) and (F, q) be two normed spaces and $T \in L(E, F)$. The operator norm $\beta(T)$ is defined by

$$\beta(T) = \sup\{q(T(x)) : x \in E, p(x) \leq 1\}.$$

Definition 1.1.2. Let E and F be two lcs and $T \in L(E, F)$.

The dual operator (also named transposed operator) corresponding to T is the linear operator T' from F' to E' defined by

$$\langle T'b, x\rangle = \langle b, Tx\rangle$$

for $b \in F'$, $x \in E$.

If E and F are Hilbert spaces, the adjoint operator of T defined in [1] will be written T^*.

Proposition 1.1.3. *If E and F are normed space, we have the identity*

$$\beta(T) = \beta(T').$$

If moreover E and F are Hilbert spaces, then

$$\beta(T) = \beta(T^*).$$

1.2 Duality theory

We will now briefly discuss about the notion of dual systems and state two important results in this topic. The definitions and results of this subsection comes from [8].

Definition 1.2.1. Let E be a $\mathbb{K}$-linear space and F a linear subset of the algebraic dual E^* of E. The pair (E, F) is said to be a dual pair if F separates the points of E, i.e. if for all $x, y \in E$ with $x \neq y$, there exists $a \in F$ such that $a(x) \neq a(y)$. Equivalently, (E, F) is a dual pair if they satisfy the following property : if $x \in E$ is a point such that $a(x) = 0$ for all $a \in F$ then $x = 0$.

Remark 1.2.2. Given a locally convex space E, the two most important dual pairs associated to E are (E, E') and (E', E) where E is viewed as a subset of E' by the mapping $E \to E'^* : e \mapsto \langle \cdot, e\rangle$. Similarly, given a dual pair (E, F), the pair (F, E) can also be viewed as a dual pair by the mapping $E \to F^* : e \mapsto \langle \cdot, e\rangle$.

Definition 1.2.3. Given a dual pair (E, F), a topology $\mathscr{T}$ on E is said to be (E, F) admissible if $(E, \mathscr{T})' = F$.

An important theorem of duality theorem concerns the compact sets of E given a (E, F)-admissibe topology on E.

Theorem 1.2.4 (Mackey theorem)**.** *Let (E, F) be a dual pair and let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two (E, F)-admissible topologies on E. Then a subset of E is bounded in $(E, \mathscr{T}_1)$ if and only if it is bounded in $(E, \mathscr{T}_2)$.*

Proof. A proof can be find in [8]. □

Given a dual pair (E, F), there exists several ways to endow E or F with a natural locally convex structure. We will explore three of them. By the preceding remark, each construction of a locally concex structure on E induce a locally convex structure on F by considering the dual pair (F, E) and reciprocally.

Definition 1.2.5. Let (E, F) be a dual pair. If for all finite subset A of F we consider the seminorm p_A defined on E by

$$p_A(x) = \max_{T \in F} |\langle T, x \rangle|$$

then E can be endowed by the fundamental system of seminorms $\{p_A : A \subset F, \#A < +\infty\}$. The set E endowed by this system is written $(E, \sigma(E, F))$.

Definition 1.2.6. Let (E, F) be a dual pair. If for all absolutely convex $\sigma(F, E)-$compact subset M of F we consider the seminorm p_M defined on E by

$$p_M(x) = \max_{T \in F} |\langle T, x \rangle|,$$

then E can be endowed by the fundamental system of seminorms $\{p_M : M \subset F, M \text{ absolutely convex and } \sigma(F, E) - \text{compact}\}$. The set E endowed by this system if written $(E, \tau(E, F))$.

Given a dual pair (E, F) and a topology $\mathscr{T}$ on E it is natural to ask in which condition on $\mathscr{T}$ is $\mathscr{T}$ a (E, F)-admissible topology. This question is totally answered by the following result by Mackey and Arens (see [8] for a proof of this result).

Proposition 1.2.7 (Mackey-Arens)**.** *Let (E, F) be a dual pair. For a topology $\mathscr{T}$ on E to be such that $(E, \mathscr{T})' = F$ it is necessary and sufficient that $\sigma(E, F) \subset \mathscr{T} \subset \tau(E, F)$.*

We will be interested in one more locally convex structure on F.

Definition 1.2.8. Let (E, F) be a dual pair. If for all compact subset K of E we consider the seminorm p_K defined on F by

$$p_M(T) = \max_{x \in K} |\langle T, x \rangle|$$

then F can be endowed by the fundamental system of seminorms $\{p_K : K \subset E, K \text{ compact}\}$. The set F endowed by this system if written $(F, c) = F_c$.

We need the following lemma from [5].

Lemma 1.2.9. *Let E be a locally convex space and A en equicontinuous subset of E'. Then the topology induced on A by $(E')_c := (E'_c)$ coincides with the topology induced by $\sigma(E', E)$ on A.*

With this lemma we can prove the following proposition that will be used when studying ε-product in section 3.

Proposition 1.2.10. *Let E be a locally convex space. If $p \in csn(E)$ then b_p° is compact in E'_c.*

Proof. It is a direct consequence of the previous lemma and the fact that b_p° is compact in $(E', \sigma(E', E))$ by Alaoglu's theorem. □

1.3 Completion of nuclear locally convex spaces

Definition 1.3.1. Let E and F be two locally convex spaces. We write $E \simeq F$ if there exists a linear bijective homeomorphism $T : E \mapsto F$.

In this case, for all $p \in \mathrm{csn}(E)$, there exists $q \in \mathrm{csn}(F)$ such that $q \circ T = p$. Indeed, the continuous seminorm $q = p \circ T^{-1}$ verifies the relation.
Similarly, for all $q \in \mathrm{csn}(F)$, there exists $p \in \mathrm{csn}(E)$ such that $p \circ T^{-1} = q$.

An important notion that will be used in the next section is the following,

Definition 1.3.2. Let E be a lcs, a completion $\widehat{E}$ of E is a complete lcs that admits a dense locally convex subspace F such that $E \simeq F$.

We will prove that all normed spaces have a completion.

Proposition 1.3.3. *Every normed space $(X, \|.\|)$ has a completion $\widehat{X}$.*

Proof. Intuitively, we would like to add to X elements representing the limits of the Cauchy sequences of X, these limits being considered as distinct as soon as the difference of the associated sequences does not tend towards 0 at infinity.

Formally, let C be the set of all Cauchy sequences of X and let c_0 be the set of all sequences of X that vanishes at infinity. Finally, let $\widehat{X}$ be the set C/c_0.

By the completeness of $\mathbb{R}^+$, a seminorm p is naturally defined on C by

$$p((x_j)_{j\in\mathbb{N}}) = \lim_{j\to\infty} \|x_j\|.$$

With this definition, it is clear that $c_0 = \{\mathbf{x} \in C : p(\mathbf{x}) = 0\}$ is a closed linear subspace of C. We can then endow $\widehat{X} = C/c_0$ with the natural seminorm of the quotient. For simplicity, this seminorm will also be denoted by p.

It is clear that p is a norm on $\widehat{X}$. Moreover, if we set $j : X \to \widehat{X} : x \mapsto (x, x, x, \ldots) + c_0$, then it is easy to prove that j is an isometry and that $j(X)$ is dense in $\widehat{X}$.

It remains to prove that $\widehat{X}$ is complete. Let then $(\mathbf{x}^N)_{N\in\mathbb{N}}$ be a Cauchy sequence of C. Let N be an arbitrary natural number, since $\mathbf{x}^N$ is a Cauchy sequence of X, there exists $J_N \in \mathbb{N}$ such that

$$\|x_k^N - x_j^N\| < \frac{1}{N} \text{ for all } k, j \geq J_N.$$

We will now prove that if $\mathbf{y} = (y_N)_{N\in\mathbb{N}}$ is the sequence of X defined by $y_N = x_{J_N}^N$ for all $N \in \mathbb{N}$, then $\mathbf{y}$ is in C and $\lim_N \mathbf{x}^N = \mathbf{y}$ in C.

- $\mathbf{y} \in C$: Let $\varepsilon > 0$, since the sequence $(\mathbf{x}^N)_{N\in\mathbb{N}}$ is a Cauchy sequence of C, there exists $N_0 > 1/\varepsilon$ such that

  $$p(\mathbf{x}^M - \mathbf{x}^N) < \varepsilon \text{ for all } M, N \geq N_0.$$

  In particular, if $M, N \geq N_0 > 1/\varepsilon$ are fixed, there exists $J \in \mathbb{N}$ such that

  $$\|x_j^M - x_j^N\| < 2\,\varepsilon \text{ for all } j \geq J.$$

  For these M and N, we have for $j = \max\{J_M, J_N, J\}$ that

  $$\|y_M - y_N\| \leq \|y_M - x_j^M\| + \|x_j^M - x_j^N\| + \|x_j^N - y_N\| < \frac{1}{M} + 2\,\varepsilon + \frac{1}{N} < 4\,\varepsilon\,.$$

  Since $M, N \geq N_0$ are chosen arbitrarily, this proves that $y \in C$.

- $\lim_N \mathbf{x}^N = \mathbf{y}$: Let $\varepsilon > 0$ and let $N_0 > 1/\varepsilon$ be such that

  $$\|y_M - y_N\| < \varepsilon \text{ for all } M, N \geq N_0.$$

  If $N > N_0$, we have

  $$\|y_j - x_j^N\| \leq \|y_j - y_N\| + \|y_N - x_j^N\| < 2\,\varepsilon \text{ for all } j \geq \max\{N_0, J_N\}$$

  hence $p(\mathbf{y} - \mathbf{x}^N) < 2\varepsilon$ for all $N \geq N_0$.

To obtain the conclusion, we simply need to compose with the projection on the quotient. □

As stated earlier, we have the more general theorem, whose a proof can be found in [8]

Theorem 1.3.4. *Each locally convex space E has a completion $\widehat{E}$.*

We will now prove that the completion of a locally convex spaces is essentially unique. The following lemma will be the key argument to prove this result.

Lemma 1.3.5. *Let E and F be two lcs and let $\widehat{E}$ and $\widehat{F}$ be completion of E and F respectively. For any operator $T \in L(E,F)$, there is a unique operator $\tilde{T} \in L(\widehat{E},\widehat{F})$ such that $\widetilde{T}|_E = T$.*

Proof. It is a direct consequence of the fact that the image by T of a Cauchy sequence of E is a Cauchy sequence of F. □

Proposition 1.3.6. *If E_1 and E_2 are two completions of the same locally convex space E, then $E_1 \simeq E_2$.*

Proof. Let F_1 and F_2 be dense subsets of E_1 and E_2 respectively such that $E \simeq F_1$ and $E \simeq F_2$, we have then $F_1 \simeq F_2$ and by the preceding lemma and the definition of $\simeq$, we have $E_1 \simeq E_2$. □

From now on, we will say that $\widehat{E}$ is the completion of E and not a completion of E. Since $E \simeq F$ for a dense subspace F of $\widehat{E}$, we will always assume that E is a (dense) subset of its completion $\widehat{E}$.

Since the seminorms defined on the completion $\widehat{E}$ of E extends the one of E, we will use the same notation for both of them.

2 Nuclear spaces

The first part of this master's thesis will focus on the study of the so-called nuclear spaces. We will define those spaces then give an important characterization of them. Finally we will prove some of their properties and a simple yet non trivial family of examples.

This section is mainly based on Pietsch's book [11].

2.1 Definition

Definition 2.1.1. Let E be a locally convex space and $p \in \operatorname{csn}(E)$, we denote by E_p the Banach space obtained by completing the normed space $\tilde{E}_p := (E,p)/p^{-1}(0)$. E_p is refered as the local Banach space of E associated to p.

If π_p represents the canonical projection from E to $\tilde{E}_p$ and j_p represents the canonical inclusion from $\tilde{E}_p$ to E_p, we set

$$\iota^p = j_p \circ \pi_p.$$

Using the lemma 1.3.5, it is easy to see that if $q, p \in \operatorname{csn}(E)$ are such that $q \geq p$, then there exists a unique linear continuous operator ι_q^p from E_q to E_p such that

$$\iota^p = \iota_q^p \circ \iota^q.$$

We can now define the nuclear operators which will be used to define the nuclear spaces.

Definition 2.1.2. Let E, F be two Banach spaces, a linear operator $T : E \to F$ is said to be nuclear if there exists sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ we have $a_n \in E'$, $b_n \in F$,

$$\sum_{n=0}^{+\infty} \|a_n\|_{E'} \|b_n\|_F < +\infty$$

and

$$T(x) = \sum_{n\in\mathbb{N}} \langle a_n, x\rangle b_n \quad \text{for all } x \in E.$$

The space $\mathscr{N}(E,F)$ of all nuclear operators between E and F is linear and we endow this space with the norm defined naturally by

$$\nu(T) = \inf \left\{ \sum_{n=0}^{+\infty} \|a_n\|_{E'} \|b_n\|_F \right\}$$

where the infimum is taken over all the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ representing T.

The proof that $\mathscr{N}(E,F)$ is linear and that ν defines a seminorm on this space can be found in [11]. The fact that ν is a norm comes then directly from the observation that we always have $\beta(T) \leq \nu(T)$ so $\nu(T) = 0$ implies $\beta(T) = 0$ and then $T = 0$.

Proposition 2.1.3. *Let E and F be two normed spaces. If $(T_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence on $\mathscr{N}(E,F)$ and if there exists an operator $T \in L(E,F)$ such that $\lim_{n\to\infty} T_n x = Tx$ for all $x \in E$, then T is nuclear and $\lim_{n\to\infty} T_n = T$ in $\mathscr{N}(E,F)$.*

Proof. Let $(T_{n_j})_{j\in\mathbb{N}}$ a subsequence of $(T_n)_{n\in\mathbb{N}}$ such that for all $j \in \mathbb{N}_0$ we have

$$\nu(T_{n_j} - T_{n_{j-1}}) < 2^{-n}.$$

For each $j \in \mathbb{N}_0$, the previous relation gives sequences $\mathbf{a}^j$ on E' and $\mathbf{b}^j$ on F such that for all $x \in E$ we have

$$T_{n_j}(x) - T_{n_{j-1}}(x) = \sum_{k=0}^{+\infty} \langle a_k^j, x\rangle b_k^j$$

and

$$\sum_{k=0}^{+\infty} \|a_k^j\|_{E'} \|b_k^j\|_F < 2^{-n}.$$

Let also $\mathbf{a}^0$ and $\mathbf{b}^0$ be sequences on E' and F respectively such that for all $x \in E$, we have

$$T_{n_0}(x) = \sum_{k=0}^{+\infty} \langle a_k^0, x\rangle b_k^0$$

and

$$\sum_{k=0}^{+\infty} \|a_k^0\|_{E'} \|b_k^0\|_F < \nu(T_{n_0}) + 1.$$

Let $(c_k, d_k)_{k\in\mathbb{N}}$ be an enumeration of $(a_k^j, b_k^j)_{j,k\in\mathbb{N}}$ and $x \in E$. For every $J \in \mathbb{N}$ and $j \in \mathbb{N}$ let K_j^J be the set of indices $k \le j$ such that the couple (c_k, d_k) originate from a couple of the form $(a_{k_0}^{j_0}, b_{k_0}^{j_0})$ with $j_0 \le J, k_0 \in \mathbb{N}$. For each $J \in \mathbb{N}, j \in \mathbb{N}$, we have

$$\begin{aligned}\|T(x) - \sum_{k=0}^{j} \langle c_k, x\rangle d_k\|_F &\le \|T(x) - T_{n_J}(x)\|_F \\ &+\|T_{n_J}(x) - \sum_{k\in K_j^J} \langle c_k, x\rangle d_k\|_F + \sum_{j_0=J+1}^{+\infty} \sum_{k=0}^{+\infty} \|a_k^{j_0}\|_E' \|b_k^{j_0}\|_F.\end{aligned}$$

The first and last term can be made arbitrary small by choosing J large enough and then the second term can be made arbitrary small by choosing j large enough. This proves that for all $x \in E$ we have

$$T(x) = \sum_{k=0}^{+\infty} \langle c_k, x\rangle d_k.$$

Moreover we have

$$\sum_{k=0}^{+\infty} \|c_k\|'_E \|d_k\|_F \leq \nu(T_{n_0}) + 3 < +\infty$$

so $T \in \mathscr{N}(E, F)$.

Finally, similar reasoning proves that for every $J \in \mathbb{N}$, we have

$$\nu(T - T_{n_J}) \leq \sum_{j=J+1}^{+\infty} \sum_{k=0}^{+\infty} \|a_k^j\|'_E \|b_k^j\|_F \leq 2^{-J}$$

which proves that $(T_{n_j})_{j\in\mathbb{N}}$ converges to T in $\mathscr{N}(E, F)$ hence $(T_n)_{n\in\mathbb{N}}$ converges to T in $\mathscr{N}(E, F)$. □

The proof by Pietsch in ([11] lemma 3.1.3) of the previous resut proves the nuclearity of T by saying that it can be written as

$$T(x) = T_0(x) + \sum_{j=1}^{+\infty} \sum_{k=0}^{+\infty} \langle a_k^j, x\rangle b_k^j.$$

Details have been added here to proves that the right hand side can be expressed with a single series to match the definition of nuclear operators.

Nuclear operator have stability properties under composition with continuous linear operators.

Proposition 2.1.4. *Let E, F, G, H be Banach spaces and $T \in L(E, F)$, $S \in \mathscr{N}(F, G)$, $R \in L(G, H)$ be linear operators. Then the linear operator RST from E to G is nuclear and satisfy $\nu(RST) \leq \beta(R)\nu(S)\beta(T)$.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences such that $a_n \in F'$, $b_n \in G$ for all $n \in \mathbb{N}$,

$$\sum_{n=0}^{+\infty} \|a_n\|_{F'} \|b_n\|_G < +\infty$$

and

$$S(y) = \sum_{n\in\mathbb{N}} \langle a_n, y\rangle b_n \quad \text{for all } y \in F.$$

By continuity and linearity of R and S we have

$$RST(x) = \sum_{n\in\mathbb{N}} \langle a_n, Tx\rangle Rb_n = \sum_{n\in\mathbb{N}} \langle T' a_n, x\rangle Rb_n \quad \text{for all } x \in E.$$

By proposition 1.1.3, this representation satisfy

$$\sum_{n=0}^{+\infty} \|T' a_n\|_{E'} \|Rb_n\|_H \leq \sum_{n=0}^{+\infty} \beta(T)\|a_n\|_{F'} \beta(R)\|b_n\|_H < +\infty$$

which proves that RST is nuclear and satisfy $\nu(RST) \leq \beta(R)\nu(S)\beta(T)$. □

Definition 2.1.5. A locally convex space E is nuclear if for all $p \in \operatorname{csn}(E)$, there exists $q \in \operatorname{csn}(E)$ such that $q \geq p$ and ι_q^p is nuclear.

Using the preceding proposition, this definition can be simplified by imposing the condition only on a fundamental system of seminorms.

Proposition 2.1.6. *Let E a lcs and $\mathscr{P}$ a fundamental system of seminorms of E. The locally convex space E is nuclear if and only if for all $p \in \mathscr{P}$, there exists $q \in \mathscr{P}$ such that $q \geq p$ and ι_q^p is nuclear.*

Proof. The necessary condition is trivial.

For the sufficient condition, let $p \in \operatorname{csn}(E)$ be arbitrary. Since $\mathscr{P}$ is a fundamental system of seminorms of E, there exists $p_0 \in \mathscr{P}$ such that $p \leq p_0$. By assumption, there exists a seminorm $q_0 \in \mathscr{P}$ such that $q_0 \geq p$ and $\iota_{q_0}^{p_0}$ is nuclear. By the preceding proposition, the operator $\iota_{q_0}^p = \iota_{q_0}^{p_0} \circ \iota_{p_0}^p$ is therefore nuclear. □

Using proposition 2.1.4, one can prove that nuclearity is preserved by linear homeomorphisms.

Proposition 2.1.7. *Let E and F be two lcs that and T a linear bijective isomorphism between E and F, the space F is nuclear if and only if E is nuclear.*

Proof. We suppose that E is nuclear. Let $q_1 \in \operatorname{csn}(F)$ and $p_1 \in \operatorname{csn}(E)$ such that $q_1 \circ T \prec p_1$. By nuclearity of E there exists $p_2 \in \operatorname{csn}(E)$ such that $p_2 \geq p_1$ and $\iota_{p_2}^{p_1} \in \mathscr{N}(E_{p_2}, E_{p_1})$. If $q_2 \in \operatorname{csn}(F)$ is chosen such that $p_2 \circ T^{-1} \prec q_2$, then we can prove that. □

We can prove using similar arguments the following proposition.

Lemma 2.1.8. *Let E be a nuclear lcs and F a topological linear subspace of E. Then F is nuclear.*

2.2 Connection with weakly summable and absolutely summable families

The goal of this section is to prove a characterization of certain nuclear spaces in terms of summability of sequences. To achieve this goal we will define and study absolutely summing operators and prove that if we replace the nuclear mappings in the definition of nuclear spaces by absolutely summing mappings, then the spaces defined are also nuclear spaces.

In this section, E and F will always be lcs and I will always designate an index set (not necessarily countable).

Notations 2.2.1.
- Let X be a set. If $\mathbf{x} \in X^I$ and if $i \in I$, the i^{th} component of $\mathbf{x}$ will always be written x_i.

- Let X be a set. If $\mathbf{x}$ is an element of X^I and if $J \subset I$, we set $\mathbf{x}_J = (\tilde{x}_i)_{i \in J}$ the family defined by
$$\tilde{x}_i = \begin{cases} x_i \text{ if } i \in J \\ 0 \text{ else} \end{cases} .$$

- The set of all finite subsets of I is denoted $\mathscr{F}(I)$.

- If T is a linear operator on E and $\mathbf{x} \in E^I$, we set $T(\mathbf{x}) = (Tx_i)_{i \in I}$.
  Similarly, if a is a linear form on E and $\mathbf{x} \in E^I$, we set $\langle a, \mathbf{x} \rangle = (\langle a, x_i \rangle)_{i \in I} = a(\mathbf{x})$.

Definition 2.2.2. A directed set A is a set endowed with a reflexive and transitive binary relation $\leq$ such that each finite subset $\{a_1, \ldots, a_n\}$ of A has an upper bond $a \in A$, i.e. an element of A such that $a \geq a_1, \ldots, a \geq a_n$.

Definition 2.2.3. Let A be a directed set and X a topological space. A *directed system* of X indexed by A (or simply a directed system if A and X are fixed) is an element of X^A.

A directed system $\mathbf{x} \in X^A$ is a *directed Cauchy-system* if for each zero neighborhood U, there exists an index $a_0 \in A$ such that $x_a - x_b \in U$ for each $a, b \geq a_0$

A directed system $\mathbf{x} \in X^A$ is said to converge to $x_0 \in X$ if for each neighborhood U of x_0 there exists an index $a_0 \in A$ such that $x_a \in U$ for each $a \geq a_0$. We write $\lim_{a \in A} x_a = x_0$.

We start by defining some notion of convergence of general series or sequences of scalars.

Definition 2.2.4. Let α be a sequence of scalars indexed by I.

For $\mathfrak{i} \in \mathscr{F}(I)$, let $s_{\mathfrak{i}} = \sum_{i \in \mathfrak{i}} \alpha_i$. The sequence $(s_{\mathfrak{i}})_{\mathfrak{i} \in \mathscr{F}(I)}$ is a directed system indexed by $\mathscr{F}(I)$ endowed with the inclusion order. If this system has a limit $S \in \mathbb{K}$, we say that α is summable and we write
$$S = \sum_{i \in I} \alpha_i.$$

An important observation is that the value of this series is independent of the order in which we "add the elements".

Definition 2.2.5. We can define a norm $\| \cdot \|_{\ell^1_I}$ on the linear space ℓ^1_I of all families α of scalars such that the family $(|\alpha_i|)_{i \in I}$ is summable by
$$\|\alpha\|_{\ell^1_I} = \sum_{i \in I} |\alpha_i|.$$

We extend naturally the application $\| \cdot \|_{\ell^1_I}$ on the space of all families of scalars by setting $\|\alpha\|_{\ell^1_I} = +\infty$ if $\alpha \notin \ell^1_I$.

Definition 2.2.6. A sequence α of scalars indexed by I is said to converge to 0 if for each $\delta > 0$, there exists a finite set $\mathfrak{i} \in \mathscr{F}(I)$ such that

$$|\alpha_i| < \delta \text{ for all } i \in I \backslash \mathfrak{i}.$$

The set of these sequences is written $\mathbf{c}_I$.

Proposition 2.2.7. *Let β be a sequence of scalars indexed by I. If*

$$\sum_{i \in I} |\alpha_i \beta_i| < +\infty$$

for all $\alpha \in \boldsymbol{c}_I$ then

$$\sum_{i \in I} |\beta_i| < +\infty.$$

Proof. The demonstration is classic and then omitted. □

We will now be interested in series in arbitrary lcs or normed spaces. We will define three notions of summability for series of elements of a lcs.

Definition 2.2.8. A family $\mathbf{x} \in E^I$ of elements of E is called weakly summable if it satisfies $\langle a, \mathbf{x} \rangle \in l_I^1$ for each $a \in E'$. The space of weakly summable families on E indexed by I is denoted $\ell_I^1[E]$.

From this definition, it directly follows that a sequence $\mathbf{x}$ is weakly summable if and only if the set

$$A_x := \left\{ \sum_{i \in \mathfrak{i}} \alpha_i x_i : \mathfrak{i} \in \mathscr{F}(I), \forall i \in \mathfrak{i}\ |\alpha_i| \leq 1 \right\}$$

is weakly bounded (i.e. bounded in $((E, \sigma(E, E'))$. By Mackey's theorem (theorem 1.2.4), this happens exactly when this set is bounded in E.

Let $p \in csn(E)$ and $a \in b_p^\circ$. For every $i \in I$, let α_i be a scalar such that $|\langle a, x_i \rangle| = \langle a, x_i \rangle \alpha_i$. If $\mathfrak{i} \in \mathscr{F}(I)$, we obtains

$$\sum_{i \in \mathfrak{i}} |\langle a, x_i \rangle| = \left\langle a, \sum_{i \in \mathfrak{i}} \alpha_i x_i \right\rangle \leq p\left(\sum_{i \in \mathfrak{i}} \alpha_i x_i \right) \leq \sup_{y \in A_x} p(y) < +\infty.$$

We can therefore consider the seminorm p_ε defined for $\mathbf{x}$ in $\ell_I^1[E]$ by

$$p_\varepsilon(\mathbf{x}) = \sup\{ \| \langle a, \mathbf{x} \rangle \|_{\ell_I^1} : a \in b_p^\circ \}.$$

The linear space $\ell_I^1[E]$ is endowed with the locally convex space structure induced by the fundamental system of seminorms

$$\{ p_\varepsilon : p \in \text{csn}(E) \}.$$

Definition 2.2.9. An element $\mathbf{x}$ of $\ell^1_I[E]$ is *summable* if the directed system $(\mathbf{x}_{\mathfrak{i}})_{\mathfrak{i}\in\mathscr{F}(I)}$ converges to $\mathbf{x}$ in $\ell^1_I[E]$. The set of all summable families on E indexed by I is denoted $\ell^1_I(E)$.

The following propositions are direct consequences of the definitions.

Proposition 2.2.10. *A sequence $\boldsymbol{x} \in \ell^1_I[E]$ is summable if it satisfies the following property : for each $p \in csn(E)$ and $\delta > 0$, there exists a finite set $\mathfrak{i} \in \mathscr{F}(I)$ such that*

$$p_\varepsilon(\boldsymbol{x}_{I\setminus\mathfrak{i}}) < \delta.$$

Proposition 2.2.11. *If $\boldsymbol{x} \in \ell^1_I[E]$ and $\alpha \in \boldsymbol{c}_I$, then $\alpha\,\boldsymbol{x} \in \ell^1_I(E)$.*

Definition 2.2.12. A sequence $\mathbf{x} \in E^I$ is absolutely summable if for each $p \in \operatorname{csn}(E)$ we have

$$\sum_{i\in I} p(x_i) < +\infty.$$

For each $p \in \operatorname{csn}(E)$, a seminorm p_π is defined on the space $\ell^1_I\{E\}$ of all absolutely summable sequences by

$$p_\pi(\mathbf{x}) = \sum_{i\in I} p(x_i).$$

We endow the space $\ell^1_I\{E\}$ by the fundamental system of seminorms $\{p_\pi : p \in \operatorname{csn}(E)\}$.

The seminorms p_ε and p_π can naturally be extended on the set of all sequences on E if we accept that their value can be infinite.

The absolute summability implies the summability.

Proposition 2.2.13. *The lcs $\ell^1_I\{E\}$ is continuously included in $\ell^1_I(E)$.*

Proof. Let $\mathbf{x} \in \ell^1_I\{E\}, p \in \operatorname{csn}(E)$ and $\delta > 0$. Since $\sum_{i\in I} p(x_i) < +\infty$, there exists a finite set $\mathfrak{i} \in \mathscr{F}(I)$ such that $p_\pi(\mathbf{x}_{I\setminus\mathfrak{i}}) < \delta$. The conclusion comes from the fact that $p_\varepsilon \le p_\pi$ on $\ell^1_I\{E\}$. □

We can now define a new family of operators between locally convex spaces.

Definition 2.2.14. An operator $T \in L(E,F)$ is absolutely summing if it maps each summable countable family $\mathbf{x} \in \ell^1_{\mathbb{N}}(E)$ of E to an absolutely summable family $T(\mathbf{x}) \in \ell^1_{\mathbb{N}}\{F\}$ of F.

Proposition 2.2.15. *Each abolutely summing mapping $T \in L(E,F)$ maps $\ell^1_I[E]$ into $\ell^1_I\{F\}$ and this mapping is bounded.*

Proof. If it was not the case we could find a bounded set B of $\ell^1_I[E]$ and a continuous seminorm $q \in \operatorname{csn}(F)$ satisfying

$$\sup\{q_\pi(T\,\mathbf{x}) : \mathbf{x} \in B\} = +\infty.$$

In this condition, for each $n \in \mathbb{N}_0$ we can find a family $\mathbf{x}^{(n)} \in B$ and a finite set $\mathfrak{i}_n \in \mathscr{F}(I)$ such that $q_\pi(\mathbf{x}^{(n)}_{\mathfrak{i}_n}) > 2^n$.

Since B is bounded in $\ell^1_I[E]$, for each $a \in E'$ there exists a positive number C such that

$$\|\langle a, \mathbf{x}\rangle\|_1 < C \text{ for all } \mathbf{x} \in B.$$

We then obtain

$$\sum_{n\in\mathbb{N}_0} \sum_{i\in\mathfrak{i}_n} 2^{-n} |\langle a, x^{(n)}_i\rangle| < C.$$

In other words, if P is the set $P = \{(n,i) : n \in \mathbb{N}_0, i \in \mathfrak{i}_n\}$, then the family $(2^{-n} x^{(n)}_i)_{(n,i)\in P}$ is weakly summable in E. Let $(\alpha^{(n)}_i)_{(n,i)\in P} \in \mathbf{c}_P$ be arbitrary. By the proposition 2.2.11, the family $(2^{-n}\alpha^{(n)}_i x^{(n)}_i)_{(n,i)\in P}$ is a countable summable family of E. The absolute summability of T gives then

$$\sum_{(n,i)\in P} 2^{-n} |\alpha^{(n)}_i| q(x^{(n)}_i) < +\infty.$$

Since this relation is true for all $\alpha \in \mathbf{c}_P$, the proposition 2.2.7 gives

$$\sum_{(n,i)\in P} 2^{-n} q(x^{(n)}_i) < +\infty,$$

which is false by choice of the sequences $\mathbf{x}^{(n)}$ $(n \in \mathbb{N})$. □

If the lcs E and F are normed spaces, absolutely summing mapping can be easily described and a seminorm can be naturally defined on the set of all absolutely summing operators.

Proposition 2.2.16. *If (E, p) and (F, q) are two normed spaces then an operator $T \in L(E, F)$ is absolutely summing if and only if there exists a positive number C such that*

$$q_\pi(Tx) \le C p_\varepsilon(x)$$

for all $\mathfrak{i} \in \mathscr{F}(\mathbb{N})$ and all finite family $\boldsymbol{x} = (x_i)_{i\in\mathfrak{i}}$ of E.

Proof. The necessary condition is a direct consequence of the previous proposition and the sufficient condition is clear. □

If $\pi(T)$ denotes the infimum of the constant $C > 0$ such that the proposition holds, then π is a norm on the linear space of all absolutely summing operators from E to F.

The following technical lemma will be used in the next theorem.

Lemma 2.2.17. *Let (E, p) a normed space and Δ the unit disk of $\mathbb{K}$. If b°_p is endowed with the weak topology induced by $\sigma(E', E)$, then each $\boldsymbol{x} \in \ell^1_I[E]$ can be associated with the continuous function $\Phi_{\boldsymbol{x}}$ on $\Delta^I \times b^\circ_p$ defined for $\alpha \in \Delta^I, a \in b^\circ_p$ by*

$$\Phi_{\boldsymbol{x}}(\alpha, a) = \sum_{i\in I} \alpha_i \langle a, x_i\rangle.$$

Moreover, the mapping $\boldsymbol{x} \mapsto \Phi_{\boldsymbol{x}}$ is an isometry as a mapping from $\ell^1_I[E]$ to $\mathscr{C}(\Delta^I \times b^\circ_p)$.

Proof. We begin to observa that by Alaoglu's theorem, b_p° is a compact set for the topology $\sigma(E', E)$ and by Tikhonov theorem $\Delta^I \times b_p^\circ$ is compact as a product of compact sets. The space $C(\Delta^I \times b_p^\circ)$ is then a Banach space so the lemma makes sense.

Let $\mathbf{x} \in \ell_I^1[E]$, by definition, for every positive natural number $n > 0$, there exists a finite set $\mathfrak{i}_n \in \mathscr{F}(I)$ such that the relation

$$\sum_{i \in I \setminus \mathfrak{i}_n} |\langle a, x_i \rangle| < 1/n$$

holds for all $a \in b_p^\circ$. For such number n, if $\Phi_{\mathbf{x}}^{(n)}$ denotes the continuous function defined for $\alpha \in \Delta^I$ and $a \in b_p^\circ$ by

$$\Phi_{\mathbf{x}}^{(n)}(\alpha, a) = \sum_{i \in \mathfrak{i}_n} \alpha_i \langle a, x_i \rangle,$$

then we have

$$|\Phi_{\mathbf{x}}(\alpha, a) - \Phi_{\mathbf{x}}^{(n)}(\alpha, a)| \leq \sum_{i \in I \setminus \mathfrak{i}_n} |\langle a, x_i \rangle| < 1/n$$

for all $\alpha \in \Delta^I$ and $a \in b_p^\circ$. $\Phi_{\mathbf{x}}$ is therefore the uniform limit of the continuous functions $\Phi_{\mathbf{x}}^{(n)}$ and is therefore continuous itself.

If $\|.\|$ is the norm on $C(\Delta^i \times b_p^\circ)$ then the relation

$$\|\Phi_{\mathbf{x}}\| \leq p_\varepsilon(\mathbf{x})$$

is clear.

For every $a \in b_p^\circ$ we can choose a sequence $\alpha \in \Delta^I$ such that $|\langle a, x_i \rangle| = \alpha_i \langle a, x_i \rangle$ for all $i \in I$. For this a we have

$$\sum_{i \in I} |\langle a, x_i \rangle| = \Phi_{\mathbf{x}}(\alpha, a)| \leq \|\Phi_{\mathbf{x}}\|.$$

Since this relation is true for all $a \in b_p^\circ$, we have proved

$$p_\varepsilon(\mathbf{x}) \leq \|\Phi_{\mathbf{x}}\|$$

hence

$$p_\varepsilon(\mathbf{x}) = \|\Phi_{\mathbf{x}}\|.$$

□

We can now prove the following theorem which will be used later to prove that the composition of two absolutely summing operators is nuclear.

Theorem 2.2.18. *If (E, p) and (F, q) are two normed spaces then an operator $T \in L(E, F)$ is absolutely summing if and only if there exists a borelian positive measure μ on $(b_p^\circ, \sigma(E', E))$ such that $\mu(b_p^\circ) = \pi(T)$ and*

$$q(Tx) \leq \int_{b_p^\circ} |\langle a, x \rangle| d\mu(a) \text{ for all } x \in E$$

Proof. Suppose that T is absolutely summing, we can define a linear continuous form A on $\ell^1_{b_q^\circ}(E)$ by

$$\langle A, \mathbf{x}\rangle := \sum_{b\in b_q^\circ} \langle b, Tx_b\rangle \text{ for } \mathbf{x} \in \ell^1_{b_q^\circ}(E).$$

Indeed, we have

$$\sum_{b\in b_q^\circ} |\langle b, Tx_b\rangle| \leq q_\pi(T\mathbf{x}) \leq \pi(T)p_\varepsilon(\mathbf{x}) \text{ for } \mathbf{x} \in \ell^1_{b_q^\circ}(E).$$

By the previous lemma, we can identify each summable family $\mathbf{x} \in \ell^1_{b_q^\circ}(E)$ with a function $\Phi_{\mathbf{x}} \in C\left(\Delta^{b_q^\circ} \times b_p^\circ\right)$. By the Hahn-Banach theorem, there exists a linear continuous form M_0 on $C\left(\Delta^{b_q^\circ} \times b_p^\circ\right)$ such that

$$\langle M_0, \Phi_{\mathbf{x}}\rangle = \langle A, \mathbf{x}\rangle \text{ for all } \mathbf{x} \in \ell^1_{b_q^\circ}(E)$$

and whose norm is bounded by $\pi(T)$.

By the Riesz representation theorem from measure theory [10], there exists a complex regular borel measure μ_0 on the compact set $\Delta^{b_q^\circ} \times b_p^\circ$ such that

$$\langle M_0, \Phi\rangle = \int_{\Delta^{b_q^\circ}\times b_p^\circ} \Phi d\mu_0 \text{ for all } \Phi \in C\left(\Delta^{b_q^\circ} \times b_p^\circ\right).$$

Moreover, we have $\|M_0\| = |\mu_0|\left(\Delta^{b_q^\circ} \times b_p^\circ\right)$

Let's define a positive linear form M on $C(b_p^\circ)$ by

$$\langle M, \varphi\rangle = \int_{\Delta^{b_q^\circ}\times b_p^\circ} \varphi(a)d|\mu_0|(\alpha, a) \text{ for all } \varphi \in C(b_p^\circ).$$

Clearly, we have $\|M\| \leq |\mu_0|\left(\Delta^{b_q^\circ} \times b_p^\circ\right) = \|M_0\| \leq \pi(T)$. Another application of the Riesz representation theorem gives a regular Borel measure μ on b_p° such that

$$\langle M, \varphi\rangle = \int_{b_p^\circ} \varphi d\mu \text{ for all } \varphi \in C(b_p^\circ).$$

Once again, we have $\mu(b_p^\circ) = \|M\| \leq \pi(T)$.

We can now prove the theorem. Let $x \in E$ and $b_0 \in b_q^\circ$ be such that $\langle b_0, Tx\rangle = q(Tx)$. If $y = (x\delta_{b_0,b}))_{b\in b_q^\circ}$, then we have

$$q(Tx) = \langle b_0, x\rangle = \langle A, y\rangle = \langle M_0, \Phi_y\rangle = \int_{\Delta^{b_q^\circ}\times b_p^\circ} \Phi_y(\alpha, a)d\mu_0(\alpha, a).$$

Since $q(Tx) \geq 0$, we obtain

$$\begin{aligned} q(Tx) &\leq \int_{\Delta^{b_q^\circ}\times b_p^\circ} |\Phi_y(\alpha, a)|d|\mu_0|(\alpha, a) \\ &= \int_{\Delta^{b_q^\circ}\times b_p^\circ} |\alpha_b||\langle a, x\rangle|d|\mu_0|(\alpha, a) \\ &\leq \langle M, |\varphi_x|\rangle, \end{aligned}$$

where φ_x is the continuous linear function defined on b_p° by $\varphi_x(a) = \langle a, x\rangle$ for all $a \in b_p^\circ$. This proves the necessary condition since

$$\langle M, |\varphi_x|\rangle = \int_{b_p^\circ} |\langle a, x\rangle| d\mu(a).$$

Conversely, suppose that $T \in L(E, F)$ satisfies for every $x \in E$ that

$$q(Tx) \leq \int_{b_p^\circ} |\langle a, x\rangle| d\mu(a)$$

for a borelian positive measure μ on $(b_p^\circ, \sigma(E', E))$. In this condition, if $\mathbf{x} \in \ell_I^1[E]$ and $\mathfrak{i} \in \mathscr{F}(I)$ then we have

$$\sum_{i\in\mathfrak{i}} q(T(x_i)) \leq \int_{b_p^\circ} \sum_{i\in\mathfrak{i}} |\langle a, x_i\rangle| d\mu(a) \leq p_\varepsilon(\mathbf{x})\mu(b_p^\circ)$$

which proves the theorem by proposition 2.2.16. □

Given two normed spaces E and F and an absolutely summing operator $T \in L(E, F)$, the preceding theorem allows us to consider the Hilbert space $L_\mu^2(b_p^\circ)$. We will then study briefly the properties of Hilbert spaces and of operators between Hilbert spaces.

Definition 2.2.19. Let E a Hilbert space. A family $\mathbf{e}$ of E indexed by I is an orthonormal basis of E if it is an orthonormal family whose linear span is dense in E.

Proposition 2.2.20. *Every Hilbert space E has an orthonormal basis.*

Proof. Let I be an ordinal number for which there exists a bijective sequence $(e_i)_{i\in I}$ from I to E.

For $i \in I$, let $E_i = \overline{\rangle e_j : j < i\langle}$ and P_i be the orthogonal projection from E onto E_i. The sequence $(f_i)_{i\in I} := e_i - P_i(e_i)$ is an orthogonal sequence of E. A transfinite induction proves directly that

$$\rangle e_j : j < i\langle = \rangle f_j : j < i\langle$$

for all $i \in I$ so that the span of the sequence $(f_i)_{i\in I}$ is dense in E.

If $J = \{j \in I : f_j \neq 0\}$, then the family

$$(f_j\|f_j\|^{-1})_{j\in J}$$

is an orthonormal basis of E. □

Definition 2.2.21. Let E and F be two Hilbert spaces. An operator $T \in L(E, F)$ is a Hilbert-Schmidt operator if there exists orthonormal bases $\mathbf{e} \in E^I$ on E and $\mathbf{f} \in F^J$ of F such that

$$\sigma^2(T) := \sum_{i\in I, j\in J} |\langle Te_i, f_j\rangle|^2 < +\infty.$$

This definition does not actually depends on the orthonormal bases $\mathbf{e}$ and $\mathbf{f}$ because Parseval idendity gives

$$\sigma^2(T) = \sum_{i\in I} ||Te_i\|^2 = \sum_{j\in J} \|T^* f_j\|^2.$$

One can easily prove that $\beta(T) \leq \sigma(T)$ for all Hilbert-Schmidt operator T.

Lemma 2.2.22. *Let H be a Hilbert space, M be a compact Haussdorf space and μ a positive Radon measure on M with $\mu(M) = 1$. Let K be the canonical mapping from $C(M)$ into $L^2_\mu(M)$. If $T \in L(H, C(M))$, then KT is a Hilbert-Schmidt operator from H into $L^2_\mu(M)$ and $\sigma(KT) \leq \beta(T)$.*

Proof. Let $\mathbf{h}$ be a basis of H indexed by an index set I and $\varphi_i = Th_i$ for all $i \in I$. Let $x \in M$, if δ_x is the continuous linear form defined on $C(M)$ by

$$\langle \delta_x, \varphi \rangle = \varphi(x) \text{ for } \varphi \in C(M),$$

we have

$$\sum_{i \in I} |(Th_i)(x)|^2 = \sum_{i \in I} |\langle \delta_x, \varphi_i \rangle|^2 = \sum_{i \in I} |\langle T'\delta_x, h_i \rangle|^2 = \|T'\delta_x\|^2_{E'} \leq \|T\|^2.$$

Where the last equality is obtained by an application of the Riesz representation theorem for continouus linear forms on a Hilbert space followed by an application of Parseval idendity. If $\mathfrak{i} \in \mathscr{F}(I)$, integration over M gives

$$\sum_{i \in \mathfrak{i}} \|Th_i\|^2_{L^2_\mu(M)} \leq \|T\|$$

and then

$$\sum_{i \in I} \|Th_i\|^2_{L^2_\mu(M)} \leq \|T\|$$

which proves the theorem. □

Lemma 2.2.23. *Let H be a Hilbert space, M be a compact Haussdorf space and μ a positive borelian measure on M with $\mu(M) = 1$. Let K be the canonical mapping from $C(M)$ into $L^2_\mu(M)$. If T is a Hilbert-Schmidt operator from $L^2_\mu(M)$ into H, then TK is a nuclear operator from $C(M)$ into H and $\nu(TK) \leq \sigma(T)$.*

Proof. In a first time, we will prove the theorem in the case where T has the form

$$T\hat{f} = \sum_{k=1}^{n} \left\langle \hat{f}, \hat{f}_k \right\rangle y_k \text{ for all } \hat{f} \in L^2_\mu(M)$$

where $n \in \mathbb{N}_0$, $\hat{f}_1, \ldots \hat{f}_n \in L^2_\mu(M)$ are step functions and $y_1, \ldots y_n \in H$.

For such an operator T there exist an $m \in \mathbb{N}_0$, disjoints $\mu-$measurable sets $M_1, \ldots, M_m \subset M$ and vectors $z_1, \ldots, z_m \in H$ such that for all $\hat{f} \in L^2_\mu(M)$ we have

$$T\hat{f} = \sum_{k=1}^{m} \left\langle \hat{f}, K\chi_{M_k} \right\rangle z_k.$$

Specifically, for $f \in C(M)$, we have

$$TKf = \sum_{k=1}^{m} \langle Kf, K\chi_{M_k} \rangle z_k.$$

This gives

$$\nu^2(TK) \leq \left(\sum_{k=1}^{m} \mu(M_k)\|z_k\| \right)^2 \leq \left(\sum_{k=1}^{m} \mu(M_k)\|z_k\|^2 \right) \left(\sum_{k=1}^{m} \mu(M_k) \right) \leq \sum_{k=1}^{m} \mu(M_k)\|z_k\|^2.$$

If we set $e_k = K(\chi_{M_k})\mu(M_k)^{-1/2}$ for $k = 1, \ldots m$, then $(e_1, \ldots e_m)$ is an orthonormal family of $L^2_\mu(M)$, thus we have

$$\sigma^2(T) \geq \sum_{k=1}^{m} \|Te_i\|^2 = \sum_{k=1}^{m} \mu(M_k)\|z_k\|^2 \geq \nu^2(TK)$$

which proves the theorem in this setting.

We will now prove the theorem for arbitrary Hilbert-Schmidt operator T. Let $\mathbf{h}$ be an orthonormal basis of H indexed by I. Since T is an Hilbert-Schmidt operator, we have

$$\sum_{i \in I} \|T^* h_i\| < +\infty.$$

Hence for all $n \in \mathbb{N}$ there exists $\mathfrak{i}_n \in \mathscr{F}(I)$ such that

$$\sum_{i \in I \setminus \mathfrak{i}_n} \|T^* h_i\| < (2n)^{-1}.$$

For $n \in \mathbb{N}, i \in \mathfrak{i}_n$, let $f_i^{(n)}$ be a step function on M such that

$$\|T^* h_i - f_i^{(n)}\| < (2|\mathfrak{i}_n|n)^{-1}.$$

For $n \in \mathbb{N}$, we then define the Hilbert-Schmidt operator $T_n \in L(L^2_\mu(M), H)$ by

$$T_n \hat{f} = \sum_{i \in \mathfrak{i}_n} \left\langle \hat{f}, Kf_i^{(n)} \right\rangle h_i \text{ for } \hat{f} \in L^2_\mu(M).$$

Since the relation $\left\langle T_n \hat{f}, h_i \right\rangle = \left\langle \hat{f}, Kf_i^{(n)} \right\rangle$ holds for all $\hat{f} \in L^2_\mu(M)$ and $i \in \mathfrak{i}_n$ and $\left\langle T_n \hat{f}, h_i \right\rangle = 0$ if $i \notin \mathfrak{i}_n$, we have $T_n^* h_i = Kf_i^{(n)}$ and $\sigma(T - T_n) < n^{-1}$.

For $\varphi \in C(M)$, we have

$$\|(T - T_n)K\varphi\| \leq \beta(T - T_n)\beta(K)\|\varphi\| = \beta((T - T_n)^*)\|\varphi\| < n^{-1}\|\varphi\|$$

so $(T_nK)\varphi \to (TK)\varphi$ for all $\varphi \in C(M)$. Moreover, for $N \in \mathbb{N}$ and $p, q \geq N$, we have already proved that $(T_m - T_n)K$ is nuclear and satisfy $\nu(T_mK - T_nK) \leq \sigma(T_m - T_n) \leq 2N^{-1}$.

By proposition 2.1.3, T is nuclear and the $\nu-$Cauchy-sequence $(T_n)_{n\in\mathbb{N}}$ converges to T in $\mathscr{N}(C(M), H)$. Finally, we have

$$\nu(T) = \lim_{n\to\infty} \nu(T_n) \leq \lim_{n\to\infty} \sigma(T_n) = \sigma(T).$$

□

Lemma 2.2.24. *Let (E, p) be a normed space and (F, q) a Banach space. For every absolutely summing operator $T \in L(E, F)$, there exists operators $T_1 \in L(E, C(M))$ and $T_2 \in L(L^2_\mu(M), F)$ such that $T = T_2KT_1$ where $M = b_p^\circ$ endowed with a positive Radon measure and K is the canonical projection from $C(M)$ on $L^2_\mu(M)$.*

Proof. By proposition 2.2.18, there exists a measure μ on $M = b_p^\circ$ such that $\mu(M) = 1$ and

$$q(Tx) \leq \pi(T) \int_M |\langle a, x\rangle| d\mu(a) \text{ for all } x \in E.$$

For $x \in E$ let $\varphi_x \in C(M)$ be the function defined by $\varphi_x(a) = \langle a, x\rangle$ for all $a \in M$. We define the linear continuous operator

$$T_1 : E \to C(M) : x \mapsto \varphi_x.$$

Let S be the operator defined on the linear space $KT_1(E) \subset L^2_\mu(M)$ by $S(K\varphi_x) = Tx$ for $x \in E$. It is continuous because

$$\|Tx\| \leq \pi(M) \int_M |\varphi_x(a)| d\mu(a) \leq \pi(M)\|K\varphi_x\|$$

If $\tilde{S}$ is the unique extension of S on $\overline{KT_1(E)}$ and P the orthogonal projection from $L^2_\mu(M)$ on $\overline{KT_1(E)}$ then the linear operator $T_2 = P\tilde{S} \in L(L^2_\mu(M), F)$ satisfy $T = T_2KT_1$. □

Theorem 2.2.25. *If $(E, p), (F, q), (G, s)$ are Banach spaces and $T \in L(E, F), S \in L(F, G)$ are absolutely summing operators, then the continuous linear operator ST is nuclear.*

Proof. Let $M = b_p^\circ$ and $N = b_q^\circ$. By the preceding proposition, there exists measures and continuous linear operators

$$E \xrightarrow{T_1} C(M) \xrightarrow{K_1} L^2_\mu(M) \xrightarrow{T_2} F \xrightarrow{S_1} C(N) \xrightarrow{K_2} L^2_\nu(N) \xrightarrow{S_2} G.$$

By lemma 2.2.22, $K_2S_1T_2$ is a Hilbert-Schmidt operator so, by the lemma 2.2.23 , $K_2S_1T_2K_1$ is nuclear. Finally, proposition 2.1.4 gives that $ST = S_2(K_2S_1T_2K_1)T_1$ is nuclear. □

As a direct corollary we have the following sufficient condition for the nuclearity of a space

Theorem 2.2.26. *A lcs E is nuclear if for all $p \in csn(E)$, there exists $q \in csn(E)$ such that $q \geq p$ and the canonical mapping ι_q^p from E_q to E_p is absolutely summing.*

Proof. If E satisfy this property, then for every $p \in \text{csn}(E)$, there exists $q \in \text{csn}(E)$ and $r \in \text{csn}(E)$ such that $r \geq q \geq p$ and such that the mappings ι_q^p and ι_r^q are absolutely summing. Since we have $\iota_s^p = \iota_r^q \circ \iota_q^p$, the preceding theorem proves that ι_s^p is nuclear. $\square$

It can be proved that nuclear operators are automatically absolutely summing (see [11] propositions 3.2.5 and 3.2.13). The above property is then a characterization of nuclear spaces,

Lemma 2.2.27. *Let E be a nuclear space. For each $p \in csn(E)$ there exists $q \in csn(E)$ with $q \geq p$, a sequence of positive numbers $(\lambda_n)_{n\in\mathbb{N}} \in \ell^1$ and a sequence of continuous linear forms $(a_n)_{n\in\mathbb{N}}$ with $a_n \in b_q^\circ$ for all $n \in \mathbb{N}$ such that*

$$p(x) \leq \sum_{n=0}^{+\infty} \lambda_n |\langle a_n, x\rangle| \textit{ for all } x \in \mathbb{R}\,.$$

Proof. Let $p \in \text{csn}(E)$, since E is nuclear there exists $q \in \text{csn}(E)$ such that $\iota_q^p : E_q \to E_p$ is nuclear. By definition, there exists a sequence $(\tilde{a}_n)_{n\in\mathbb{N}}$ of continuous forms on E_q and a sequence $(\tilde{b}_n)_{n\in\mathbb{N}}$ of elements of E_p with $p(\tilde{b}_n) = 1$ for all $n \in \mathbb{N}$ such that

$$\sum_{n=0}^{+\infty} \|a_n\|_{E_q'} < +\infty$$

and

$$\iota_q^p(\tilde{x}) = \sum_{n=0}^{+\infty} \langle \tilde{a}_n, \tilde{x}\rangle \tilde{b}_n \text{ for all } \tilde{x} \in E_q.$$

For $x \in E$, if π_q is the canonical projection from E to E_q this relation gives

$$p(x) \leq \sum_{n=0}^{+\infty} |\langle \tilde{a}_n, \pi_q(x)\rangle|. \tag{1}$$

For $n \in \mathbb{N}$, let $\lambda_n = \|a_n\|_{E_q'}$ and a_n the linear form on E defined by $\langle a_n, x\rangle = \lambda_n^{-1}\langle a_n, \pi_q(x)\rangle$. We have $|\langle a_n, x\rangle| \leq q(x)$ so $a_n \in b_q^\circ$. Finally, relation (1) gives

$$p(x) \leq \sum_{n=0}^{+\infty} \lambda_n |\langle a_n, x\rangle|.$$

$\square$

Proposition 2.2.28. *If E is a nuclear lcs then $\ell^1[E] = \ell^1\{E\}$ as lcs.*

Proof. Let $p \in \operatorname{csn}(E)$ and let $q \in \operatorname{csn}(E), \lambda \in \ell^1$ and $(a_n)_{n\in\mathbb{N}}$ be the seminorm and sequences defined by the preceding lemma. Let $\mathbf{x} \in \ell^1[E]$, we have

$$\sum_{j=0}^{+\infty} p(x_j) \leq \sum_{n=0}^{+\infty}\sum_{j=0}^{+\infty} \lambda_n |\langle a_n, x_j\rangle| \leq \sum_{n=0}^{+\infty} \lambda_n q_\varepsilon(\mathbf{x}).$$

Since this relation is true for all weakly summable sequence $\mathbf{x}$, the space $\ell^1[E]$ is continuously included in $\ell^1\{E\}$ which proves the theorem as the other inclusion is always true and continuous. □

To prove the reciprocal, we will need the following lemma concerning absolutely summing mappings.

Lemma 2.2.29. *Let (E, p) and (F, q) be two normed spaces and let E_1 be a dense subset of E. An operator $T \in L(E, F)$ is absolutely summing if its restriction on E_1 is absolutely summing.*

Proof. Let $T \in L(E, F)$ be an operator whose restriction $T|_{E'}$ is absolutely summing. Let $\mathbf{x} \in \ell^1[E]$ be arbitrary, since E_1 is dense in E, for every $n \in \mathbb{N}$ there exists $y_n \in E_1$ such that $p(x_n - y_n) < 2^{-n}$. We have $(y_n)_{n\in\mathbb{N}} \in \ell^1[E_1]$ because for every $a \in E_1'$, we have

$$\sum_{n=0}^{+\infty} |\langle a, y_n\rangle| \leq \sum_{n=0}^{+\infty} |\langle \tilde{a}, x_n\rangle| + \sum_{n=0}^{+\infty} |\langle \tilde{a}, y_n - x_n\rangle| \leq \beta(a)p_\varepsilon(\mathbf{x}) + 2\beta(a) < +\infty$$

where $\tilde{a} \in E'$ is a continuous linear extension of a given by the Hahn Banach theorem. Since $T|_{E'}$ is absolutely summing, this implies that $\sum_{n=0}^{+\infty} q(Ty_n) < +\infty$.

Finally, we have

$$\sum_{n=0}^{+\infty} q(Tx_n) \leq \sum_{n=0}^{+\infty} q(Ty_n) + \sum_{n=0}^{+\infty} q(T(x_n - y_n)) \leq \sum_{n=0}^{+\infty} q(Ty_n) + 2\beta(T) < +\infty$$

and thus $T(\mathbf{x}) \in \ell^1\{F\}$. □

Proposition 2.2.30. *Let E be a lcs. If $\ell^1[E] = \ell^1\{E\}$ as lcs then E is nuclear.*

Proof. Let $p \in \operatorname{csn}(E)$. Given the hypothesis, there exists $q \in \operatorname{csn}(E)$ such that for all $\mathbf{x} \in \ell^1[E]$, we have

$$p_\pi(\mathbf{x}) \leq q_\varepsilon(\mathbf{x}).$$

Since this relation is true for finite families, it must be true for arbitrary family on E if we accept that the seminorms can take the value $+\infty$. Note that we must have $q \geq p$.

By choosing $\mathbf{x}$ such that $q_\varepsilon(\mathbf{x}) < +\infty$ and setting $\mathbf{y} = \pi_q(\mathbf{x})$, we obtains the relation

$$p_\pi(\iota_q^p \mathbf{y}) \leq q_\varepsilon(\mathbf{y})$$

that holds for every $\mathbf{y} \in \ell^1[\tilde{E}_q]$. Hence, the mapping ι_q^p is absolutely summing from $\tilde{E}_q$ to E_p and is then absolutely summing from E_q to E_p by the preceding lemma. □

If E is a Fréchet space, then we can improve this result.

Proposition 2.2.31. *A Fréchet space E is nuclear if and only if $\ell^1[E] = \ell^1\{E\}$ as sets.*

Proof. We only have to prove that if $\ell^1[E] = \ell^1\{E\}$ as sets then E is nuclear. In such condition, the idendity map on E is absolutely summing and by proposition 2.2.15 it is a bounded operator between $\ell^1[E]$ and $\ell^1\{E\}$ and since those two spaces are metric spaces, the idendity map on E is a continuous operator between $\ell^1[E]$ and $\ell^1\{E\}$. Since it is also a continuous operator between $\ell^1\{E\}$ and $\ell^1[E]$ those spaces are equals as lcs and we have already proved that in this condition, E is nuclear. $\square$

2.3 The Dynin-Mityagin theorem

In this subsection, we will be interested by nuclear Fréchet space that admit a Schauder basis. We will prove the powerful Dynin-Mityagin theorem.

Definition 2.3.1. Let E be a locally convex space and $(e_j)_{j\in\mathbb{N}}$ a sequence of elements of E. The sequence $(e_j)_{j\in\mathbb{N}}$ is a Schauder basis of E if for all $x \in E$ there exists an unique sequence $(c_j(x))_{j\in\mathbb{N}}$ such that

$$x = \sum_{j=0}^{+\infty} c_j(x)e_j.$$

By the uniqueness assumption, for all $j \in \mathbb{N}$, the coefficient c_j is a linear form on E. The sequence $(c_j)_{j\in\mathbb{N}}$ is the sequence of coefficient functional associated to $(e_j)_{j\in\mathbb{N}}$. If the basis is clear, we will always write $(c_j)_{j\in\mathbb{N}}$ for the sequence of coefficient functional associated to $(e_j)_{j\in\mathbb{N}}$.

The proof of the Dynin-Mityagin will need the following concerning orthonormal basis.

Lemma 2.3.2. *Let G, H be two Hilbert spaces and $T \in L(G, H)$. If there exist orthonormal basis (indexed by $\mathbb{N}$) $\boldsymbol{g}$ and $\boldsymbol{h}$ of G and H respectively and a scalar sequence $\lambda \in \ell^1$ such that*

$$T(g) = \sum_{i=0}^{+\infty} \lambda_i h_i \langle g, g_i \rangle$$

then T is nuclear and

$$\nu(T) = \sum_{i=0}^{+\infty} |\lambda_i|.$$

Proof. Since the elements of G can be interpreted as element of G' by $g \mapsto \langle \cdot, g \rangle$ it is clear that T is nuclear and satisfy

$$\nu(T) \leq \sum_{i=0}^{+\infty} |\lambda_i|.$$

To prove the converse inequality, by the Riesz representation theorem, we have to prove that for all $\mathbb{N}-$indexed sequences $\mathbf{x}$ and $\mathbf{y}$ of G and H respectively, that satisfies

$$T(g) = \sum_{j=0}^{+\infty} \langle g, x_j \rangle y_j$$

and

$$\sum_{j=0}^{+\infty} \|x_j\| \|y_j\| < +\infty,$$

we have

$$\sum_{i=0}^{+\infty} |\lambda_i| \leq \sum_{j=0}^{+\infty} \|x_j\| \|y_j\|.$$

For such sequences, we do have

$$\begin{aligned}
\sum_{i=0}^{+\infty} |\lambda_i| &= \sum_{i=0}^{+\infty} \langle T g_i, h_i \rangle \\
&\leq \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} |\langle g_i, x_j \rangle \langle y_j, h_i \rangle| \\
&\leq \sum_{j=0}^{+\infty} \left(\sum_{i=0}^{+\infty} |\langle g_i, x_j \rangle|^2 \right)^{1/2} \left(\sum_{i=0}^{+\infty} |\langle y_j, h_i \rangle|^2 \right)^{1/2} \\
&= \sum_{j=0}^{+\infty} \|x_j\| \|y_j\|.
\end{aligned}$$

□

Lemma 2.3.3. *Let E be a Fréchet space and $\mathscr{P} = (p_m : m \in \mathbb{N})$ an increasing fundamental system of seminorms of E. If $(e_j)_{j\in\mathbb{N}}$ is a Schauder basis of E, then for all $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ and $C > 0$ such that for all $x \in E$ and $k \in \mathbb{N}$ we have*

$$p_m \left(\sum_{j=0}^{k} c_j(x) e_j \right) \leq C p_n(x).$$

Proof. For all $m \in \mathbb{N}$, let q_m the seminorm on E defined by

$$q_m(x) = \sup_{k\in\mathbb{N}} p_m \left(\sum_{j=0}^{k} c_j(x) e_j \right).$$

□

We set $\mathscr{Q} = \{q_m : m \in \mathbb{N}\}$. If we prove that $(E, \mathscr{Q})$ is a Fréchet space, then by the isomorphism's theorem of ... the theorem will be proved since it is clear that $\mathscr{P} \prec \mathscr{Q}$.

Let then $(x_\nu)_{\nu\in\mathbb{N}}$ be a Cauchy sequence of $(E, \mathscr{Q})$. For all $\mu, \nu, k, m \in \mathbb{N}$, we have

$$\begin{aligned} p_m(e_k)|c_j(x_\nu) - c_j(x_\mu)| &\leq p_m\left(\sum_{j=0}^{k} c_j(x_\mu - x_\nu)e_j\right) + p_m\left(\sum_{j=0}^{k-1} c_j(x_\mu - x_\nu)e_j\right) \\ &\leq 2q_m(x_\mu - x_\nu) \end{aligned}$$

For all $k \in \mathbb{N}$, choosing $m \in \mathbb{N}$ such that $p_m(e_j) \neq 0$ in the previous relation implies that $(c_k(x_\nu))_{\nu\in\mathbb{N}}$ is a Cauchy sequence, let c_k be its limit.

Let $m \in \mathbb{N}$, $\varepsilon > 0$ and $\nu_0 \in \mathbb{N}$ such that $q_m(x_\nu - x_\mu) < \varepsilon$ for all $\nu, \mu \geq \nu_0$, for all $k \in \mathbb{N}$ we have

$$p_m\left(\sum_{j=0}^{k} c_j(x_\nu)e_j - \sum_{j=0}^{k} c_j(x_\mu)e_j\right) < \varepsilon$$

and then

$$p_m\left(\sum_{j=0}^{k} c_j(x_\nu)e_j - \sum_{j=0}^{k} c_je_j\right) \leq \varepsilon\,.$$

Hence, for all $l > k \in \mathbb{N}$ by an argument used previously we have

$$p_m\left(\sum_{j=k}^{l} c_je_j\right) \leq 2\,\varepsilon + p_m\left(\sum_{j=k}^{l} c_j(x_\nu)e_j\right).$$

Since the series $\sum_{j=0}^{+\infty} c_j(x_\nu)e_j$ converges in $(E, \mathscr{P})$, this implies that $\sum_{j=0}^{+\infty} c_je_j$ is a Cauchy series in $(E, \mathscr{P})$. Since $(E, \mathscr{P})$ is Fréchet, this series has a limit x in $(E, \mathscr{P})$. By the relations obtained previously it is clear that x_ν converges to x in $(E, \mathscr{Q})$ when ν goes to infinity. The space $(E, \mathscr{Q})$ is then complete and therefore, a Fréchet space.

We have the following direct corollary.

Corollary 2.3.4. *Let E be a Fréchet space and $(p_m : m \in \mathbb{N})$ an increasing fundamental system of seminorms of E. If $(e_j)_{j\in\mathbb{N}}$ is a Schauder basis of E, then for all $m \in \mathbb{N}$ there exist $n \in \mathbb{N}$ and $C > 0$ such that for all $x \in E$ and $j \in \mathbb{N}$ we have*

$$|c_j(x)|p_m(e_j) \leq Cp_n(x).$$

Since for all $j \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $p_m(e_j) \neq 0$, this implies that the coefficient functionnals are then continuous.

We can now prove the Dynin-Mityagin theorem. Note that, compare to the verion in the book [8] we have added the condition that the topology of E comes from a system of norm in the assumption of the theorem. We will see that this assumption is needed to the proof presented is this thesis and in the book [8].

Theorem 2.3.5 (Dynin-Mityagin theorem)**.** *Let E a nuclear Fréchet space whose topology comes from an increasing fundamental system of norms $\mathscr{P} = (p_m : m \in \mathbb{N})$ and let $(e_j)_{j\in\mathbb{N}}$ be a schauder basis of E. Then for all $m \in \mathbb{N}$, there exist $N \in \mathbb{N}$ and $C > 0$ such that for all $x \in E$ we have*

$$\sum_{j=0}^{+\infty} |c_j(x)| p_m(e_j) < C p_N(x).$$

Proof. Let $m \in \mathbb{N}$, by the previous corollary, there exist $n \in \mathbb{N}$ and $C > 0$ such that for all $x \in E$ and $j \in \mathbb{N}$ we have

$$|c_j(x)| p_m(e_j) \leq C p_n(x).$$

By nuclearity of E, there exists $M \geq n$ such that ι_M^n is nuclear. By the previous corollary, there exist $N \in \mathbb{N}$ and $C' > 0$ such that for all $x \in E$ and $j \in \mathbb{N}$ we have

$$|c_j(x)| p_M(e_j) \leq C p_N(x).$$

Let $x_0 \in$ be arbitrary, since p_n is a norm, for all $\lambda \in \ell^2$ we can consider the following linear operators :

$$\begin{aligned} T_\lambda : \ell^2 \to E_M : \mu \quad &\mapsto \sum_{j=0}^{+\infty} \lambda_j \mu_j c_j(x_0) \iota_M(e_j) \\ S_\lambda : E_n \to \ell^2 : \iota^n(x) &\mapsto (\lambda_j c_j(x) p_m(e_j))_{j\in\mathbb{N}}. \end{aligned}$$

Those operators are well defined and continuous because, if $\mu \in \ell^2$ and $x \in E$, we have

$$\sum_{j=0}^{+\infty} |\lambda_j||\mu_j||c_j(x_0)| p_M(\iota^M(e_j)) \leq C_2 p_N(x_0) \sum_{j=0}^{+\infty} |\lambda_j||\mu_j| \leq C_2 p_N(x_0) \|\lambda\|_2 \|\mu\|_2,$$

$$\left(\sum_{j=0}^{+\infty} |\lambda_j c_j(x) p_m(e_j)|^2 \right)^{1/2} \leq C_1 p_n(x) \left(\sum_{j=0}^{+\infty} |\lambda_j|^2 \right)^{1/2} = C_1 p_n(x) \|\lambda\|_2.$$

We note that if p_n was not a norm but only a seminorm, S_λ would not be well defined. Indeed, if we suppose that $x, y \in E$ satisfies $p_n(x - y) = 0$, for S_λ to be well defined we must have $c_j(x) = c_j(y)$ for all $j \in \mathbb{N}$, but then $x = y$.

By proposition 2.1.4, the operator $K := S_\lambda \circ \iota_M^n \circ T_\lambda$ is nuclear and satisfies

$$\nu(K) \leq \beta(S_\lambda) \nu(\iota_M^n) \beta(T_\lambda) \leq C p_N(x_0) \|\lambda\|_2^2$$

where $C = C_1 C_2 \nu(\iota_M^n)$. Since $c_j(e_k) = \delta_{j,k}$ holds for all $j, k \in \mathbb{N}$, we have

$$K(\mu) = (\lambda_j^2 \mu_j c_j(x_0) p_m(e_j))_{j\in\mathbb{N}}$$

or equivalently, if $f_j = (\delta_{j,k})_{k\in\mathbb{N}}$ $(j \in \mathbb{N})$ denotes the canonical basis of ℓ^2,

$$K(\mu) = \sum_{j=0}^{+\infty} \lambda_j^2 c_j(x_0) p_m(e_j) f_j \langle \mu, f_j \rangle.$$

By the lemma 2.3.2, we obtains

$$\sum_{j=0}^{+\infty} |\lambda_j|^2 |c_j(x_0)| p_m(e_j) = \nu(K) \leq C p_N(x_0) \|\lambda\|_2^2.$$

One can easily prove that since this relation holds for all $\lambda \in \ell^2$, then we must also have

$$\sum_{j=0}^{+\infty} |c_j(x_0)| p_m(e_j) \leq C p_N(x_0).$$

□

2.4 Köthe spaces

In this section we will provide a simple family of spaces (the Köthe spaces) for which we know exactly when they are nuclear.

In this section, p will always be a real number in $\{0\} \cup [1; +\infty]$ and q will be his conjugated, i.e. the real number such that $\frac{1}{p} + \frac{1}{q} = 1$ if $p \in]1, +\infty[$ and $q = \infty$ (resp. $q = 1$) if $p = 1$ (resp. $p = 0, \infty$).

To simplify the notations, we will assume that the sequences are indexed by $\mathbb{N}$ but the results needs little to no adjustment to work if the sequences are indexed by another countable family.

Definition 2.4.1. Let $\mathbf{a} \in \mathbb{K}^{\mathbb{N}}$ and $p \in [1, +\infty]$, the Banach space $\ell^p(\mathbf{a})$ is the set

$$\ell^p(\mathbf{a}) = \{\mathbf{x} \in \mathbb{K}^{\mathbb{N}} : \| \mathbf{x} \|_{\ell^p(\mathbf{a})} := \| \mathbf{a}\, \mathbf{x} \|_p < +\infty\}$$

endowed with the seminorm $\| \cdot \|_{\ell^p(\mathbf{a})}$.

Moreover the Banach space $l^0(\mathbf{a})$ is the linear subspace of $\ell^\infty(\mathbf{a})$ of sequences that vanish at infinity.

We can already define the Köthe spaces.

Definition 2.4.2. A Köthe set $A = \{\mathbf{a}^\lambda : \lambda > 0\} \subset \mathbb{K}^{\mathbb{N}}$ is a non-increasing family of sequences of positive real numbers. More specifically, this means that if $0 < \lambda < \mu$, then $0 < \mathbf{a}^\mu \leq \mathbf{a}^\lambda$ as sequences.

If $A = \{\mathbf{a}^\lambda : \lambda > 0\}$ is a Köthe set and if $p \in \{0\} \cup [1, +\infty]$, we define the Köthe space

$$\ell^p(A) = \{x \in \mathbb{K}^{\mathbb{N}} : x \in \ell^p(\mathbf{a}^\lambda), \forall \lambda > 0\}.$$

It is a locally convex space under the system of seminorms $(\| \cdot \|_{\ell^p(\mathbf{a}^\lambda)})_{\lambda > 0}$.

We will prove that the local Banach space associated to the seminorms defining $\ell^p(A)$ can be seen as closed subspaces of the classical Banach space ℓ^p. If $p \in \{0\} \cup [1; +\infty[$, it is known that the dual of ℓ^p is isometric to the Banach space ℓ^q. This result does not hold if $p = +\infty$ but in this case we have the following weaker relation between $(\ell^\infty)'$ and ℓ^1.

Lemma 2.4.3. *Let E be a linear subspace of ℓ^∞ containing the basis vectors $e_k = (\delta_{j,k})_{j\in\mathbb{N}}$ ($k \in \mathbb{N}$) and let $a \in E'$. For all $k \in \mathbb{N}$ let $a_k := \langle a, e_k\rangle$. In this condition, the sequence $(a_k)_{k\in\mathbb{N}}$ is absolutely summable and satisfy*

$$\|(a_k)_{k\in\mathbb{N}}\|_1 \leq \|a\|_{E'}$$

Proof. For $k \in \mathbb{N}$ let $\rho_k \in \mathbb{K}$ be such that $|\rho_k| = 1$ and $|a_k| = \rho_k a_k$.

Let $N \in \mathbb{N}$, we have

$$\sum_{k=0}^{N} |a_k| = \left| a\left(\sum_{k=0}^{N} \rho_k e_k\right)\right| \leq \|a\|_{E'} \left\|\sum_{k=0}^{N} \rho_k e_k\right\|_{\ell^\infty} = \|a\|_{E'}.$$

This conclues the proof since $N \in \mathbb{N}$ is arbitrary □

Lemma 2.4.4. *Let E be a closed subspace of ℓ^p such that $e_k \in E$ for all $k \in \mathbb{N}_0$ (if $p \in [1, \infty[$, this means that $E = \ell^p$). Let $T \in L(E, \ell^p)$ be an operator for which there exists a $y \in \mathbb{K}^{\mathbb{N}}$ such that $T(x) = xy$ for all $x \in E$. Then T is nuclear if and only if $y \in \ell^1$.*

Proof.

- If T is nuclear, there exists $a^{(j)} \in E'$ and $b^{(j)} \in E$ ($j \in \mathbb{N}$) such that

$$\sum_{j\geq 0} \|a^{(j)}\|_{E'} \|b^{(j)}\|_E < +\infty$$

and

$$T(x) = \sum_{j\geq 0} \left\langle a^{(j)}, x\right\rangle b^{(j)} = xy \quad \forall x \in E.$$

For $k \in \mathbb{N}$, choosing $x = e_k$ in the last equality gives

$$y_k = \sum_{j\geq 0} a_k^{(j)} b_k^{(j)},$$

where we set $a_k^{(j)} = \left\langle a^{(j)}, e_k\right\rangle$ for all $j, k \in \mathbb{N}$. The duality between ℓ^p and ℓ^q if $p \neq \infty$ and the preceding lemma else gives $\|(a_k^{(j)})_{k\in\mathbb{N}}\|_q \leq \|a\|_{E'}$. Hölder inequality finally gives

$$\sum_{k=0}^{+\infty} |y_k| \leq \sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty} |a_k^{(j)} b_k^{(j)}| = \sum_{j=0}^{+\infty} \|(a_k^{(j)} b_k^{(j)})_{k\in\mathbb{N}_0}\|_1 \leq \sum_{j=0}^{+\infty} \|(a_k^{(j)})_{k\in\mathbb{N}_0}\|_q \|b^{(j)}\|_p < +\infty.$$

- Let now suppose that $y \in \ell^1$. We have

$$T(x) = \sum_{j=0}^{+\infty} x_j y_j e_j.$$

Indeed it is obvious if $p \in \{0\} \cup [1;+\infty[$ and if $p = \infty$ we have for all $J \in \mathbb{N}_0$ that

$$\|xy - \sum_{j=0}^{J} x_j y_j e_j\|_\infty = \sup_{j>J} |x_j y_j| \leq \|x\|_\infty \sup_{j>J} |y_j|,$$

which converges to 0 when J goes to infinity since $y \in \ell^1$.
For each $j \in \mathbb{N}$ it seems then natural to define $a^{(j)}$ to be the continuous linear operator defined on E by $a^{(j)}(x) = x_j y_j$ and to set $b_j = e_j$. Clearly, we have $\|a^{(j)}\|_{E'} = |y_j|$. Finally

$$\sum_{j\geq 0} \|a^{(j)}\|_{E'} \|b^{(j)}\|_E = \|y\|_{\ell^1} < +\infty$$

and

$$T(x) = \sum_{j\geq 0} \langle a^{(j)}, x\rangle b^{(j)} \quad \forall x \in E.$$

Hence T is nuclear.

□

We can now determine under which condition on the Köthe set A and the exponent p is the associated Köthe space nuclear. The proof given is a generalization of the one given by Pietsch in [11] to arbitrary exponent p.

Theorem 2.4.5. *Let A be a Köthe set and $p \in \{0\} \cup [1;+\infty]$. The space $E = \lambda^p(A)$ is nuclear if and anly if for all $a \in A$ there exists a $b \in A$ such that $b \geq a$ and $a/b \in \ell^1$.*

Proof. Let $a \in A$ and $p_a = \|\cdot\|_{\ell^q(a)}$ the seminorm on E associated to a.

Let us start by proving that E_{p_a} can be viewed as a closed subspace of ℓ^p such that that $e_k \in E_{p_a}$ for all $k \in \mathbb{N}_0$. Since the elements of a are non-zero, we have $(E/p_a^{-1}(0), p_a) = (E, p_a)$. To find the completion of this space let consider the operator

$$T_a : (E, p_a) \to \ell^p : x \mapsto ax.$$

It is easy to check that this operator is well defined, linear, continuous and that it is an isometry. Moreover the closure of the image of T_a is a closed subspace of the Banach space ℓ^p and is then a Banach space. From this we deduce that $E_{p_a} = \overline{\text{im}(T_a)}$ and $\iota_{p_a}(x) = ax$ for all $x \in E$. Let notice that if $k \in \mathbb{N}_0$, then $a_k^{-1} e_k \in E$ and $T_a(a_k^{-1} e_k) = e_k$.

Let $b \in A$ such that $b \geq a$. For all $x \in E$, we have $T_a(x) = ax = \frac{a}{b}(bx) = \frac{a}{b} T_b(x)$, then $\iota_b^a(x) = \frac{a}{b} x$. By the preceding lemma, this operator is nuclear if ond only if $\frac{a}{b}$ belongs to l^1. □

It is worth noting that the condition does not depends on the value of p. It can even be showed that the condition given in the theorem is a necessary and sufficient condition under which the space $\ell^p(A)$ is independent of p.

We have thus the proposition

Proposition 2.4.6. *Let A be a Köthe set. The following are equivalent :*

- *For all $a \in A$, there exists $b \in A$ such that $b \geq a$ and $a/b \in \ell^1$,*
- *$\ell^p(A)$ is nuclear for all $p \in [1; +\infty]$,*
- *$\ell^p(A)$ is nuclear for a $p \in [1; +\infty]$,*
- *$\ell^p(A) = \ell^q(A)$ for all $p, q \in [1; +\infty]$.*

3 Topological tensor product

This section is dedicated to the study of the tensor product of locally convex spaces and on the topologies on this tensor product. The two topologies studied are the π-topology and the ε-topology. We will see that those two topologies coincide if at least one of the two spaces considered is nuclear.

Notations 3.0.1. In this section, the notation $\mathscr{L}(E, F)$ will be used to describe the set of all linear mappings from the linear space E to the linear space F and as always, the notation $L(E, F)$ will describe the set of all continuous linear mappings from the locally convex space E to the locally convex space F. Similarly, if G is another linear space (resp. another locally convex space), then $\mathscr{B}(E, F)$ (resp. $B(E, F)$) will describe the set of all bilinear mapping (resp. continuous bilinear mappings) from $E \times F$ to G . All the topological spaces E and F will be supposed complete.

3.1 Algebraic tensor product

Definition 3.1.1. Let E and F be two linear spaces, a tensor product $(E \bigotimes F, \varphi)$ of (E, F) consist of a linear space $E \bigotimes F$ and a bilinear operator $\varphi : E \times F \to E \bigotimes F$ whose range span $E \bigotimes F$ and is such that for all linear space G, the mapping

$$\mathscr{L}(E \bigotimes F, G) \to \mathscr{B}(E, F; G) : T \mapsto T \circ \varphi$$

defines a bijection between $\mathscr{B}(E, F; G)$ and $\mathscr{L}(E \bigotimes F, G)$.

The tensor product of two linear spaces is unique up to linear bijections.

Proposition 3.1.2. *Let E and F be two linear spaces and let (H_1, φ_1) and (H_2, φ_2) be two tensor products of (E, F). Then there exists a linear bijection $T : H_1 \to H_2$ such that $\varphi_2 = T \circ \varphi_1$.*

Proof. Since φ_2 is a bilinear operator between $E \times F$ and H_2, there exists an unique linear operator $T \in \mathscr{L}(H_1, H_2)$ such that $\varphi_2 = T \circ \varphi_1$. Similarly, there exists an unique linear operator $S \in \mathscr{L}(H_2, H_1)$ such that $\varphi_1 = S \circ \varphi_2$. The mappings $T \circ S$ and $S \circ T$ are the identical mapping on the range of φ_1 and φ_2 respectively and then on H_1 and H_2 respectively. □

In what follows we will always use the following representation of the tensor product of (E, F).

Definition 3.1.3. Let E and F be two locally convex spaces. The linear space of all formal finite sums

$$\sum_{j=1}^{r} e_j \otimes f_j \quad \text{for } e_j \in E \text{ and } f_j \in Y \ (j \in \{1, \dots r\})$$

with identification of expressions of the form

1. $(e_1 + e_2) \otimes f = e_1 \otimes f + e_2 \otimes f$,

2. $e \otimes (f_1 + f_2) = e \otimes f_1 + e \otimes f_2$,

3. $\alpha(e \otimes f) = (\alpha e) \otimes f = e \otimes (\alpha f)$,

is the canonical tensor product of (E, F).

For $T \in \mathscr{L}(E \bigotimes F, G)$ we denote by b_T the bilinear operator on $E \times F$ that maps (e, f) on $T(e \bigotimes f)$. Reciprocally, for $b \in \mathscr{B}(E, F; G)$, we denote by T_b the linear form on $E \bigotimes F$ that maps $\sum_{j=1}^{r} e_j \otimes f_j$ to $\sum_{j=1}^{r} b(e_j, f_j)$.

There are several ways to endow the tensor product of locally convex spaces with a locally convex space structure, the one that we are interested in will be described in the two following subsections.

3.2 The π-topology

By analogy with the algebraic definition of tensor product one could define a topology on $E \bigotimes F$ in such a way that the bilinear continuous operator on $E \times F$ correspond to the linear continuous operator on $E \bigotimes F$. This leads to the following definition.

Definition 3.2.1. Let E and F be two lcs. For $p \in \operatorname{csn}(E), q \in \operatorname{csn}(F)$ the seminorm $p \otimes_\pi q$ on $E \otimes F$ is defined by

$$(p \otimes_\pi q)(x) = \inf \left\{ \sum_{j=1}^{r} p(e_j) q(f_j) \right\}$$

where the infimum is taken on all the possible representation $x = \sum_{r=1}^{j} e_j \otimes f_j$ of $x \in E \bigotimes F$. The space $E \otimes F$ endowed by the system of seminorms $\{p \otimes_\pi q : p \in \operatorname{csn}(E), q \in \operatorname{csn}(F)\}$ is denoted $E \bigotimes_\pi F$.

Proposition 3.2.2. *Let E, F, G be locally convex spaces. The mapping $T \mapsto b_T$ defines a bijection between the space $L(E \bigotimes_\pi F, G)$ of continuous linear operator on $E \bigotimes_\pi F$ with values in G and the space $B(E, F; G)$ of continuous bilinear operator on $E \times F$ with values in G.*

Proof. It is clear that $\otimes$ is a continuous operator from $E \times F$ to $E \bigotimes_\pi F$. Therefore, if $T \in L(E \bigotimes_\pi F)$, then $b_T = T \circ (\cdot \otimes \cdot)$ is continuous.

Let $b \in B(E, F)$ and $s \in \operatorname{csn}(G)$, there exists $p \in \operatorname{csn}(E), q \in \operatorname{csn}(F)$ such that the relation $s(b(e, f)) \leq p(e) q(f)$ holds for all $e \in E$ and $f \in F$. If $x \in E \bigotimes F$ has the representation $x = \sum_{j=1}^{r} e_j \otimes f_j$, then we have

$$s(T_b(x)) = s \left(\sum_{j=1}^{r} b(e_j, f_j) \right) \leq \sum_{j=1}^{r} p(e_j) q(f_j),$$

since the representation of x was arbitrary this proves that $s(T_b(x)) \leq (p \otimes_\pi q)(x)$ and proves the continuity of T_b. □

3.3 The ε-topology and the ε-product

Another way to define a locally convex space structure on $E \bigotimes F$ comes from the observation that the elements $\sum_{j=1}^{r} e_j \otimes f_j$ of the tensor product $E \bigotimes F$ define naturally a bilinear form on $E' \times F'$ by

$$(e', f') \mapsto \sum_{j=1}^{r} \langle e', e_j \rangle \langle f', f_j \rangle.$$

To explore this idea we will define a space $E \,\varepsilon\, F$ of bilinear forms on $E' \times F'$ that contains all the preceding bilinear forms.

We will need the following definition.

Definition 3.3.1. Let E be a complete lcs, we set E'_c the topological dual of E endowed with the topology of uniform convergence on compact subset of E, i.e. endowed with the system of seminorms of the form

$$p_K(e') = \sup_{e \in K} |\langle e', e \rangle|$$

for all $e' \in E'$ with K a compact subset of E.

If E is complete, the Mackey-Arens theorem (theorem 1.2.7) proves that $(E'_c)'$ can be identified with E via the bijective mapping $e \mapsto \langle \cdot, e \rangle$ between E and $(E'_c)'$. Indeed, finite set of E are compact and compact sets on E are bounded.

Let E and F be two complete lcs. Since compact sets are bounded, if T is a linear continuous operator between E'_c and F, then for all $p \in \mathrm{csn}(E)$ and $q \in \mathrm{csn}(F)$, the quantity

$$(p \,\varepsilon\, q)(T) := \sup_{e' \in b_p^\circ} q(Te')$$

is finite.

We can then endow $L(E'_c, F)$ with the system of seminorms $\{p \,\varepsilon\, q : p \in \mathrm{csn}(E), q \in \mathrm{csn}(F)\}$, we denote by $L_\varepsilon(E'_c, F)$ the locally convex space obtained.

Since $(E'_c)'$ can be identified with E, if $T \in L(E'_c, F)$, then the dual operator T' can be seen as an operator from F' to E defined uniquely by the relation $\langle e', T'f' \rangle = \langle f', Te' \rangle$ for $e' \in E', f' \in F'$.

Proposition 3.3.2. *Let E and F be two complete lcs. If $T \in L(E'_c, F)$ then T' belongs to $L(F'_c, E)$ and for all $p \in csn(E), q \in csn(F)$, we have*

$$(p \,\varepsilon\, q)(T) = (q \,\varepsilon\, p)(T').$$

Proof. For the first part, let's note that by the remark 1.2.10, the set b_p° is a compact set of E'_c for all $p \in \mathrm{csn}(E)$. Let $p \in \mathrm{csn}(E)$ and $f' \in F'$, we have

$$\begin{aligned} p(T'f') &= \sup_{e' \in b_p^\circ} |\langle e', T'f' \rangle| \\ &= \sup_{e' \in b_p^\circ} |\langle f', Te' \rangle| \\ &= \sup_{f \in T(b_p^\circ)} |\langle f', f \rangle|. \end{aligned}$$

By continuity of T on E'_c, the image $T(b_p^\circ)$ of the compact subset b_p° of E'_c is a compact set of F, the continuity of T' is then proved.

For the second part, for $T \in L(E'_c, F), p \in \mathrm{csn}(E)$ and $q \in \mathrm{csn}(F)$, we have

$$\begin{aligned} (p \,\varepsilon\, q)(T) &= \sup_{e' \in b_p^\circ} q(Te') \\ &= \sup_{e' \in b_p^\circ} \sup_{f' \in b_q^\circ} |\langle f', Te' \rangle| \\ &= \sup_{f' \in b_q^\circ} \sup_{e' \in b_p^\circ} \langle e', T'f' \rangle = (q \,\varepsilon\, p)(T') \end{aligned}$$

□

Definition 3.3.3. Let E and F be two complete locally convex spaces, the space $E \,\varepsilon\, F$ is defined as either the space $L_\varepsilon(E'_c, F)$ or $L_\varepsilon(F'_c, E)$ as those two spaces can be identified as locally convex spaces.

Alternatively, $E \,\varepsilon\, F$ can be defined as the space of all bilinear forms $u : E'_c \times F'_c \to \mathbb{K}$ that are separately continuous and that satisfy one of the two following equivalent conditions

- the map $T : E'_c \to (F'_c)'_\varepsilon = F : e' \mapsto u(e', \cdot)$ is continuous,
- the map $S : F'_c \to (E'_c)'_\varepsilon = E : f' \mapsto u(\cdot, f')$ is continuous.

We have clearly $S = T'$ and the preceding result proves that the two conditions are equivalent. To obtain the same locally convex space as previously, the space is endowed with the system of seminorms of the form

$$(p \,\varepsilon\, q)(u) = \sup_{e' \in b_p^\circ} \sup_{f' \in b_q^\circ} |u(e', f')|$$

for $p \in \mathrm{csn}(E), q \in \mathrm{csn}(F)$.

It is easy to prove that if $\mathbf{x} = \sum_{j=1}^r e_j \otimes f_j \in E \bigotimes F$ then the bilinear mapping u on $E'_c \times F'_c$ defined by $u(e', f') = \sum_{j=1}^r \langle e', e_j \rangle \langle f', f_j \rangle$ is an element of $E \,\varepsilon\, F$. The space $E \bigotimes F$ can then be seen as a subspace of $E \,\varepsilon\, F$ which gives rise to the following definition.

Definition 3.3.4. Let E and F be two complete lcs. For $p \in \operatorname{csn}(E), q \in \operatorname{csn}(F)$ the seminorm $p \otimes_\varepsilon q$ on $E \bigotimes F$ is defined by

$$(p \otimes_\varepsilon q)(x) = \sup \left\{ \left| \sum_{j=1}^{r} \langle a, e_j \rangle \langle b, f_j \rangle \right| : a \in b_p^\circ, b \in b_q^\circ \right\}$$

where we suppose that $x \in E \otimes F$ can be expressed as $x = \sum_{r=1}^{j} e_j \otimes f_j$ (the chosen decomposition does not impact the result). The space $E \bigotimes F$ endowed by the system of seminorms $\{p \otimes_\varepsilon q : p \in \operatorname{csn}(E), q \in \operatorname{csn}(F)\}$ is denoted $E \bigotimes_\varepsilon F$.

We write $E \widehat{\bigotimes}_\varepsilon F$ for the completion of $E \bigotimes_\varepsilon F$. If E or F is nuclear we will prove that $E \widehat{\bigotimes}_\varepsilon F = E \,\varepsilon\, F$. To prove it we will use the fact that if E is nuclear, then it has the so-called weak approximation property and if E satisfy the weak approximation property and E, F are complete, then $E \bigotimes_\varepsilon F$ is a dense subset of $E \,\varepsilon\, F$. Since Schwartz has proven that $E \,\varepsilon\, F$ is complete for all complete lcs E and F this will prove the result.

Definition 3.3.5. Let E a complete lcs, the space $E'_c \bigotimes E$ can be interpreted as a subspace of $L(E, E)$ by the mapping

$$E'_c \bigotimes E \to L(E, E) : \mathbf{x} = \sum_{j=1}^{r} e'_j \otimes e_j \mapsto \left[T : Te = \sum_{j=1}^{r} \langle e'_j, e \rangle e_j \right].$$

The space E is said to have the weak approximation property if the idendity operator on E is in the closure of $E' \bigotimes E$ in $L_c(E, E)$ where an element $\mathbf{x} = \sum_{j=1}^{r} e'_j \otimes e_j$ of $E' \bigotimes E$ is interpreted as the linear operator $T : E \to E$ defined by

$$Te = \sum_{j=1}^{r} \langle e'_j, e \rangle e_j.$$

Proposition 3.3.6. *If E is a complete nuclear lcs, then E has the weak approximation property.*

Proof. Let $p \in \operatorname{csn}(E)$ and K a compact subset of E. By nuclearity of E there exists $q \in \operatorname{csn}(E)$ such that $q \geq p$ and ι_q^p is nuclear. Let then $(a_n)_{n \in \mathbb{N}}$ and $(e_n)_{n \in \mathbb{N}}$ be sequences on b_q° and E respectively such that

$$\sum_{n=0}^{+\infty} p(e_n) < +\infty$$

and, for all $e \in E$,

$$e = \sum_{n=0}^{+\infty} \langle a_n, e \rangle e_n \text{ in } (E, p).$$

For $e \in K$ and $N \in \mathbb{N}$, we have

$$p\left(e - \sum_{j=0}^{N} \langle a_n, e \rangle e'\right) \leq q(e) \sum_{n=N+1}^{+\infty} p(e_n).$$

Since K is compact, it is bounded and the preceding quantity converge to 0 uniformly on K. This proves the proposition since $p \in \operatorname{csn}(E)$ and K are arbitrary. □

Proposition 3.3.7. *If E and F are two complete lcs and E has the weak approximation property, then $E \bigotimes_\varepsilon F$ is a dense subset of $E \,\varepsilon\, F$.*

Proof. Let $L \in L_\varepsilon(F'_c, E)$. The map $S : L_c(E, E) \to L_\varepsilon(F'_c, E) : T \mapsto T \circ L$ is continuous, indeed, let $p \in \operatorname{csn}(E)$ and $q \in \operatorname{csn}(F)$, by the remark 1.2.10, the set b_q° is relatively compact in F'_c and then the set $K := L(b_q^\circ)$ is compact in E as the continuous image of a compact, this gives

$$(p \,\varepsilon\, q)(TL) = \sup_{f' \in b_q^\circ} p(TLf) = \sup_{e \in K} p(Te) = p_K(T)$$

which proves the continuity of S. We obtain $L = id \circ L \in S(\overline{E' \bigotimes E}) \subset \overline{S(E' \bigotimes E)}$ where the first closure is taken in $L_c(E, E)$ and the second in $E \,\varepsilon\, F$.

□

It is proven in [12] that if E and F are complete then $E \,\varepsilon\, F$ is complete, the preceding proposition proves then that $E \widehat{\bigotimes}_\varepsilon F = E \,\varepsilon\, F$.

3.4 Relation between ϵ-topology and π-topology

It is clear that for $p \in \operatorname{csn}(E)$ and $q \in \operatorname{csn}(F)$ we have $p \otimes_\varepsilon q \leq p \otimes_\pi q$, therefore the topology on $E \bigotimes_\pi F$ is finer than the one on $E \bigotimes_\varepsilon F$.

We will now prove that the two topologies coincide when at least one of the space E or F is nuclear.

Proposition 3.4.1. *Let E a complete nuclear space and F an arbitrary complete lcs. We have $E \bigotimes_\varepsilon F = E \bigotimes_\pi F$.*

Proof. Let $p \in \operatorname{csn}(E)$ and $q \in \operatorname{csn}(F)$. Since E is nuclear, there exist sequences $(a_n)_{n \in \mathbb{N}} \subset b_p^\circ$ and $(x_n)_{n \in \mathbb{N}} \subset E$ such that

$$\sum_{n=0}^{+\infty} p(x_n) < +\infty$$

and for all $e \in E$,

$$\sum_{n=0}^{+\infty} \langle a_n, e \rangle x_n = e \text{ in } (E, p). \qquad (\star)$$

Let $\mathbf{x} \in E \bigotimes F$. An application of the Hahn-Banach theorem gives a continuous linear form c on $E \otimes_\pi F$ such that $\langle c, \mathbf{x} \rangle = q(\mathbf{x})$ and $|c| \leq q$ on $E \otimes F$, this last relation and the relation $(\star)$ proves that $\langle c, e \otimes f \rangle = \sum_{n=0}^{+\infty} \langle a_n, e \rangle \langle c, x_n \otimes f \rangle$ for all $(e, f) \in E \times F$. If $\mathbf{x} = \sum_{j=1}^{r} e_j \otimes f_j$, we have

$$(p \otimes_\pi q)(\mathbf{x}) = \sum_{j=1}^{r} \langle c, e_j \otimes f_j \rangle = \sum_{n=0}^{+\infty} \sum_{j=1}^{r} \langle a_n, e_j \rangle \langle c, x_n \otimes f_j \rangle = \sum_{n=0}^{+\infty} \left\langle c, x_n \otimes \left(\sum_{j=1}^{r} \langle a_n, e_j \rangle f_j \right) \right\rangle.$$

For $n \in \mathbb{N}$, let $b_n \in b_q^\circ$ be such that $q\left(\sum_{j=1}^{r} \langle a_n, e_j \rangle f_j\right) = \left\langle b_n, \sum_{j=1}^{r} \langle a_n, e_j \rangle f_j \right\rangle$. We have

$$\begin{aligned} \left| \left\langle c, x_n \otimes \left(\sum_{j=1}^{r} \langle a_n, e_j \rangle f_j \right) \right\rangle \right| &\leq p(x_n) \left| \left\langle b_n, \sum_{j=1}^{r} \langle a_n, e_j \rangle f_j \right\rangle \right| \\ &= p(x_n) \left| \sum_{j=1}^{r} \langle a_n, e_j \rangle \langle b_n, f_j \rangle \right| \\ &\leq p(x_n) (p \otimes_\varepsilon q)(\mathbf{x}). \end{aligned}$$

Summing over n gives

$$(p \otimes_\pi q)(\mathbf{x}) \leq \sum_{n=0}^{+\infty} p(x_n) (p \otimes_\varepsilon q)(\mathbf{x}).$$

Since $\mathbf{x} \in E \bigotimes F$ is arbitrary and the relation $p \otimes_\pi q \leq p \otimes_\varepsilon q$ is clear, this concludes the proof. $\square$

If E is a complete nuclear space and F a complete space, we write $E \bigotimes F$ for the tensor product endowed with one of the two equivalent topology.

4 Gelfand-Shilov spaces

In this section, the letters μ and λ will always be natural numbers which can always be supposed to be such that $\lambda < \mu$, d will be an integer representing a dimension, the basis vectors of $\mathbb{R}^d$ will be designated $e_1, \dots, e_d$, finally we will designate by e the vector $e = (1, 1, \dots, 1) \in \mathbb{R}^d$.

4.1 Definition

4.1.1 Weight function systems

Definition 4.1.1. A weight function on $\mathbb{R}^d$ is a continuous function from $\mathbb{R}^d$ to $\mathbb{R}_0^+$.

Definition 4.1.2. A weight function system $\mathscr{W} = (w^\lambda)_{\lambda\in\mathbb{N}}$ on $\mathbb{R}^d$ is a family of weight functions on $\mathbb{R}^d$ satisfying $w^\lambda \le w^\mu$ for all indices $\lambda < \mu$.

We consider the following conditions on weight function systems.

[wM] $\forall K \Subset \mathbb{R}^n \;\; \forall \lambda \in \mathbb{N} \;\; \exists \mu \in \mathbb{N} \;\; \exists C > 0 \; \forall x \in \mathbb{R}^n \;\; \forall y \in K : w^\lambda(x+y) \le C w^\mu(x)$,

[M] $\forall \lambda \in \mathbb{N} \;\; \exists \mu \in \mathbb{N} \;\; \exists C > 0 \; \forall x, y \in \mathbb{R}^n : w^\lambda(x+y) \le C w^\mu(x) w^\mu(y)$,

[N] $\forall \lambda \in \mathbb{N} \;\; \exists \mu \in \mathbb{N} : w^\lambda / w^\mu \in L^1(\mathbb{R}^d)$,

[N′] $\forall \lambda \in \mathbb{N} \;\; \exists \mu \in \mathbb{N} : (w^\lambda(j)/w^\mu(j))_{j\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$.

An order is naturally defined on weight function systems by
• $\mathscr{W} \le \mathscr{V}$ if $\forall \lambda \in \mathbb{N} \;\; \exists \mu \in \mathbb{N} \;\; \exists C > 0 : w^\lambda \le C v^\mu$,
• $\mathscr{W} \simeq \mathscr{V}$ if $\mathscr{W} \le \mathscr{V}$ and $\mathscr{V} \le \mathscr{W}$.
We define naturally the tensor product of two weight function systems as follows.

Definition 4.1.3. Let $\mathscr{W}$ and $\mathscr{V}$ be weight function systems on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ respectively. We set

$$\mathscr{W} \otimes \mathscr{V} = \{w^\lambda \otimes v^\lambda : \lambda \in \mathbb{N}^d\}.$$

Definition 4.1.4. Associated to a weight function system $\mathscr{W}$ we will consider the Köthe set $A_{\mathscr{W}}$ defined by

$$A_{\mathscr{W}} = \{(w^\lambda(j))_{j\in\mathbb{Z}^d} : \lambda \in \mathbb{N}\}.$$

By proposition 2.4.6, for any $q \in [1; +\infty]$, $A_{\mathscr{W}}$ is nuclear if and only if $\mathscr{W}$ satisfies [N'].

4.1.2 Weight sequence systems

Definition 4.1.5. A weight sequence $M = (M_\alpha)_{\alpha\in\mathbb{N}^d}$ is a multi-indexed sequence of positive numbers satisfying

$$M_\alpha \le M_\beta \text{ if } \beta \le \alpha$$

and

$$\lim_{\alpha\to\infty}\left(\frac{M_\alpha}{M_0}\right)^{1/|\alpha|} = 0$$

We will always assume the following condition on weight sequences (where $e_n = (\delta_{n,j})_{1\le j\le d}$ are the basis vectors of $\mathbb{N}^d$).

[M.1] $\forall\alpha\in\mathbb{N}^d\ \ \forall j\in\{1,\dots,d\} : M_\alpha/M_{\alpha+e_j} \le M_{\alpha+e_j}/M_{\alpha+2e_j}$

We can define tensor products of weight sequences.

Definition 4.1.6. Let $(M^1,\dots,M^n)$ be a finite collection of weight sequences on $\mathbb{N}^{d_1},\dots,\mathbb{N}^{d_n}$ respectively. The tensor product $M^1\otimes\cdots\otimes M^n$ is the weight sequence on $\mathbb{N}^{d_1+\cdots d_n}$ defined by

$$(M^1\otimes\cdots\otimes M^n)_{(\alpha_1,\dots,\alpha_d)} = M^1_{\alpha_1}\cdots M^d_{\alpha_n} \text{ for all } (\alpha_1,\dots,\alpha_d)\in\mathbb{N}^{d_1+\cdots+d_n}.$$

Definition 4.1.7. Let M be a weight sequence on $\mathbb{N}^d$ and σ a permutation of $\{1,\dots,d\}$. We write $\sigma(M) = \left(M_{(\alpha_{\sigma_1},\dots,\alpha_{\sigma_d})}\right)_{(\alpha_1,\cdots,\alpha_d)\in\mathbb{N}^d}$.

Definition 4.1.8. Let M be a weight sequence on $\mathbb{N}^d$. M is isotropic if $M_\alpha = M_\beta$ for all $\alpha,\beta\in\mathbb{N}^d$ with $|\alpha| = |\beta|$. A weight sequence M on $\mathbb{N}^d$ is isotropically decomposable if there exist a permutation σ of $\{1,\dots,d\}$ such that $\sigma(M)$ is the tensor product of a finite number of weight sequences.

Definition 4.1.9. Given a weight sequence M, we define the associated weight function ω_M on $\mathbb{R}^d$ defined by

$$\omega_M(x) = \sup_{\alpha\in\mathbb{N}^d} M_\alpha|x^\alpha|.$$

We admit that for isotropically decomposable weight sequences, property [M.1] can be characterized as follows,

Proposition 4.1.10. *Let M be an isotropically decomposable weight sequence. Then M satisfies* [M.1] *if and only if, for all $\alpha\in\mathbb{N}^d$ we have*

$$M^\alpha = \sup_{x\in\mathbb{R}^d}\frac{\omega_M(x)}{|x^\alpha|}.$$

Moreover, for isotropically decomposable weight sequence we have the following strengthened version of the condition [M.1].

Lemma 4.1.11. *Let M be an isotropically decomposable weight sequence satisfying* [M.1]. *Then M has the following property:*

$$\forall\alpha\le\beta\in\mathbb{N}^d : M_0M_\alpha \le M_\beta M_{\alpha-\beta}.$$

Proof. We do the proof in the case when M is isotropic. We write $N = (M_{(n,\dots,0)})_{n\in\mathbb{N}}$. For $n \in \mathbb{N}$ we set $\Delta_n = N_n/n_{n+1}$. With this notation, [M.1] implies that $\Delta_n \leq \Delta_{n+1}$ for all $n \in \mathbb{N}$. We then have for all natural numbers $a < b$,

$$N_0/N_b = \Delta_0 \cdots \Delta_{b-1} \leq \Delta_{a-b} \cdots \Delta_{a-1} = N_{a-b}/N_a$$

and then $M_0 M_\alpha \leq M_\beta M_{\alpha-\beta}$ holds for all $\beta \leq \alpha \in \mathbb{N}^d$. □

We will now define weight sequence systems.

Definition 4.1.12. Let X be a topological space. A weight sequence system on X is a family $\mathfrak{M} = (M^\lambda)_{\lambda\in\mathbb{N}}$ of weight sequences on X satisfying $w^\lambda \leq w^\mu$ for all indices $\lambda < \mu$.

A weight sequence system $\mathfrak{M}$ is isotropically decomposable if for all $\lambda \in \mathbb{N}$, M^λ is isotropically decomposable. We consider the following conditions on the weight sequence systems.

[L] $\forall R > 0\ \forall \lambda \in \mathbb{N}\ \exists \mu \in \mathbb{N}\ \exists C > 0\ \forall \alpha \in \mathbb{N}^d : R^{|\alpha|} M^\lambda_\alpha \leq C M^\mu_\alpha$,

[M.2′] $\forall \lambda \in \mathbb{N}\ \exists \mu \in \mathbb{N}\ \exists C, H > 0\ \forall \alpha \in \mathbb{N}^d\ \ \forall j \in \{1, \dots, d\} : M^\lambda_{\alpha+e_j} \leq C H^{|\alpha|} M^\mu_\alpha$.

A weight sequence system is accelerating if for all $\lambda < \mu \in \mathbb{N}, \alpha \in \mathbb{N}^d$ and $j \in \{1, \dots, d\}$ we have

$$M^\lambda_\alpha / M^\lambda_{\alpha+e_j} \leq M^\mu_\alpha / M^\mu_{\alpha+e_j}.$$

An order is naturally defined on weight sequence systems by

- $\mathfrak{M} \leq \mathfrak{N}$ if $\forall \lambda \in \mathbb{N}\ \exists \mu \in \mathbb{N} : M^\lambda \leq N^\mu$,
- $\mathfrak{M} \simeq \mathfrak{N}$ if $\mathfrak{M} \leq \mathfrak{N}$ and $\mathfrak{N} \leq \mathfrak{M}$.

Definition 4.1.13. Given a weight sequence system $\mathfrak{M}$, we can naturally define a weight function system $\mathscr{W}_{\mathfrak{M}}$ by

$$\mathscr{W}_{\mathfrak{M}} = \{\omega_{M^\lambda} : \lambda \in \mathbb{N}\}.$$

For accelerating isotropically decomposable weight sequence we will admit that condition [M.2'] can be characterized as follows.

Lemma 4.1.14. *Let $\mathfrak{M}$ be an accelerating isotropically decomposable weight sequence system satisfying* [L]. *Then $\mathfrak{M}$ satisfies* [M.2′] *if and only if $\ell^1(A_{\mathscr{W}_{\mathfrak{M}}})$ is nuclear.*

4.1.3 Gelfand-Shilov spaces

From now on, we will always consider that the space X considered is $\mathbb{R}^d$.

Definition 4.1.15. Let $M = (M_\alpha)_{\alpha\in\mathbb{N}^d}$ a multi indexed sequence of positive numbers, w a positive function on $\mathbb{R}^d$ and $q \in [1; +\infty]$. The set $S^M_{w,q}$ is the set of functions $\varphi \in C^\infty(\mathbb{R}^d)$ satisfying

$$\|\varphi\|_{S^M_{w,q}} := \sup_{\alpha\in N^d} M_\alpha \|\varphi^\alpha w\|_q < +\infty.$$

It is a normed space under the norm $\|\cdot\|_{S^M_{w,q}}$.

To prove the completeness of $S^M_{w,q}$, we will use the following lemma. The original version of this lemma comes directly from the proof of lemma 3.3.3 in Lenny Neyt's thesis [9] but the validity of this original version is not clear if $d \neq 1$.

Lemma 4.1.16. *Let $I_d = \{0,1\}^d$, there exists functions $(\psi_i)_{i\in I_d}$ in $L^\infty(\mathbb{R}^d)$ with support in $[-\frac{1}{2}, \frac{1}{2}]^d$ such that for all $\varphi \in C^\infty(\mathbb{R}^d)$, we have*

$$\varphi = \sum_{i\in I_d} \varphi^{(i)} * \psi_i$$

We note directly that since the function ψ_i ($i \in I_d$) is compactly supported and bounded, it belongs to L^q for all $q \in [1;+\infty]$.

Proof. Let $\psi \in D([-\frac{1}{2}, \frac{1}{2}])$ such that $\psi = 1$ on a neighborhood of 0. We will prove the proposition by induction over the dimension d.

- If $d = 1$, let $H = \chi_{]0,+\infty[}$ the Heaviside function. We have $DH = \delta$ (here δ is the delta distribution of Dirac) which implies $D(H\psi) - \delta = D(H(\psi - 1)) = HD\psi$. Therefore, if $\varphi \in C^\infty(\mathbb{R})$, we have

  $$\varphi = \varphi * \delta = \varphi * (D(H\psi)) - \varphi * (HD\psi) = D\varphi * (H\psi) - \varphi * (HD\psi)$$

  which has the announced form if we set $\psi_0 = -HD\psi$ and $\psi_1 = H\psi$.

- If $d = 2$, using the same notation as in the previous case, we have

  $$\begin{aligned} \varphi(x_1, x_2) &= \int_{\mathbb{R}} \varphi^{(0,1)}(x_1, t_2)\psi_1(x_2 - t_2)dt_2 + \int_{\mathbb{R}} \varphi^{(0,0)}(x_1, t_2)\psi_0(x_2 - t_2)dt_2 \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi^{(1,1)}(t_1, t_2)\psi_1(x_1 - t_1)\psi_1(x_2 - t_2)dt_1dt_2 \\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi^{(0,1)}(t_1, t_2)\psi_0(x_1 - t_1)\psi_1(x_2 - t_2)dt_1dt_2 \\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi^{(1,0)}(t_1, t_2)\psi_1(x_1 - t_1)\psi_0(x_2 - t_2)dt_1dt_2 \\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi^{(0,0)}(t_1, t_2)\psi_0(x_1 - t_1)\psi_0(x_2 - t_2)dt_1dt_2 \end{aligned}$$

  which has the announced form if we set $\psi_{(i,j)} = \psi_i \otimes \psi_j$.

- The induction step is proven in the same way as the case $d = 2$.

□

Proposition 4.1.17. *Let $M = (M_\alpha)_{\alpha\in\mathbb{N}^d}$ be a multi-indexed sequence of positive numbers, w a positive function on $\mathbb{R}^d$ and $q \in [1;+\infty]$. The space $S^M_{w,q}$ is a Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of smooth functions of $S^M_{w,q}$ such that

$$\sum_{n=0}^{+\infty} \|f_n\|_{S^M_{w,q}} < +\infty.$$

Let $\alpha \in \mathbb{N}^d$ be fixed and $K \Subset \mathbb{R}^d$ a compact. If $M > N$ are two arbitrary natural numbers, we have for all $x \in K$

$$\begin{aligned}\left|\sum_{n=M}^{N} f_n^{(\alpha)}(x)\right| &= \left|\sum_{n=M}^{N}\sum_{i\in I_d}\int_{[-1/2,1/2]^d} f_n^{(\alpha+i)}(x-y)\psi_i(y)dy\right| \\ &\leq C\sum_{i\in I_d}\sum_{n=M}^{N}\left(\int_{[-1/2,1/2]^d} |f_n^{(\alpha+i)}(x-y)w(x-y)\psi_i(y)|dy\right) \\ &\leq C\sum_{i\in I_d}\|\psi_i\|_{q'}\sum_{n=M}^{N}\|f_n^{(\alpha+i)}w\|_q \\ &\leq C\sum_{i\in I_d}M_{\alpha+i}^{-1}\|\psi_i\|_{q'}\sum_{n=M}^{N}\|f_n w\|_{S^M_{w,q}}\end{aligned}$$

where $C = \sup\{w^{-1}(z) : z \in K + [-1/2;1/2]^d\}$. This proves that the derivatives of any order of the series $\sum_{n=0}^{+\infty} f_n$ are uniformly Cauchy on all compact $K \Subset \mathbb{R}^d$ and it is well known that in this case there exist a smooth function $f \in C^\infty(\mathbb{R}^d)$ such that $\sum_{n=0}^{+\infty} f_n^{(\alpha)}$ converges uniformly to $f^{(\alpha)}$ on every compact $K \Subset \mathbb{R}^d$ for all $\alpha \in \mathbb{N}^d$. As a direct consequence we have that $\sum_{n=0}^{+\infty} f_n^{(\alpha)}w$ converges ponctually to $f^\alpha w$ for all $\alpha \in \mathbb{N}^d$ and since the series $\sum_{n=0}^{+\infty} f_n^{(\alpha)}w$ is a Cauchy series in L^q this means that $f^{(\alpha)}w$ is q-integrable and that $\sum_{n=0}^{+\infty} f_n^{(\alpha)}w$ converges to $f^{(\alpha)}w$ in L^q for all $\alpha \in \mathbb{N}^d$. We then have, for all $\alpha \in \mathbb{N}^d$ and $N \in \mathbb{N}$,

$$M_\alpha\left\|\left(f^{(\alpha)} - \sum_{n=0}^{N} f_n^{(\alpha)}\right) w\right\|_q \leq \sum_{n=N+1}^{+\infty}\|f_n\|_{S^M_{w,q}}$$

which proves that $\sum_{n=0}^{+\infty} f_n$ converges to f in $S^M_{w,q}$.

□

The following result follows directly from the definitions.

Proposition 4.1.18. *Let $\mathscr{W}$ be a weight function system, $\mathfrak{M}$ a weight sequence system and $q \in [1;+\infty]$. If $\lambda < \mu$ are natural numbers then the space $S^{M^\mu}_{w^\mu,q}$ is continuously embedded in $S^{M^\lambda}_{w^\lambda,q}$.*

By the preceding proposition we can give the following definition.

Definition 4.1.19. Let $\mathscr{W}$ be a weight function system, $\mathfrak{M}$ a weight sequence system and $q \in [1;+\infty]$. The Gelfand-Shilov space $S^{\mathfrak{M}}_{\mathscr{W},q}$ associated to these system is

$$S^{\mathfrak{M}}_{\mathscr{W},q} := \varprojlim_{\lambda\to+\infty} S^{M^\lambda}_{w^\lambda,q}.$$

4.2 Influence of the parameters

Naturally, we get the following result.

Proposition 4.2.1. *Let $\mathfrak{M}, \mathfrak{N}$ be two weight sequence systems such that $\mathfrak{M} \leq \mathfrak{N}$ and $\mathscr{W}, \mathscr{V}$ two weight function systems such that $\mathscr{W} \leq \mathscr{V}$. For every $q \in [1; +\infty]$ the space $S^{\mathfrak{N}}_{\mathscr{V},q}$ is continuously included in $S^{\mathfrak{M}}_{\mathscr{W},q}$. Consequently, if $\mathfrak{M} \simeq \mathfrak{N}$ and $\mathscr{W} \simeq \mathscr{V}$, then $S^{\mathfrak{M}}_{\mathscr{W},q} = S^{\mathfrak{N}}_{\mathscr{V},q}$ as locally convex spaces.*

Proof. Let $\lambda \in \mathbb{N}$, by the hypothesis, there exists $\mu \in \mathbb{N}$ and $C > 0$ such that $M^\lambda \leq N^\mu$ and $w^\lambda \leq C v^\mu$, therefore we have the following continuous embeddings

$$S^{\mathfrak{N}}_{\mathscr{V},q} \subset S^{N^\mu}_{v^\mu,q} \subset S^{M^\lambda}_{w^\lambda,q}.$$

By definition of projective limits, this proves the theorem since $\lambda \in \mathbb{N}$ is arbitrary. □

Proposition 4.2.2. *Let $\mathfrak{M}$ be a weight sequence system satisfying properties* [L] *and* [M.2′] *and let $\mathscr{W}$ be a weight function system satisfying* [wM]. *In this condition, if $1 \leq q \leq r \leq \infty$, the following imbedding holds and is continuous :*

$$S^{\mathfrak{M}}_{\mathscr{W},q} \subset S^{\mathfrak{M}}_{\mathscr{W},r}.$$

Proof. We begin by the case $r = +\infty$.

Let $\varphi \in C^\infty(\mathbb{R}^n)$. By lemma 4.1.16, keeping the same notations we can write $\varphi = \sum_{i \in I_d} \varphi^{(i)} * \psi_i$ where $\psi_i \in \mathrm{L}^\infty(\mathbb{R}^d)$ and has a support included in $[-\frac{1}{2}, \frac{1}{2}]^d$ for all $i \in I_d$.

Hypohtesis gives, for all $\lambda \in \mathbb{N}$, a $\mu \in \mathbb{N}$ and a $C > 0$ such that $w^\lambda(x) \leq C w^\mu(t)$ for all $x \in \mathbb{R}^d$ and all $t \in x + [-\frac{1}{2}, \frac{1}{2}]^d$ and such that $M^\mu_{\alpha+i} \leq C M^\lambda_\alpha$ for all $i \in I_d$. Moreover, we can assume without loss of generality that $\lambda < \mu$.

For all $\varphi \in C^\infty(\mathbb{R}^d)$ and $(x, \alpha) \in \mathbb{R}^d \times \mathbb{N}^d$, the following relations hold :

$$\begin{aligned} M^\lambda_\alpha |\varphi^{(\alpha)}(x) w^\lambda(x)| &\leq M^\lambda_\alpha w^\lambda(x) \sum_{i \in I_d} \int_{x+[-\frac{1}{2},\frac{1}{2}]^d} |\psi_i(x-t)\varphi^{(\alpha+i)}(t)| dt \\ &\leq C^2 \sum_{i \in I_d} \|\psi_i\|_\infty M^\mu_{\alpha+i} \int_{x+[-\frac{1}{2},\frac{1}{2}]^d} |\varphi^{(\alpha+i)}(t)| w^\mu(t) dt \\ &\leq C^2 \sum_{i \in I_d} \|\psi_i\|_\infty M^\mu_{\alpha+i} \left(\int_{x+[-\frac{1}{2},\frac{1}{2}]^d} (|\varphi^{(\alpha+i)}(t)| w^\mu(t))^q dt \right)^{1/q} \\ &\leq C' \|\varphi\|_{S^{M^\mu}_{w^\mu,q}} \end{aligned}$$

where the penultimate inequality is obtained by Jensen's inequality and the last one is obtained by setting $C' = C^2 \sum_{i \in I_d} \|\psi_i\|_\infty$. Since this is true for all x and all α, this shows that

$$\|\varphi\|_{S^{M^\lambda}_{w^\lambda,\infty}} \leq C' \|\varphi\|_{S^{M^\mu}_{w^\mu,q}}$$

which concludes the case $r = \infty$.

The general case is obtained with the inequality $\|\varphi\|_r \leq \|\varphi\|_\infty^{(r-q)/r}\|\varphi\|_q^{q/r}$ that holds for all $\varphi \in \mathrm{L}^q \cap \mathrm{L}^\infty$. Indeed, if λ and μ are chosen as in the previous case, the inequality implies that

$$\|\varphi\|_{S^{M^\lambda}_{w^\lambda,r}} \leq \|\varphi\|^{(r-q)/r}_{S^{M^\lambda}_{w^\lambda,\infty}}\|\varphi\|^{q/r}_{S^{M^\lambda}_{w^\lambda,q}} \leq (C')^{(r-q)/r}\|\varphi\|_{S^{M^\mu}_{w^\mu,q}}.$$

□

Definition 4.2.3. Let $\mathfrak{M}$ be a weight sequence system and $\mathscr{W}$ a weight function system. The set $\widetilde{S}^{\mathfrak{M}}_{\mathscr{W}}$ is defined as

$$\widetilde{S}^{\mathfrak{M}}_{\mathscr{W}} = \bigcap_{\lambda\in\mathbb{N}}\bigcap_{k\in\mathbb{N}} S^{M^\lambda}_{(1+|\cdot|)^k w^\lambda,\infty}.$$

We will prove that the triviality of $S^{\mathfrak{M}}_{\mathscr{W},q}$ is deeply connected with the one of $\widetilde{S}^{\mathfrak{M}}_{\mathscr{W}}$. We will need the following lemma.

Lemma 4.2.4. *There exists a function $\chi \in C^\infty(\mathbb{R}^d)$ such that, for all polynomial $P : \mathbb{R} \to \mathbb{R}$, there exists $C > 0$ for which*

$$\sum_{\alpha\in\mathbb{N}^d} \|\chi^{(\alpha)}(x)P(|x|)\|_\infty < C$$

and such that $\chi(0) = 1$.

Proof. Let φ be a positive function in $\mathscr{D}(b(-1/2))$ such that $\|\varphi\|_1 = 1$ and let $\chi = \widehat{\varphi}$. For $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$, there exists $C > 0$ such that

$$|x|^k|D^\alpha_x\mathscr{F}_x\varphi| \leq C|x^{ke}||\mathscr{F}_{y\to x}(y^\alpha\varphi(y))| = C|\mathscr{F}_{y\to x}D^{k\cdot e}_y(y^\alpha\varphi(y))| \leq C\|D^{k\cdot e}_y(y^\alpha\varphi(y))\|_1.$$

We have $D^{k\cdot e}_y(y^\alpha\varphi(y)) = 0$ if $|y| > 1/2$, else we have

$$\begin{aligned}|D^{k\cdot e}_y(y^\alpha\varphi(y))| &\leq \sum_{\substack{\beta\leq k\cdot e\\ \beta\leq\alpha}}\binom{k\cdot e}{\beta}\frac{\alpha!}{(\alpha-\beta)!}|y^{\alpha-\beta}\varphi^{(k\cdot e-\beta)}(y)|\\ &\leq \sum_{\substack{\beta\leq k\cdot e\\ \beta\leq\alpha}}\binom{k\cdot e}{\beta}|\alpha|^{|\beta|}2^{|\beta|-|\alpha|}\|\varphi^{(k\cdot e-\beta)}\|_\infty\\ &\leq C'|\alpha|^{kd}2^{-|\alpha|}\end{aligned}$$

where $C' = \sum_{\beta\leq k\cdot e}\binom{k\cdot e}{\beta}2^{|\beta|}\|\varphi^{(k\cdot e-\beta)}\|_\infty$ depends only on k.
One can easily prove that

$$\sum_{\alpha\in\mathbb{N}^d}|\alpha|^{kd}2^{-|\alpha|} < +\infty$$

hence the theorem is proved. □

Proposition 4.2.5. *If $\mathfrak{M}$ satisfies* [L] *and $\mathscr{W}$ satisfies* [wM], *the following assertions are equivalent :*

1. $S^{\mathfrak{M}}_{\mathscr{W},q} \neq \{0\}$ *for all* $q > 0$,
2. $S^{\mathfrak{M}}_{\mathscr{W},q} \neq \{0\}$ *for a* $q > 0$,
3. $\widetilde{S}^{\mathfrak{M}}_{\mathscr{W}} \neq \{0\}$.

Proof. $1 \Rightarrow 2$ is obvious.

$2 \Rightarrow 3$. Let $\varphi \in S_q$ not identically equals to 0, $\psi \in \mathscr{D}(\mathbb{R}^d)$ a smooth function such that $\varphi * \psi(0) \neq 0$ and χ the function given by the previous lemma. We will show that the function $\varphi_0 := (\varphi * \psi)\chi$ is an element of $\widetilde{S}^{\mathfrak{M}}_{\mathscr{W}}$. Since $\varphi_0(0) = 1$ this will be enough. By the previous lemma and some reasoning used in proposition 4.2.2, for all $k \in \mathbb{N}$ and $\lambda \in \mathbb{N}$, there exists $\mu, \nu \in \mathbb{N}$ and $C, C', C'' > 0$ such that (where q' is the conjugate exponent of q)

$$\begin{aligned}
&M^{\lambda}_{\alpha}\|\varphi_0^{(\alpha)} w^{\lambda}(1+|.|)^k\|_\infty \\
\leq& M^{\lambda}_{\alpha}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\|(\varphi^{(\alpha-\beta)} * \psi)w^{\lambda}\|_\infty\|\chi^{(\beta)}(1+|.|)^k\|_\infty \\
\leq& C2^{|\alpha|}M^{\mu}_{\alpha}\sum_{\beta\leq\alpha}\|(\varphi^{(\alpha-\beta)}w^{\mu}) * \psi\|_\infty\|\chi^{(\beta)}(1+|.|)^k\|_\infty \\
\leq& C'M^{\nu}_{\alpha}\sum_{\beta\leq\alpha}\|\varphi^{(\alpha-\beta)}w^{\nu}\|_q\|\psi\|_{q'}\|\chi^{(\beta)}(1+|.|)^k\|_\infty \\
\leq& C''\sum_{\beta\leq\alpha}M^{\nu}_{\alpha-\beta}\|(\varphi^{(\alpha-\beta)}w^{\nu})\|_q\|\chi^{(\beta)}(1+|.|)^k\|_\infty \\
\leq& C\|\varphi\|_{S^{M^\nu}_{w^\nu,q}} < +\infty,
\end{aligned}$$

which proves that $\varphi_0 \in \widetilde{S}^{\mathfrak{M}}_{\mathscr{W}}$.

$3 \Rightarrow 1$. It suffices to observe that $\widetilde{S}^{\mathfrak{M}}_{\mathscr{W}} \subset S^{\mathfrak{M}}_{\mathscr{W},q}$ for all $q \in [1; +\infty]$. Indeed, if $\varphi \in \widetilde{S}^{\mathfrak{M}}_{\mathscr{W}}$ then for all $\lambda \in \mathbb{N}, \alpha \in \mathbb{N}^d$, we have

$$M^{\lambda}_{\alpha}\|\varphi^{(\alpha)}w^{\lambda}\|_q \leq M_{\alpha}\|\varphi^{(\alpha)}(1+|\cdot|)^{(d+1)/q}w^{\lambda}\|_\infty\|(1+|\cdot|)^{-(d+1)/q}\|_q < +\infty.$$

□

Lemma 4.2.6. *If $\mathscr{W}$ is a weight system on $\mathbb{R}^d$ satisfying* [wM] *and* [N] *then for all $\lambda \in \mathbb{N}$ there exists $\mu \in \mathbb{N}$ such that $w^{\lambda}/w^{\mu} \in L^1(R^d) \cap C^0_0(\mathbb{R}^d)$ where $C^0_0(\mathbb{R}^d)$ denotes the space of continuous functions on $\mathbb{R}^d$ that vanishes at infinity. As a direct consequence, the function w^{λ}/w^{μ} is an element of L^p for all $p \in [1; +\infty]$.*

Proof. For $\lambda \in \mathbb{N}$ let $\lambda' \in \mathbb{N}$ and $C > 0$ be such that for all $x \in \mathbb{R}^d, y \in [-1, 1]^d$ we have

$$w^{\lambda}(x) \leq Cw^{\lambda'}(x+y).$$

By [N] there exists $\mu' \in \mathbb{N}$ such that $w^{\lambda'}/w^{\mu'} \in L^1$. By [wM] there exists $\mu \in \mathbb{N}$ and $C' > 0$ such that for all $x \in \mathbb{R}^d, y \in [-1,1]^d$ we have

$$w^{\mu}(x) \geq C' w^{\mu'}(x+y).$$

For all $x \in \mathbb{R}^d$ and $y \in [-1,1]^d$ we obtains

$$\frac{w^{\lambda}(x)}{w^{\mu}(x)} \leq \frac{C}{C'} \frac{w^{\lambda'}(x+y)}{w^{\mu'}(x+y)}.$$

This relation gives immediatly the integrability of w^{λ}/w^{μ}. Moreover, since $w^{\lambda'}/w^{\mu'}$ is integrable it is clear that the quantity

$$\inf_{y \in [-1,1]^d} \frac{w^{\lambda'}(x+y)}{w^{\mu'}(x+y)}$$

goes to 0 as x approaches infinity which proves the theorem. □

Proposition 4.2.7. *Let $\mathfrak{M}$ be a weight sequence system and $\mathscr{W}$ a weight function system satisfying* [wM] *and* [N]. *If $1 \leq r \leq q \leq +\infty$ then the space $S^{\mathfrak{M}}_{\mathscr{W},q}$ is continuously included in $S^{\mathfrak{M}}_{\mathscr{W},r}$.*

Proof. It is a consequence of the Hölder inequality and of the previous lemma. □

Proposition 4.2.8. *Let $\mathfrak{M}$ be a weight sequence system, $\mathscr{W}$ a weight function system satisfying* [wM] *and $q \in [1;+\infty]$. If $S_q = S$ is the operator*

$$S : S^{\mathfrak{M}}_{\mathscr{W},q} \to \ell^q(A_{\mathscr{W}}) : \varphi \mapsto (\varphi(j))_{j \in \mathbb{Z}^d}$$

is well defined and is a continuous linear operator from $S^{\mathfrak{M}}_{\mathscr{W},q}$ to $\ell^q(A_{\mathscr{W}})$.

Proof. Let $(\psi_i)_{i \in \{0,1\}^d}$ be the functions defined by lemma 4.1.16. For $\varphi \in S^{\mathfrak{M}}_{\mathscr{W},q}$ and $\lambda \in \mathbb{N}$, we prove as in theorem 4.2.2 the existence of a constant $C > 0$ and an index $\mu \in \mathbb{N}$ such that for all $j \in \mathbb{Z}^d$ we have

$$|\varphi(j)| w^{\lambda}(j) \leq C \sum_{i \in \{0,1\}^d} \|\psi_i\|_{\infty} \left(\int_{j+[-1/2,1/2]^d} (|\varphi^{(i)}(t) w^{\mu}(t))^q dt \right)^{1/q}.$$

From this relation we obtains the existence of a constant $C' > 0$ such that

$$\|(\varphi(j) w^{\lambda}(j))_{j \in \mathbb{Z}^d}\|_{\ell^q(\mathbb{Z}^d)} \leq C \|\varphi\|_{S^{\mathfrak{M}}_{\mathscr{W},q}}.$$

□

Proposition 4.2.9. *Let $\mathfrak{M}$ be a weight sequence system, $\mathscr{W}$ a weight sequence system satisfying* [M] *and $q \in [1;+\infty]$. For all $\psi \in \widetilde{S}^{\mathfrak{M}}_{\mathscr{W}}$, the operator*

$$T_{\psi,q} = T_{\psi} : \ell^q(A_{\mathscr{W}}) \to S^{\mathfrak{M}}_{\mathscr{W},q} : (c_j)_{j \in \mathbb{Z}^d} \mapsto \sum_{j \in \mathbb{Z}^d} c_j \psi(\cdot - j)$$

is well defined and is a continuous linear operator from $\ell^q(A_{\mathscr{W}})$ to $S^{\mathfrak{M}}_{\mathscr{W},q}$.

Proof. Let $\mathbf{c} = (c_j)_{j\in\mathbb{Z}^d} \in \ell^q(A_{\mathscr{W}})$ and $\lambda \in \mathbb{N}$. By [M] there exists $\mu \in \mathbb{N}$ and $C > 0$ such that $w^\lambda(x+y) \le Cw^\mu(x)w^\mu(y)$ holds for all $x, y \in \mathbb{R}^d$.

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}^d} |c_j||\psi^{(\alpha)}(x-j)|w^\lambda(x) \\
\le& C\sum_{j\in\mathbb{Z}^d} \frac{|c_j|w^\mu(j)|}{(1+|x-j|)^{(d+1)/q}} \cdot \frac{\psi^{(\alpha)}(x-j)|w^\mu(x-j)(1+|x-j|)^{(d+1)}}{(1+|x-j|)^{(d+1)/q'}} \\
\le& C\|\psi^{(\alpha)}(1+|\cdot|)^{d+1}w^\mu\|_\infty \left(\sum_{j\in\mathbb{Z}^d} \frac{(|c_j|w^\mu(j))^q}{(1+|x-j|)^{d+1}}\right)^{1/q} \left(\sum_{j\in\mathbb{Z}^d} \frac{1}{(1+|x-j|)^{d+1}}\right)^{1/q'} \\
\le& C'\|\psi^{(\alpha)}(1+|\cdot|)^{d+1}w^\mu\|_\infty \left(\sum_{j\in\mathbb{Z}^d} \frac{(|c_j|w^\mu(j))^q}{(1+|x-j|)^{d+1}}\right)^{1/q}
\end{aligned}$$

this proves that for all $\alpha \in \mathbb{N}^d$, the series $\sum_{j\in\mathbb{Z}^d} c_j\psi^{(\alpha)}(\cdot - j)$ is normally convergent on every compact, thus $T_\psi(\mathbf{c}) \in C^\infty(\mathbb{R}^d)$. Finally, if p is the canonical norm on $S^{M^\mu}_{(1+|\cdot|)^{d+1}w^\mu,q}$, we have

$$\begin{aligned}
M^\lambda_\alpha \|T^{(\alpha)}_\psi(\mathbf{c})w^\lambda\|_q &\le Cp(\psi) \left\| \left(\sum_{j\in\mathbb{Z}^d} \frac{(|c_j|w^\mu(j))^q}{(1+|x-j|)^{d+1}}\right)^{1/q} \right\|_q \\
&\le C'p(\psi) \left(\sum_{j\in\mathbb{Z}^d} (|c_j|w^\mu(j))^q\right)^{1/q}.
\end{aligned}$$

□

We will prove that T_ψ act as a right inverse of S for a suitable $\psi \in \widetilde{S}^{\mathfrak{M}}_{\mathscr{W}}$.

Lemma 4.2.10. *Let* $\mathfrak{M}$ *a weight sequence system satisfying* L, $\mathscr{W}$ *a weight function system satisfying* [wM] *and* $q \in [1; +\infty]$. *If* $\widetilde{S}^{\mathfrak{M}}_{\mathscr{W}}$ *is non trivial then there exists* $\psi \in \widetilde{S}^{\mathfrak{M}}_{\mathscr{W}}$ *such that* $\psi(j) = \delta_{0,j}$ *for all* $j \in \mathbb{Z}^d$.

Proof. Let $\chi \in L^\infty(\mathbb{R}^d)$ defined by

$$\chi(x) = \frac{1}{2\pi} \cdot \mathscr{F}_x(\mathbb{1}_{[-\pi,\pi]^d}).$$

One can prove as in lemma 4.2.4 that $\|\chi^{(\beta)}\|_\infty \le \pi^{|\beta|}$ for all $\beta \in \mathbb{N}^d$. Let $\varphi \in \widetilde{S}^{\mathfrak{M}}_{\mathscr{W}}$ such that $\varphi(0) = 1$ be given by prop ... The function $\psi = \varphi\chi$ has the desired property. Indeed for all $\lambda, k \in \mathbb{N}$, there exists $\mu > \nu > \lambda$ and $C, C' > 0$ such that for all $\alpha \in \mathbb{N}^d$ we have

$$\begin{aligned} M_\alpha^\lambda \|\psi^{(\alpha)} w^\lambda (1 + |\cdot\|)^k\|_\infty &\leq M_\alpha^\lambda \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \|\varphi^{(\beta)}(1+|\cdot|)^k w^\lambda\|_\infty \|\psi^{(\alpha-\beta)}\|_\infty \\ &\leq C M_\alpha^\mu \sum_{\beta\leq\alpha} \|\varphi^{(\beta)}(1+|\cdot|)^k w^\mu\|_\infty (2\pi)^{|\alpha|-|\beta|} \\ &\leq \left(C' \sum_{\beta \in \mathbb{N}^d} (2\pi)^{-|\beta|} \right) \|\varphi\|_{S^{M^\nu}_{(1+|\cdot|)^k w^\nu, \infty}} \end{aligned}$$

and we have trivially $\psi(j) = \delta_{0,j}$ for all $j \in \mathbb{Z}^d$. □

For this ψ we have that $S \circ T_\psi = \mathrm{id}_{\ell^q(A_{\mathscr{W}})}$ which gives the following link between $\ell^q(A_{\mathscr{W}})$ and $S^{\mathfrak{M}}_{\mathscr{W},q}$.

Proposition 4.2.11. *Let $\mathfrak{M}$ be a weight sequence system satisfying* [L]*, $\mathscr{W}$ a weight function system satisfying* [M] *and $q \in [1; +\infty]$. If $\widetilde{S}^{\mathfrak{M}}_{\mathscr{W}}$ is non-trivial then the space $\ell^q(A_{\mathscr{W}})$ is isomorphic to a subspace of $S^{\mathfrak{M}}_{\mathscr{W},q}$.*

Proposition 4.2.12. *Let $\mathfrak{M}$ be a weight sequence system satisfying* [L] *and* [M.2′] *and let $\mathscr{W}$ be a weight function system satisfying* [wM]*. Let us also assume that $\widetilde{S}^{\mathfrak{M}}_{\mathscr{W}}$ is non-trivial. Consider the following assertions :*

1. *$\mathscr{W}$ satisfies [N].*
2. *$S^{\mathfrak{M}}_{\mathscr{W},q} = S^{\mathfrak{M}}_{\mathscr{W},r}$ as sets, for all $q, r \in [1; +\infty]$,*
3. *$S^{\mathfrak{M}}_{\mathscr{W},q} = S^{\mathfrak{M}}_{\mathscr{W},r}$ as locally convex spaces, for specific $q, r \in [1; +\infty]$,*

Then the implications $1 \Rightarrow 2 \Rightarrow 3$ are always true and $3 \Rightarrow 1$ is true if $\mathscr{W}$ satisfies [M]*.*

Proof. $1 \Rightarrow 2$ is a combination of proposition 4.2.2 and proposition 4.2.7.
$2 \Rightarrow 3$ is obvious.
$3 \Rightarrow 1$ (if $\mathscr{W}$ satisfies [M]). We can suppose that $q \leq r$. Let $\psi \in \widetilde{S}^{\mathfrak{M}}_{\mathscr{W}}$ given by lemma 4.2.10. We know that $S_r \circ T_{\psi,r} = \mathrm{id}_{\ell^r(A_{\mathscr{W}})}$ but since $S^{\mathfrak{M}}_{\mathscr{W},r} = S^{\mathfrak{M}}_{\mathscr{W},q}$ as sets, algebraically, we have $S_r \circ T_{\psi,r} = S_q \circ T_{\psi,r}$ which is a continuous map from $\ell^r(A_{\mathscr{W}})$ to $\ell^q(A_{\mathscr{W}})$. This proves that $\ell^r(A_{\mathscr{W}})$ is continuously included in $\ell^q(A_{\mathscr{W}})$. Since the inclusion $\ell^q(A_{\mathscr{W}}) \subset \ell^r(A_{\mathscr{W}})$ always holds and is continuous this proves that $\ell^q(A_{\mathscr{W}}) \subset \ell^r(A_{\mathscr{W}})$ as locally convex spaces. By the proposition 2.4.6, this implies that $\mathscr{W}$ satisfies [N']. Since W satisfies also [wM], similar arguments as the one used in lemma 4.2.6 proves that in this case $\mathscr{W}$ satisfies [N]. □

Notation 4.2.13. If $\mathfrak{M}$ satisfies [L] and [M.2'] and $\mathscr{W}$ satisfies [wM] and [N], we will denote by $\mathrm{S}^{\mathfrak{M}}_{\mathscr{W}}$ the space $S^{\mathfrak{M}}_{\mathscr{W},q}$ for any q.

4.3 Nuclearity of Gelfand-Shilov spaces

We can now study the nuclearity of Gelfand Shilov spaces.

Proposition 4.3.1. *Let* $\mathfrak{M}$ *be a weight sequence system satisfying* [L] *and* [M.2′] *and let* $\mathscr{W}$ *be a non-degenerate weight function system satisfying* [wM] *and* [N]*. Under these conditions, the space* $\mathrm{S}^{\mathfrak{M}}_{\mathscr{W}}$ *is nuclear.*

Proof. Let $(\varphi_j)_{j\in\mathbb{N}} \in \ell^1\left(\mathrm{S}^{\mathfrak{M}}_{\mathscr{W}}\right) = \ell^1\left(S^{\mathfrak{M}}_{\mathscr{W},\infty}\right)$. As we saw when defining weakly summable sequences, this means that for all $\lambda > 0$, there exists $C > 0$ such that for all sequence $\mathbf{c} \in \ell^\infty$ with $\|\mathbf{c}\|_\infty \leq 1$ and $k \in \mathbb{N}$, we have

$$\left\| \sum_{n=0}^{k} c_n \varphi_n \right\|_{S^{M^\lambda}_{w^\lambda,\infty}} < C.$$

For $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$, setting c_n such that $c_n\varphi^{(\alpha)}(x) = |\varphi^{(\alpha)}(x)|$ in the previous relation gives

$$M^\lambda_\alpha \sum_{n=0}^{+\infty} |\varphi_n^{(\alpha)}(x)| w^\lambda(x) < C.$$

Let $\lambda > 0$, by [L] and [N] there is a $\mu > 0$ such that $w^\lambda/w^\mu \in L^1$ and $M^\lambda_\alpha \leq C' 2^{-|\alpha|} M^\mu_\alpha$ for all $\alpha \in \mathbb{N}^d$. In this condition we have

$$\begin{aligned}
\sum_{n=0}^{+\infty} \|\varphi_n\|_{S^{M^\lambda}_{w^\lambda,1}} &\leq \sum_{n=0}^{+\infty} \sum_{\alpha\in\mathbb{N}^d} M^\lambda_\alpha \|\varphi_n^{(\alpha)}(x) w^\lambda(x)\|_1 \\
&\leq C' \sum_{\alpha\in\mathbb{N}^d} \sum_{n=0}^{+\infty} 2^{-|\alpha|} M^\mu_\alpha \|\varphi_n^{(\alpha)} w^\mu\|_\infty \|w^\lambda/w^\mu\|_1 \\
&\leq CC' 2^d \|w^\lambda/w^\mu\|_1 < +\infty
\end{aligned}$$

This proves that $(\varphi_j)_{j\in\mathbb{N}} \in \ell^1\left\{S^{\mathfrak{M}}_{\mathscr{W},1}\right\} = \ell^1\left\{\mathrm{S}^{\mathfrak{M}}_{\mathscr{W}}\right\}$. □

Proposition 4.3.2. *Let* $\mathfrak{M}$ *be a weight sequence system satisfying* [L]*,* $\mathscr{W}$ *be a weight function system satisfying* [M] *and* $q \in [1;+\infty]$*. If we suppose additionally that the space* $S^{\mathfrak{M}}_{\mathscr{W},q}$ *is non-trivial and nuclear, then* $\mathscr{W}$ *satisfies* [N]*.*

Proof. By proposition 4.2.11, the space $\ell^q(A_{\mathscr{W}})$ can be embedded in $S^{\mathfrak{M}}_{\mathscr{W},q}$ by a linear homeomorphism T. By proposition 2.1.7 and proposition 2.1.8, $\ell^q(A_{\mathscr{W}})$ is then nuclear and thus $\mathscr{W}$ satisfies [N']. Since $\mathscr{W}$ satisfies also [M] and thus [wM], this implies that $S^{\mathfrak{M}}_{\mathscr{W},q}$ satisfies [N]. □

In a restricted case, we can prove that the condition [M.2'] is also a necessary condition for nuclearity. To prove this result we will need some lemmas.

Lemma 4.3.3. *Let E be a nuclear Fréchet space whose topology originates from a dundamental set of norm and $A = \{a^\lambda : \lambda \in \mathbb{N}\}$ a Köthe set. If $T : \ell^1(A) \to E$ and $S : E \to \ell^\infty(A)$ are continuous mapping such that $S \circ T$ is the canonical mapping from $\ell^1(A)$ to $\ell^\infty(A)$, then $\ell^1(A)$ is nuclear.*

Proof. Since $T(\ell^1(A))$ is a topological subset of the nuclear space E, by lemma ... it is sufficient to prove that T induces a linear homeomorphism between $\ell^1(A)$ ans its image. Since $S \circ T$ is one-to-one, T must be one-to-one too.

In this prove we will designate by $e_j = (\delta_{j,k})_{k\in\mathbb{Z}^d}$ ($j \in \mathbb{Z}^d$) the canonical "base" of $\mathbb{Z}^d$. $(e_j)_{j\in\mathbb{Z}^d}$ is a Schauder basis of $\ell^1(A)$ whose coefficient functionnals $(c_j)_{j\in\mathbb{Z}^d}$ are given, for $j \in \mathbb{Z}^d$, by

$$c_j : \ell^1(A) \to \mathbb{K} : (b_k)_{k\in\mathbb{Z}^d} \mapsto b_j.$$

The family $(Te_j)_{j\in\mathbb{Z}^d}$ is a Schauder basis of $T(\ell^1(A))$. Indeed if $\mathbf{b} \in \ell^1(A)$ can be expressed as $\mathbf{b} = \sum_{j=0}^{+\infty} c_j(\mathbf{b})e_j$, then $T\mathbf{b} = \sum_{j=0}^{+\infty} c_j(\mathbf{b})T(e_j)$ and if $x, y \in T(\ell^1(A))$ have a different decomposition, their image by S will be different and x must be different from y. We denote by $(\tilde{c}_j)_{j\in\mathbb{Z}^d}$ the coefficient functionnals associated to the basis $(Te_j)_{j\in\mathbb{Z}^d}$ in $T(\ell^1(A))$, we have $\tilde{c}_j = c_j \circ T^{-1}$.

By the Dynin-Mityagin theorem (theorem 2.3.5), for all norm p on E there exists a norm q on E such that for all $x \in E$ we have

$$\sum_{j=0}^{+\infty} |\tilde{c}_j(x)| p(Te_j) \le q(x). \tag{2}$$

Let $\lambda \in \mathbb{N}$, by continuity of S, there exists a norm p on E such that $\|S(\cdot)\|_{\ell^\infty(a^\lambda)} \le p(\cdot)$. For this norm, let q the norm on E given by the relation (2). For all $\mathbf{b} \in \ell^1(A)$, we have

$$\begin{aligned}
\|\mathbf{b}\|_{\ell^1(a^\lambda)} &= \sum_{j=0}^{+\infty} |b_j| \|e_j\|_{\ell^1(a^\lambda)} \\
&= \sum_{j=0}^{+\infty} \tilde{c}_j(T\mathbf{b}) \|S(Te_j)\|_{\ell^\infty(a^\lambda)} \\
&\le \sum_{j=0}^{+\infty} \tilde{c}_j(T\mathbf{b}) p(Te_j) \\
&\le q(T\mathbf{b}).
\end{aligned}$$

Hence, $T^{-1} : T(\ell^1(A)) \to \ell^1(A)$ is continuous. □

Lemma 4.3.4. *Let $\mathfrak{M}$ be a weight sequence system satisfying* [wM] *and let $\mathscr{W}$ be a weight function system satisfying* [wM]*. If $\widetilde{S}^{\mathfrak{M}}_{\mathscr{W}} \neq \{0\}$, then there exists $\psi \in \widetilde{S}^{\mathfrak{M}}_{\mathscr{W}}$ such that*

$$\sum_{j \in \mathbb{Z}^d} \psi(x-j) = 1$$

for all $x \in \mathbb{R}^d$.

Proof. Let $\varphi \in \widetilde{S}^{\mathfrak{M}}_{\mathscr{W}}$ such that $\int_{\mathbb{R}^d} \varphi(x)dx = 1$. We will prove that the function ψ defined on $\mathbb{R}^d$ by

$$\psi(x) = \int_{[-\frac{1}{2},\frac{1}{2}]^d} \varphi(x-t)dt$$

satisfies the requirements. It is clear that

$$\sum_{j \in \mathbb{Z}^d} \psi(x-j) = 1$$

for all $x \in \mathbb{R}^d$. Since $\psi = \varphi * \chi_{[-1/2,1/2]^d}$, we have for all $\alpha \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$ that

$$\psi^{(\alpha)}(x) = \int_{[-\frac{1}{2},\frac{1}{2}]^d} \varphi^{(\alpha)}(x-t)dt.$$

Let $\lambda \in \mathbb{N}$ and $k \in \mathbb{N}$, since $\mathscr{W}$ satisfies [wM], there exists a natural number $\mu > \lambda$ and a $C > 0$ such that $w^\lambda(x) \leq C w^\mu(x-t)$ for all $t \in [-1/2, 1/2]^d$. Moreover, for all $x \in \mathbb{R}^d$ and $t \in [-1/2, 1/2]^d$, we have

$$(1+|x|)^k \leq (1+|x-t|)^k(1+|t|)^k \leq (3/2)^k(1+|x-t|)^k.$$

Hence, for all $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$, if $C' = (3/2)^k C$, we have

$$\begin{aligned} M^\lambda_\alpha |\psi^{(\alpha)}(x)| w^\lambda(x)(1+|x|)^k &\leq C' \int_{[-\frac{1}{2},\frac{1}{2}]^d} M^\mu_\alpha |\varphi^{(\alpha)}(x-t)| w^\mu(x-t)(1+|x-t|)^k dt \\ &\leq C' \|\varphi\|_{S^{M^\mu}_{(1+|\cdot|)^k w^\mu, \infty}}. \end{aligned}$$

□

Proposition 4.3.5. *Let $\mathfrak{M}$ be an isotropically decomposable weight sequence satisfying* [L]*, $\mathscr{W}$ a non-degenerate weight function system satisfying* [M] *and let $q \in [1; +\infty]$. If $S^{\mathfrak{M}}_{\mathscr{W},q}$ is non-trivial and nuclear then $\ell^1(A_{\mathscr{W}_{\mathfrak{M}}})$ is nuclear.*

Proof. For $r \in \{1, \infty\}$ we define the auxiliary space $\mathscr{E}^{\mathfrak{M}}_{\mathrm{per},r}$ of all $\mathbb{Z}^d$ periodic smooth function $\varphi \in C^\infty(\mathbb{R}^d)$ satisfying for all $\lambda > 0$,

$$\|\varphi\|_{\mathscr{E}^{M^\lambda}_{\mathrm{per},r}} := \sup_{\alpha \in \mathbb{N}^d} M^\lambda_\alpha \|\varphi^{(\alpha)}\|_{L^r([-\frac{1}{2},\frac{1}{2}]^d)} < +\infty.$$

It is endowed by its natural Fréchet structure.

We consider the following linear operators :

$$T_0 : \ell^1(A_{\mathscr{W}_{\mathfrak{M}}}) \to \mathscr{E}^{\mathfrak{M}}_{\mathrm{per},\infty} : (c_j)_{j\in\mathbb{Z}^d} \mapsto \left[\varphi : x \mapsto \sum_{j=0}^{+\infty} c_j e^{-2\pi i\langle j,x\rangle}\right],$$

$$S_0 : \mathscr{E}^{\mathfrak{M}}_{\mathrm{per},1} \to \ell^\infty(A_{\mathscr{W}_{\mathfrak{M}}}) : \varphi \mapsto \left(\int_{[-\frac{1}{2},\frac{1}{2}]^d} \varphi(x)e^{2\pi i\langle j,x\rangle}\right)_{j\in\mathbb{Z}^d}.$$

We won't prove the continuity of these mappings but the key argument is that $|D_x^\alpha e^{\pm 2i\pi\langle j,x\rangle}| = |j^\alpha e^{\pm 2i\pi\langle j,x\rangle}|$, the continuity of the second mapping must be proven using integration by parts. Now, let ψ be given by the preceding lemma. We consider two more linear mappings :

$$T_1 : \mathscr{E}^{\mathfrak{M}}_{\mathrm{per},\infty} \to S^{\mathfrak{M}}_{\mathscr{W},\infty} : \varphi \mapsto \varphi\psi,$$

$$S_1 : S^{\mathfrak{M}}_{\mathscr{W},1} \to \mathscr{E}^{\mathfrak{M}}_{\mathrm{per},1} : \varphi \mapsto \sum_{j=0}^{+\infty} \varphi(\cdot - j).$$

To prove the continuity of T_1 we will use lemma 4.1.11 which is why we required $\mathfrak{M}$ to be isotropically decomposable (unlike in the thesis of Lenny Neyt). Let $\lambda \in \mathbb{N}$, by conditions [L] and lemma 4.1.11, there exists $\mu \in \mathbb{N}$ and $C > 0$ such that for all $\alpha \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$ we have

$$\begin{aligned} M_0^\lambda M_\alpha^\lambda |(\psi\varphi)^{(\alpha)}(x)|w^\lambda(x) &\leq 2^{-|\alpha|} M_0^\mu M_\alpha^\mu \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} |\psi^{(\beta-\alpha)}(x)||\varphi^{(\beta)}(x)|w^\lambda(x) \\ &\leq 2^{-|\alpha|} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \left(M_{\beta-\alpha}^\mu |\psi^{(\alpha-\beta)}(x)|w^\lambda(x)\right) M_\beta^\mu |\varphi^{(\beta)}(x)| \\ &\leq \|\psi\|_{S^{M^\lambda}_{w^\lambda,\infty}} \|\varphi\|_{\mathscr{E}^{M^\lambda}_{\mathrm{per},\infty}}. \end{aligned}$$

Before proving the continuity of S_1, we have to prove that it is well defined, this step seems to require that $\mathscr{W}$ is non-degenerate. Let then $\lambda > 0$ and $(\psi_i)_{i\in I_d}$ be the sequence of functions given by lemma 4.1.16. For all $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$ we have

$$\begin{aligned} \sum_{j=0}^{+\infty} |\varphi^{(\alpha)}(x-j)|w^\lambda(x-j) &= \sum_{j=0}^{+\infty} \left|\sum_{i\in I_d} \int_{[-1/2,1/2]^d} \varphi^{(\alpha+i)}(x-j-t)\psi_i(t)dt\right| w^\lambda(x-j) \\ &\leq C \sum_{i\in I_d}\sum_{j=0}^{+\infty} \int_{[-1/2,1/2]^d} |\varphi^{(\alpha+i)}(x-j-t)|w^\mu(x-j-t)\psi_i(t)dt \\ &\leq C\|\psi_i\|_\infty \int_{\mathbb{R}^d} |\varphi^{(\alpha+i)}(x-t)|w^\mu(x-t)dt \\ &\leq C\|\varphi\|_{S^{M^\mu}_{w^\mu,1}} \sum_{i\in I_d} \frac{\|\psi_i\|_\infty}{M^\mu_{\alpha+i}}. \end{aligned}$$

Since $\mathscr{W}$ is non-degenerate, there exists $C > 0$ such that

$$\sum_{j=0}^{+\infty} |\varphi^{(\alpha)}(x-j)| \leq C \sum_{j=0}^{+\infty} |\varphi^{(\alpha)}(x-j)| w^{\lambda}(x-j)$$

and since the latter series is uniformy bounded on $\mathbb{R}^d$, S_1 is well defined and we can derive the series term by term, the continuity of S_1 is then easy to obtain by using again the character non-degenerate of $\mathscr{W}$.

By the proposition 4.3.2, $\mathscr{W}$ has property [N] and by lemma 4.2.7, there exists continuous linear embeddings $T_2 : S^{\mathfrak{M}}_{\mathscr{W},\infty} \to S^{\mathfrak{M}}_{\mathscr{W},q}$ and $S_2 : S^{\mathfrak{M}}_{\mathscr{W},q} \to S^{\mathfrak{M}}_{\mathscr{W},1}$.

Now if we define $T = T_2 \circ T_1 \circ T_0$ and $S = S_0 \circ S_1 \circ S_2$, then by choice of ψ, $S \circ T$ is the canonical mapping from $\ell^1(A_{\mathscr{W}_{\mathfrak{M}}})$ to $\ell^\infty(A_{\mathscr{W}_{\mathfrak{M}}})$ and lemma 4.3.3 concludes the proof. □

We obtain the following theorem.

Theorem 4.3.6. *Let $\mathfrak{M}$ be an isotropically decomposable weight sequence satisfying* [L]*, $\mathscr{W}$ a non-degenerate weight function system satisfying* [M] *and let $q \in [1; +\infty]$. Suppose that $\widetilde{S}^{\mathfrak{M}}_{\mathscr{W}} \neq \{0\}$. Then the following are equivalent :*

1. *$\mathfrak{M}$ satisfies [M.2'] and $\mathscr{W}$ satisfies [N],*
2. *$S^{\mathfrak{M}}_{\mathscr{W},q}$ is nuclear for all $q \in [1; +\infty]$,*
3. *$S^{\mathfrak{M}}_{\mathscr{W},q}$ is nuclear for some $q \in [1; +\infty]$.*

Proof. It is a direct consequence of the prop 4.3.1, of proposition 4.3.2 and of the combination of the preceding proposition and of lemma 4.1.14. □

By proposition 4.3.1 and proposition 4.3.2, we have also the following theorem.

Theorem 4.3.7. *Let M be a weight sequence system satisfying* [L] *and* [M.2′]. *Let W be a weight function system satisfying* [M]. *Suppose that $\widetilde{S}^{\mathfrak{M}}_{\mathscr{W}} \neq \{0\}$. Then, the following statements are equivalent :*

1. *$\mathscr{W}$ satisfies* [N],
2. *$S^{\mathfrak{M}}_{\mathscr{W},q}$ is nuclear for all $q \in [1; +\infty]$,*
3. *$S^{\mathfrak{M}}_{\mathscr{W},q}$ is nuclear for some $q \in [1; +\infty]$.*

4.4 Tensor product of Gelfand-Shilov spaces

In this subsection, we will always fix the value of the parameter q to ∞. Given a weight sequence system $\mathfrak{M}$ and a weight function system $\mathscr{W}$, we will write $S^{\mathfrak{M}}_{\mathscr{W}}$ instead of $S^{\mathfrak{M}}_{\mathscr{W},\infty}$. This is coherent with notation 4.2.13.

The goal of this subsection is to use the nuclearity $S^{\mathfrak{M}}_{\mathscr{W}}$ or $S^{\mathfrak{N}}_{\mathscr{V}}$ to prove that the space $S^{\mathfrak{M}\otimes\mathfrak{N}}_{\mathscr{W}\otimes\mathscr{V}}$ is equals to $S^{\mathfrak{M}}_{\mathscr{W}}\widehat{\otimes}S^{\mathfrak{N}}_{\mathscr{V}}$ as locally convex spaces. To achieve this goal we introduce, for a locally convex space E, a new spaces denoted $S^{\mathfrak{M}}_{\mathscr{W}}(E)$ and prove that $S^{\mathfrak{M}\otimes\mathfrak{N}}_{\mathscr{W}\otimes\mathscr{V}}$ and $S^{\mathfrak{M}}_{\mathscr{W}}\widehat{\otimes}S^{\mathfrak{N}}_{\mathscr{V}}$ can both be associated to $S^{\mathfrak{M}}_{\mathscr{W}}(S^{\mathfrak{N}}_{\mathscr{V}})$.

The following definition and proposition comes from the book *The Convenient Setting of Global Analysis* [7].

Definition 4.4.1. Let E be a locally convex spaces and $j \in \{1, \cdots, d\}$. A continuous function $\varphi \in C(\mathbb{R}^d, E)$ is differentiable in the j direction if for all $x \in \mathbb{R}^d$, the quantity

$$\frac{\varphi(x + te_j) - \varphi(x)}{t}$$

converges to a locally bounded limit $D_j\varphi(x) \in E$ as t converges to 0. The function φ is continuously differentiable if D_j is continuous.

We define $C^0(\mathbb{R}^d, E) = C(\mathbb{R}^d, E)$ and recursively, if $\alpha \in \mathbb{N}^d$ $\varphi \in C(\mathbb{R}^d, E)$ is in $C^\alpha(\mathbb{R}^d, E)$ if $D_j\varphi \in C^{\alpha-e_j}(\mathbb{R}^d, E)$ for all $j \in \{1, \ldots, d\}$. Naturally we define $C^\infty(\mathbb{R}^d, E) = \cap_{\alpha\in\mathbb{N}^d} C^\alpha(\mathbb{R}^d, E)$.

One can prove that in this setting, if $\varphi \in C^\infty(\mathbb{R}^d, E)$ then for all $j, k \in \{1, \ldots, d\}$ we have $D_jD_k\varphi = D_kD_j\varphi$. We can then make sense of the expression $D^\alpha\varphi = \varphi^{(\alpha)}$ for $\alpha \in \mathbb{N}^d$.

Proposition 4.4.2. *If $\varphi \in C^\infty(\mathbb{R}^d, E)$ and $e' \in E'$, then $\langle e', \varphi\rangle$ is in $C^\infty(\mathbb{R}^d)$ and for all $\alpha \in \mathbb{N}^d$, we have $D^\alpha\langle e', \varphi\rangle = \langle e', \varphi^{(\alpha)}\rangle$.*

Definition 4.4.3. Let E be a lcs, $\mathfrak{M}$ a weight sequence system and $\mathscr{W}$ a weight function system. The space $S^{\mathfrak{M}}_{\mathscr{W}}(E)$ is the space of all functions $\varphi \in C^\infty(S^{\mathfrak{M}}_{\mathscr{W}}, E)$ such that for all $p \in \operatorname{csn}(E)$ and $\lambda \in \mathbb{N}$, the quantity

$$p_\lambda(\varphi) := \sup_{\alpha\in\mathbb{N}^d} M^\lambda_\alpha \|p\left(\varphi^{(\alpha)}\right) w^\lambda\|_\infty$$

is finite. It is a locally convex space under the system of seminorms $\{p_\lambda : p \in \operatorname{csn}(E), \lambda \in \mathbb{N}\}$.

Lemma 4.4.4. *Let $\mathfrak{M}$ be a weight sequence system and $\mathscr{W}$ a weight function system. Let also $\psi \in C^\infty(\mathbb{R}^{d_1}, S^{\mathfrak{M}}_{\mathscr{W}}(\mathbb{R}^{d_2}))$ and let φ be the function defined on $\mathbb{R}^{d_1+d_2}$ by $\varphi(x, y) = \psi(x)(y)$. The function φ is infinetely differentiable and satisfies for all $(\alpha, \beta) \in \mathbb{N}^{d_1+d_2}$ the relation*

$$\varphi^{(\alpha,\beta)} := D^\beta_y D^\alpha_x \varphi(x, y) = \left(\psi^{(\alpha)}(x)\right)^{(\beta)}(y)$$

for all $(x, y) \in \mathbb{R}^{d_1+d_2}$.

Proof. Let $j \in \{1, \ldots, d_1\}$ and $x \in \mathbb{R}^{d_1}$. Since

$$\lim_{h \to 0} \frac{\psi(x + he_j) - \psi(x)}{h} = D_j \psi(x)$$

in $S^{\mathfrak{M}}_{\mathscr{W}}$, we have for all $y \in \mathbb{R}^{d_2}$ that

$$\lim_{h \to 0} \frac{\varphi(x + he_j, y) - \varphi(x, y)}{h} = D_j \varphi(x, y).$$

We then have that $\varphi^{(e_j,0)}$ exists and is equals to $\psi^{(e_j)}$. Iterating this argument proves that for all $\alpha \in \mathbb{N}^{d_1}$, $\varphi^{(\alpha,0)}$ exists and is equals to $\psi^{(\alpha)}$.

Since $\psi^{(\alpha)}(x) \in S^{\mathfrak{M}}_{\mathscr{W}} \subset C^\infty(\mathbb{R}^{d_2})$ for all $\alpha \in \mathbb{N}^{d_1}$and $x \in \mathbb{R}^{d_1}$, it is clear that for all $\beta \in \mathbb{R}^{d_2}$, $\varphi^{(\alpha,\beta)}(x, y)$ exists and is equals to $\left(\psi^{(\alpha)}(x)\right)^{(\beta)}(y)$ for all $y \in \mathbb{R}^{d_2}$. □

Proposition 4.4.5. *For all weight sequence systems $\mathfrak{M}, \mathfrak{N}$ and weight function systems $\mathscr{W}, \mathscr{V}$, we have $S^{\mathfrak{M} \otimes \mathfrak{N}}_{\mathscr{W} \otimes \mathscr{V}} \simeq S^{\mathfrak{M}}_{\mathscr{W}}(S^{\mathfrak{N}}_{\mathscr{V}})$.*

Proof. It is enough to prove that the mappings

$$T : S^{\mathfrak{M} \otimes \mathfrak{N}}_{\mathscr{W} \otimes \mathscr{V}} \to S^{\mathfrak{M}}_{\mathscr{W}}(S^{\mathfrak{N}}_{\mathscr{V}}) : \varphi \mapsto [\psi : x \mapsto (y \mapsto \varphi(x, y))]$$

and

$$S : S^{\mathfrak{M}}_{\mathscr{W}}(S^{\mathfrak{N}}_{\mathscr{V}}) \to S^{\mathfrak{M} \otimes \mathfrak{N}}_{\mathscr{W} \otimes \mathscr{V}} : \psi \mapsto [\varphi : (x, y) \mapsto \psi(x)(y)]$$

are well defined and continuous.

We begin by proving that for all $\varphi \in S^{\mathfrak{M} \otimes \mathfrak{N}}_{\mathscr{W} \otimes \mathscr{V}}$, $\psi := T\varphi$ belongs to $S^{\mathfrak{M}}_{\mathscr{W}}(S^{\mathfrak{N}}_{\mathscr{V}})$. Let then $\varphi \in S^{\mathfrak{M} \otimes \mathfrak{N}}_{\mathscr{W} \otimes \mathscr{V}}$. Let $x \in \mathbb{R}^{d_1}$ be arbitrary, for all $\mu > 0, \beta \in \mathbb{N}^{d_2}$ and $y \in \mathbb{R}^{d_2}$, we have

$$N^\mu_\beta |D^\beta_y(\psi(x)(y)) v^\mu(y)| = \frac{1}{M^\mu_0 w^\mu(x)} M^\mu_0 N^\mu_\beta |\varphi^{(0,\beta)} w^\mu(x) v^\mu(y)| \leq \|\varphi\|_{S^{M^\mu \otimes N^\mu}_{w^\mu \otimes v^\mu}},$$

hence, $\psi(x) \in S^{\mathfrak{N}}_{\mathscr{V}}$ for all $x \in \mathbb{R}^{d_1}$. We may now prove that $\psi \in C^\infty(\mathbb{R}^d, S^{\mathfrak{N}}_{\mathscr{V}})$ and more specifically that $\psi^{(\alpha)}(x)(y) = \varphi^{(\alpha,0)}(x, y)$ for all $\alpha \in \mathbb{N}^{d_1}$ and $(x, y) \in \mathbb{R}^{d_1+d_2}$. To prove that, we consider $j \in \{1, \ldots, d_1\}, (\alpha, \beta) \in \mathbb{N}^{d_1+d_2}$, $(x, y) \in \mathbb{R}^{d_1+d_2}$. For all $h \in \mathbb{R}$ with $|h| < 1$ a double application of the mean value theorem gives a $t_h \in]0, 1[$ such that

$$\left|\left(\varphi^{(\alpha,\beta)}(x + he_j, y) - \varphi^{(\alpha,\beta)}(x, y)\right) - h\varphi^{(\alpha+e_j,\beta)}(x, y)\right| \leq h^2 \left|\varphi^{(\alpha+2e_j,\beta)}(x + t_h he_j, y)\right|.$$

Let $h \in \mathbb{R}$ be such that $|h| \leq 1$, if we set $x' = x + t_h he_j$ and $w = \inf_{x'' \in B(x,1)} w^\lambda(x)$, then we have

$$\begin{aligned}
N^\lambda_\alpha & \left| \frac{\varphi^{(\alpha,\beta)}(x + he_j, y) - \varphi^{(\alpha,\beta)}(x, y)}{|h|} - \varphi^{(\alpha+e_j)}(x, y) \right| v^\lambda(y) \\
& \leq |h| N^\lambda_\alpha |\varphi^{(\alpha+2e_j,\beta)}(x', y)| v^\lambda(y) \\
& \leq \frac{|h|}{M^\lambda_{\alpha+2e_j} w} M^\lambda_{\alpha+2e_j} N^\lambda_\alpha |\varphi^{(\alpha+2e_j,\beta)}(x', y)| w^\lambda(x') v^\lambda(y) \\
& \leq \frac{|h|}{M^\lambda_{\alpha+2e_j} w} \|\varphi\|_{S^{M^\lambda \otimes N^\lambda}_{w^\lambda \otimes v^\lambda}},
\end{aligned}$$

hence $\psi^{(\alpha)}(x)$ exists and satisfies $\psi^{(\alpha)}(x)(y) = \varphi^{(\alpha,0)}(x,y)$ for all $y \in \mathbb{R}^{d_2}$.
Finally, from the relation, true for all $(\alpha,\beta) \in \mathbb{N}^{d_1+d_2}, \lambda > 0, (x,y) \in \mathbb{R}^{d_1+d_2}$,

$$M_\alpha^\lambda \left(N_\beta^\lambda \left(\left((T\varphi)^{(\alpha)}(x) \right)^{(\beta)} (y) v^\lambda(y) \right) w^\lambda(x) \right) = \left(M^\lambda \otimes N^\lambda \right)_{(\alpha,\beta)} \varphi^{(\alpha,\beta)}(x,y) \left(w^\lambda \otimes v^\lambda \right)(x,y),$$

one can easily prove that $T\varphi$ belongs to $S^{\mathfrak{M}}_{\mathscr{W}}(S^{\mathfrak{N}}_{\mathscr{V}})$ and that T is continuous.

The fact that S is well defined and continuous is much less tedious and can be obtained using an identity similar as the one used at the end of the previous case combined with the preceding lemma. □

In order to prove that $S^{\mathfrak{M}}_{\mathscr{W}} \widehat{\otimes} S^{\mathfrak{N}}_{\mathscr{V}} = S^{\mathfrak{M}}_{\mathscr{W}}(S^{\mathfrak{N}}_{\mathscr{V}})$ as locally convex spaces, we will need the following lemma from Komatsu [6].

Lemma 4.4.6. *If E is a nuclear Fréchet space continuously included in $C(\mathbb{R}^d)$ and F a complete locally convex space, then every function $\psi \in C(\mathbb{R}^d, F)$ that satisfies*

$$\langle f', \psi \rangle = f' \circ \psi \text{ belongs to } E \text{ for all } f' \in F'. \tag{3}$$

defines an element on $E \,\varepsilon\, F$, namely, the application $F'_c \to E : f' \mapsto \langle f', \psi \rangle$. Reciprocally, each element $T \in L_\varepsilon(F'_c, G)$ can be expressed as $T(f') = \langle f', \psi \rangle$ for a ψ in $C(\mathbb{R}^d, F)$ satisfying (3)*.*

We can now prove our theorem.

Proposition 4.4.7. *Let $\mathfrak{M}$ be a weight sequence system satisfying* [L] *and* [M.2′] *and let $\mathscr{W}$ be a weight function system satisfying* [wM] *and* [N]*. Then for all complete locally convex space F, we have*

$$S^{\mathfrak{M}}_{\mathscr{W}}(F) \simeq S^{\mathfrak{M}}_{\mathscr{W}}(\mathbb{R}^d) \,\varepsilon\, F \simeq S^{\mathfrak{M}}_{\mathscr{W}}(\mathbb{R}^d) \widehat{\bigotimes} F.$$

Proof. We consider the mapping

$$T : S^{\mathfrak{M}}_{\mathscr{W}}(F) \to S^{\mathfrak{M}}_{\mathscr{W}}(\mathbb{R}^d) \,\varepsilon\, F : \psi \mapsto [f' \mapsto \langle f', \psi \rangle].$$

We have to prove that it is well defined, bijective and that it is an homeomorphism.

Let $\psi \in S^{\mathfrak{M}}_{\mathscr{W}}(F)$, by the previous lemma, to prove that $T\psi \in S^{\mathfrak{M}}_{\mathscr{W}}(\mathbb{R}^d) \,\varepsilon\, F$, we only have to prove that $\langle f', \psi \rangle \in S^{\mathfrak{M}}_{\mathscr{W}}(\mathbb{R}^d)$ for all $f' \in F'$. Let then $f' \in F', \lambda \in \mathbb{N}, x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$, if we suppose that $|f'| \leq q$ for $q \in \operatorname{csn}(F)$ we have

$$M_\alpha^\lambda |\langle f', \psi(x) \rangle^{(\alpha)}| w^\lambda(x) = M_\alpha^\lambda |\langle f', \psi^{(\alpha)}(x) \rangle^{(\alpha)}| w^\lambda(x) \leq q_\lambda(\psi).$$

Using the previous lemma again, to prove the surjectivity of T we must prove that if $\psi \in C^\infty(\mathbb{R}^d, F)$ satisfies $\langle f', \psi \rangle \in S^{\mathfrak{M}}_{\mathscr{W}}(\mathbb{R}^d)$ for all $f' \in F'$ then $\psi \in S^{\mathfrak{M}}_{\mathscr{W}}(F)$. In this condition, the set

$$\{ M_\alpha^\lambda \psi^{(\alpha)}(x) w^\lambda(x) : \alpha \in \mathbb{N}^d, x \in \mathbb{R}^d \}$$

is weakly bounded in F for all $\lambda > 0$. By Mackey's theorem it is bounded in F which proves that $\psi \in S^{\mathfrak{M}}_{\mathscr{W}}(F)$.

Finally, let $\lambda > 0$ and $q \in F$, if we set $p^\lambda = \| \cdot \|_{S^{M^\lambda}_{w^\lambda,\infty}(\mathbb{R}^d)}$ we have

$$\begin{aligned}(p^\lambda \,\varepsilon\, q)(T\psi) &= \sup_{f' \in b_q^\circ} \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} M^\lambda_\alpha |\langle f', \psi^{(\alpha)}(x) \rangle w^\lambda(x) \\ &= \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \sup_{f' \in b_q^\circ} M^\lambda_\alpha |\langle f', \psi^{(\alpha)}(x) \rangle w^\lambda(x) \\ &= \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} M^\lambda_\alpha q(\psi^{(\alpha)}(x)) w^\lambda(x) \\ &= q_\lambda(\psi).\end{aligned}$$

□

We finally obtain this last theorem.

Theorem 4.4.8. *Let $\mathfrak{M}$ be a weight sequence system on $\mathbb{N}^{d_1}$ satisfying* [L] *and* [M.2′] *and let $\mathscr{W}$ be a weight function system on $\mathbb{R}^{d_1}$ satisfying* [wM] *and* [N]. *Let also $\mathfrak{N}$ be an arbitrary weight sequence system on $\mathbb{N}^{d_2}$ and $\mathscr{V}$ an arbitrary weight function system on $\mathbb{R}^{d_2}$. We have*

$$S^{\mathfrak{M} \otimes \mathfrak{N}}_{\mathscr{W} \otimes \mathscr{V}} \simeq S^{\mathfrak{M}}_{\mathscr{W}} \widehat{\otimes} S^{\mathfrak{N}}_{\mathscr{V}}.$$

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