

## **New approach of Thermal Noise Finite Element modelling for the Einstein Telescope cryogenic payload. Centre Spatial de Liège**

**Auteur :** Houlmont, Gauthier

**Promoteur(s) :** Jacques, Lionel; Collette, Christophe

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## **New approach of Thermal Noise Finite Element modelling for the Einstein Telescope cryogenic payload**

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THESIS PRESENTED TO OBTAIN THE DEGREE OF : MASTER OF  
SCIENCE IN AEROSPACE ENGINEERING

**Thesis Supervisor**

Lionel JACQUES

**Thesis Co-Supervisor**

Christophe COLLETTE

**Author**

Gauthier HOULMONT

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# Abstract

In the context of the third-generation gravitational wave detector design, several noises must be understood and minimized. One of them is thermal noise, described theoretically by the Fluctuation and Dissipation Theorem (FDT).

This study aims to mainly describe the impact of introducing a radiative heat exchanger in terms of thermal noise. Although thermal noise has been assessed for classical conductive cooling of the ET-LF payload, this has never been done for radiative cooling. The radiative cooling of the payload requires adding a significant amount of exchange area and this amount of surface might be a source of thermal noise. Moreover, the description of such a noise has been mainly analytical and applied to simple systems.

In order to describe higher order models, this study is based on a Finite Element approach, using the state space matrices rather than using the *Levin's* method [25]. Firstly, a comparison between analytical and FEM predictions is performed to validate the approach. Then, a sensitivity analysis is carried out to assess the influence of the fin properties, from the radiative heat exchanger (RHE), on thermal noise. In the end, studies of the E-TEST and ET models are performed to evaluate the impact of considering an RHE.

Results show that this addition of fins not only modifies the inertia and mass of the system, but also introduces new resonances that come from the plate modes, leading to a higher local thermal noise response. However, no overall increase in thermal noise was observed. The ET-LF model using a radiative cooling strategy shows that it satisfies the ET-LF requirements [16] for Thermal Noise. The study also highlights the importance of the boundary conditions and the assumption made in the literature.

In addition to having an estimate of the thermal noise in complex systems, this approach is easily adaptable to any model. It offers a valuable tool for design optimization, allowing different configurations to be tested, in order to minimize the thermal noise response.

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# Acronyms and Abbreviations

|               |                                                           |              |                                                     |
|---------------|-----------------------------------------------------------|--------------|-----------------------------------------------------|
| <b>AFDT</b>   | Adapted Fluctuation & Dissipation Theorem                 | <b>ASD</b>   | Amplitude Spectral Density                          |
| <b>BC</b>     | Boundary Conditions                                       | <b>CoG</b>   | Centre of Gravity                                   |
| <b>CoM</b>    | Centre of Mass                                            | <b>CP</b>    | Cold Platform                                       |
| <b>CTE</b>    | Coefficient of Thermal Expansion                          | <b>DoF</b>   | Degree of Freedom                                   |
| <b>E-TEST</b> | Einstein Telescope Euregio-Meuse-Rhin Site and Technology | <b>EoM</b>   | Equation of Motion                                  |
| <b>Eq</b>     | Equation                                                  | <b>ET</b>    | Einstein Telescope                                  |
| <b>ET-HF</b>  | Einstein Telescope High Frequency                         | <b>ET-LF</b> | Einstein Telescope Low Frequency                    |
| <b>FE</b>     | Finite Element                                            | <b>FEA</b>   | Finite Element Analysis                             |
| <b>FDT</b>    | Fluctuation & Dissipation Theorem                         | <b>FEM</b>   | Finite Element Model                                |
| <b>GTM</b>    | Global Temperature Matrix                                 | <b>HL</b>    | Heat Link                                           |
| <b>KAGRA</b>  | Kamioka Gravitational Wave Detector                       | <b>LIGO</b>  | Laser Interferometer Gravitational-Wave Observatory |
| <b>MA</b>     | Marionette                                                | <b>MI</b>    | Mirror                                              |
| <b>PF</b>     | Platform                                                  | <b>PSD</b>   | Power Spectral Density                              |
| <b>RBE</b>    | Rigid Body Element                                        | <b>RHE</b>   | Radiative Heat Exchanger                            |
| <b>TE</b>     | Thermoelastic                                             | <b>TN</b>    | Thermal Noise                                       |

# Chapter 1

## Introduction

### 1.1 Historical Context

In 1916, one year after publishing his theory of general relativity, *Albert Einstein* mathematically predicted that accelerated masses, for example, rotating around each other, should produce space-time ripples [10].

The celestial objects that produce the highest-amplitude gravitational waves are black holes (since their densities are incredibly high). Moreover, such waves should be highly produced when two black holes collide, as before this collision, they rotate at high speed around each other. After the merger, the emitted signal rapidly decreases.

These waves are called Gravitational waves and they are not like electromagnetic ones as they propagate at the speed of light creating spacetime distortion and curvature (see Figure 1.1).

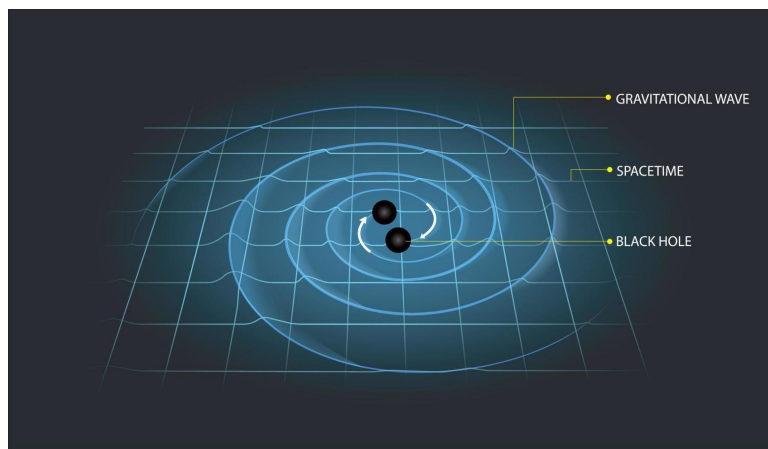


Figure 1.1: Gravitational waves illustration

At the beginning of its theory, *Einstein* himself was not sure of the results.

In 1960, *Joseph Weber* was the first to really try to measure such waves (Figure 1.2). He constructed huge aluminium cylinders that should gently vibrate at the passage of a wave. In 1969, *Weber* announced that he detected a gravitational wave. However, no one has succeeded in reproducing the experiment; this was a false alarm. [39]



Figure 1.2: Picture of *Joseph Weber* and his prototype of gravitational detector

In 1974, *Hulse* and *Taylor* discovered a binary pulsar. However, they observed that the stars were slowly approaching each other. This effect was explained by the fact that stars were losing energy by emitting gravitational waves. This was not a measurement but a proof of the existence of such waves. For this discovery, *Hulse* and *Taylor* won the Nobel prize in 1993. [17]

From 1990 to today, many interferometers have been developed (such as LIGO in the USA and VIRGO in Europe) to detect gravitational waves [2] (Figure 1.3). It was realised that the *Weber* cylinders were not sufficiently sensible to detect the waves.

In 2015, the LIGO instrument was the first to successfully measure a gravitational wave from the collision of two black holes [1]. The signal had a duration of a fraction of a second, but had confirmed one hundred years of theory.



(a) LIGO interferometer



(b) VIRGO interferometer

Figure 1.3: Pictures of actual Gravitational waves detectors

## 1.2 Measurement technique

The actual method used to measure Gravitational waves is based on optical interferometry. More particularly, the *Michelson* interferometer scheme (see Figure 1.4) is used to measure such waves.

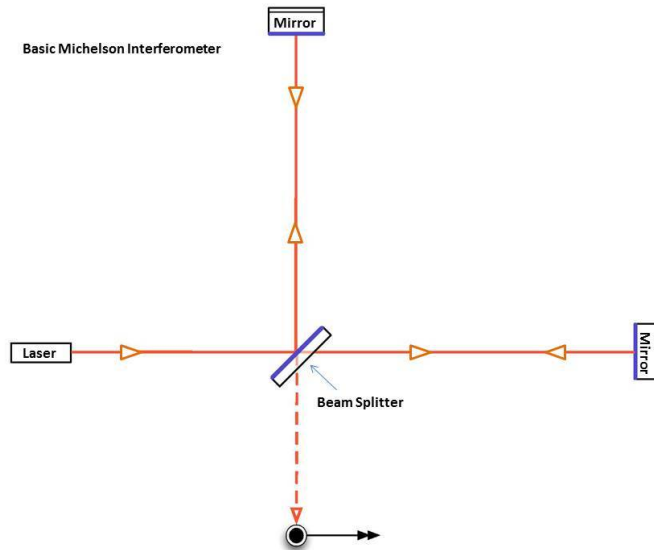


Figure 1.4: *Michelson* interferometer illustration from [26]

The principle is the following :

A light source (laser) passes through a beam splitter, half of the light beam is transmitted, and the other is reflected. The two light beams are then reflected to mirrors and get back to the beam splitter where they will be recombined.

Depending on the phase shift between the two laser beams, the waves can be constructive or destructive. Then the system measures interference fringes that are related to the relative displacement of the two mirrors. If the isolation is high enough, the interferometer might measure small relative displacements corresponding to gravitational waves. To give an order of magnitude, the LIGO interferometer is capable of measuring differential displacements of the order of  $10^{-19} \text{ m}/\sqrt{\text{Hz}}$  in its most sensitive frequency band [6].

### 1.3 Context of study

In the context of isolation, the mirrors used in the interferometer should be highly isolated from all the external perturbation to be sure that they actually measure gravitational waves. Some perturbations can be passively and actively damped, such as the ground motion using seismic isolation techniques, for example.

However, some small perturbations cannot be actively damped (since they are hard to measure). They generally depend on the structure (geometry, material) and on the environment of the system (temperature, pressure).

One of them is the **Thermal Noise** (TN) (1-10 Hz), which is a noise that is explained by the *Brownian motion* of particles at a certain temperature [36]. Any material at a certain temperature that dissipates energy will induce an excitation to the system.

As explained previously, this noise is really hard to measure; then a way to estimate it is to rely on theoretical models. The main one being the Fluctuation and Dissipation Theorem (FDT).

Thermal Noise can be separated into two fields of study :

1. The TN due to the mirror bulk, called : the **substrate thermal noise**.
2. The TN due to the suspensions of the mirror, called : the **suspension thermal noise**.

This study will only focus on the suspension thermal noise as the mirror will be considered as a rigid body.

Eventually, note that there are many other sources of noise that can be enumerated :

- **The Newtonian noise**, which has an important impact between 1 and 10 Hz. This noise arises from the local variation of the gravitational forces that are applied to the mirror. This variation

is due to density changes all around the mirror. It can come from the ground or the atmosphere. An example is the aeration in a room that creates a mass movement, modifying the local density (time-dependent) and therefore the gravitational force. [34]

- **The Quantum noise**, which has an important impact at high frequency ( $\geq 10$  Hz). This noise arises from the quantum nature of light and consists mainly of two contributions. The first one is the **Shot noise**, caused by fluctuations in the arrival time of photons at the photodetector that limits the precision of displacement measurements. The second one is the **Radiation pressure noise**, caused by fluctuations in the number of photons reflecting from the mirror, that produces random forces on the test mass. [34]

The context of this study is also based on the actual cooling strategies of the mirror. In order to minimize the thermal noise, the mirror should be cooled. To do so, two approaches exist :

- A conductive cooling strategy with Heat links (HL).
- A radiative cooling strategy with a Radiative Heat Exchanger (RHE).

The conductive cooling strategy is the most efficient to cool the mirror. However, such a connection might induce external vibration to the isolated mass.

The calculation of the TN for this strategy has already been done in [22]. The results show that such a strategy leads to an important increase in TN. That is why the study [22] proposed an alternative with a conductive cooling strategy using suspensions as thermal links rather than external wires. This strategy will not be discussed in this thesis.

The radiative cooling strategy should also be advantageous from a vibration-isolation perspective. The radiative cooling strategy was developed for the first time in the E-TEST concept.

### 1.3.1 E-TEST

The E-TEST prototype [9] is the first concept to introduce a cryogenic cooling method based on a radiative heat exchanger (see Figure 1.5).

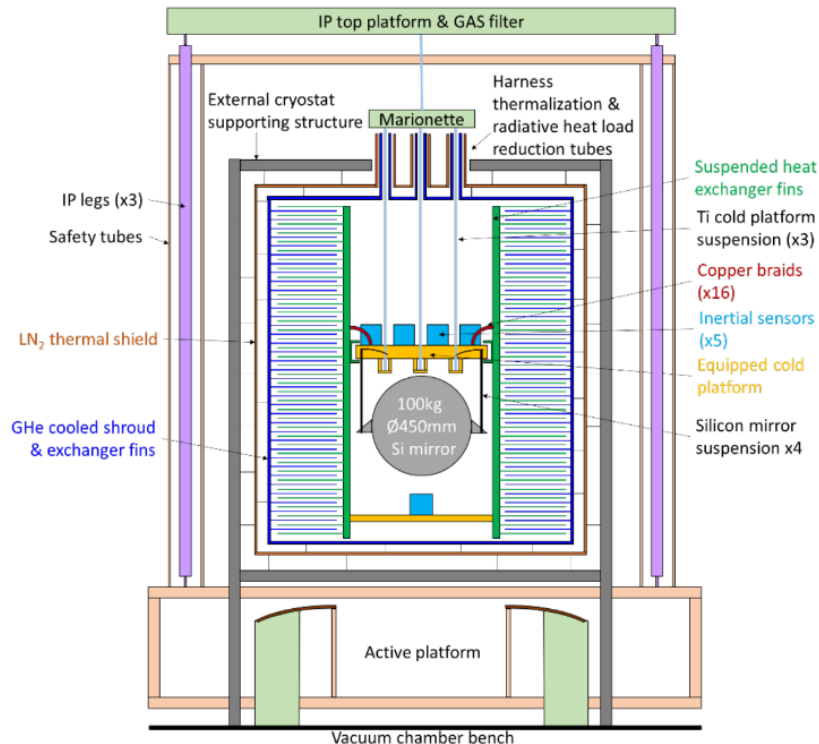


Figure 1.5: E-TEST prototype architecture from [18]

The goal of such a cryogenic cooling design is to avoid vibration transmission from Heat Links models. The RHE developed in the context of the E-TEST project [9], has been able to successfully cool the mirror to 22 K (see Figure 1.6).

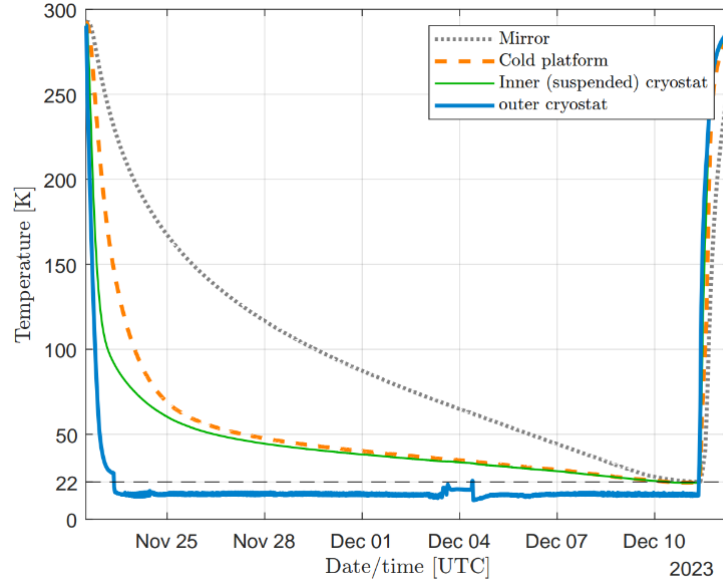


Figure 1.6: Measured temperature on the E-TEST prototype from [19]

However, there is no existing study on the TN impact of such an RHE. Although the radiative cooling strategy should not induce external vibration, nothing proves that such systems do not highly increase the thermal noise.

### 1.3.2 ET-LF

The Einstein Telescope (ET) is a concept of a third-generation gravitational wave detector. This third-generation aims to improve the actual sensitivity of the detectors (LIGO, Virgo, KAGRA). To do so, a triangular configuration (instead of the previous L-shaped configuration), with 10 km long arms, is considered (see Figure 1.7). Moreover, the detector is expected to be located between 200 and 300 meters underground in order to minimize the seismic and Newtonian noise.

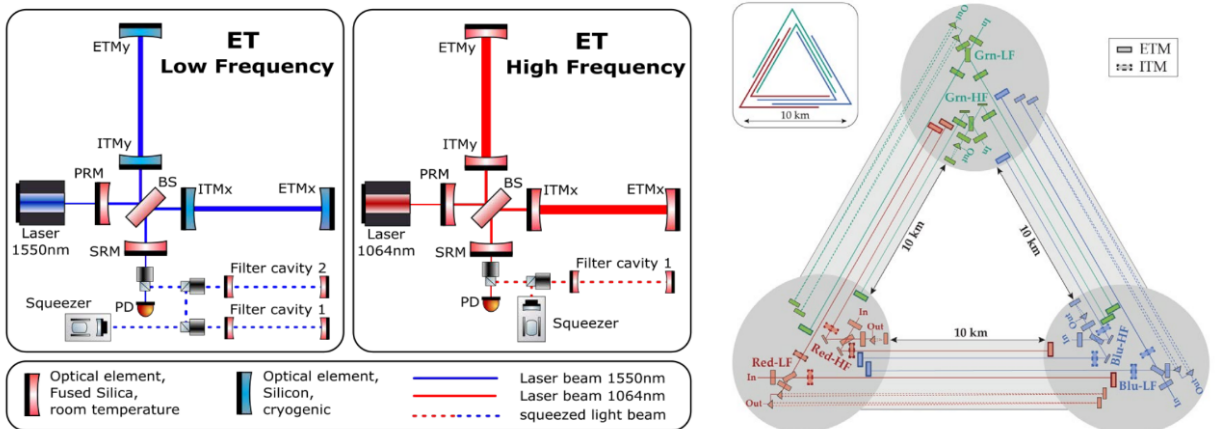


Figure 1.7: ET interferometer scheme (from the ET Design Report Update 2020)

In this triangle, two detectors will be present :

- ET-LF (Low frequency) : this part aims to operate between 2-30 Hz, where thermal noise dominates [11].

- ET-HF (High frequency) : this part aims to operate between  $30\text{-}10^4$  Hz, where quantum noise dominates [11].

The case of ET-LF is the one studied in this manuscript. In this case, the mirror should be cooled to 10-20 K (for ET-HF : 290 K) [18].

The ET-LF model is the ideal context for comparing the two cooling approaches.

Note that for the conductive cooling strategy, the isolation of the external heat link is planned using isolation towers specially designed for the cryogenic cooling system. The concept is illustrated in Figure 1.8.

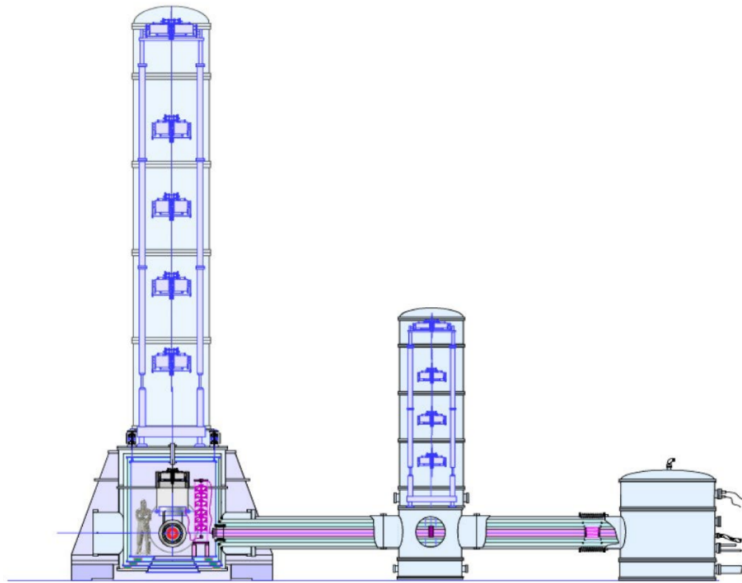


Figure 1.8: Suspensions design for the mirror and the cryogenic system of ET [18].

## 1.4 State of the art

The first approach to theoretically determine the Thermal Noise came from *Herbert B. Callen and Theodore A. Welton* in 1951 [5]. They defined the Fluctuation and Dissipation Theorem<sup>1</sup> which is the basis of all thermal noise studies. Indeed, the paper leads to an expression of the Power Spectral Density (PSD) of the TN excitation and response to such a perturbation.

However, this theorem has some assumptions that limit the study. The most limiting one is to consider that the temperature is constant for the entire system. This means that a system that has parts that are not at the same temperature (the difference is not negligible) cannot be described by this theorem<sup>2</sup>.

Then, in many TN studies, reduced systems, where the temperature is more or less constant for all parts, are studied.

The first one to apply the FDT directly to mechanical systems was *Peter R. Saulson* in his paper: "Thermal noise in mechanical experiments" [36]. *Saulson* was the first to link thermal noise not only to temperature but also to dissipation of mechanical systems. Indeed, before, the TN was just associated with the average Brownian noise following the equipartition theorem, where each degree of freedom (DoF) has an average energy only proportional to  $k_B T$  (in thermal equilibrium).  $k_B = 1.38 \times 10^{-23}$  J/K being the *Boltzmann* constant and  $T$  the temperature in K.

This study was performed in the context of the first interferometer to detect gravitational waves

<sup>1</sup>This Theorem will be derived completely in this manuscript to clearly identify the assumptions made

<sup>2</sup>An approach of a generalisation of the Fluctuation and Dissipation theorem that takes into account different temperature exists [21] and will be studied in this manuscript

(LIGO). *Saulson* described, in his paper, the application of the Fluctuation and Dissipation Theorem to basic spring mass systems.

However, the main difficulty in applying the FDT to real mechanical systems lies in the accurate description of the dissipation mechanisms.

Indeed, viscous dissipation is well known. Nevertheless, in such a study case, internal damping is the main source of dissipation and is not well described theoretically (also discussed in [13]). In many papers, by simplicity, the loss angle (which will be explained later) is considered to be a constant of the frequency. The limitations of such assumptions will be discussed later in this manuscript.

*Saulson* based the description of internal damping on the *Zener's* approach of anelastic solids [32]. He explained the different sources of dissipation and the limitation of their description. However, one of the sources has a frequency description, which is provided by *Zener*, and is the thermoelastic damping. It should be noted that a nonlinear description of this thermoelastic damping was performed in 2002 [4]; a comparison of the two approaches will be made in this thesis.

This study of dissipation is really where the theory is limited and where approximations are preferred. This means that measurements should be made to describe the actual internal dissipation.

Measurements of the internal dissipation of a sample are also really hard, and research is still ongoing to measure it. This is a full topic that is not directly discussed in this study.

Eventually, *Saulson* described, in his paper, the approach to describe the thermal noise of a 1-DOF pendulum system with an approximate solution.

In 1994, he determined (with *Gabriela I. González*) the exact solution of the 1-DoF pendulum, considering a continuous wire and an extended mass (mass + inertia), in his paper "*Brownian motion of a mass suspended by an anelastic wire*" [12]. This solution adds a description of the wire modes, called the *violin modes*. These modes are really important, as they result in high thermal noise excitation and response at high frequency.

In 2000, *Gabriela I. González* developed a more complex description of a continuous pendulum with an extended suspended mass [14]. In her paper, the assumptions made in the past are also discussed. Eventually, a LIGO-like system was described to determine the thermal noise contribution in all six Degree Of Freedom of the mirror.

In 2010, *Paola Puppo* released a study: "A thermal noise model for a branched system of harmonic oscillators" that gives a description of a 2-DOF pendulum (VIRGO-like) based on normal mode representation [33].

In 2023, she also participated in the suspension thermal noise modelling of the Cryogenic Payload for the ET-LF Project [22], where a more complex thermal noise study was performed in addition to sensitivity analyses. This study was the first to apply an adapted version of the FDT that takes into account temperature differences in the system. Indeed, they based their work on the study : "Direct approach for the fluctuation-dissipation theorem under nonequilibrium steady-state conditions" made by *Komori et al.* in 2018 [21]. This adaptation is really important as it allowed to improve the complexity of the studied systems.

At this step, all of these studies are purely analytical. Despite their good precision, they describe only basic systems. At some point, to describe the thermal noise of a more complex system, a Finite Element Method (FEM) model is needed.

In 2009, such a description was performed by *A. Cumming et al* [7]. In this study, only the description of the loss angle (in flexion) of the suspension fibres (silica) is provided. Finite Element Analysis (FEA) was used to model the distribution of elastic energy along the fibres for different neck shapes and cross sections.

Such an FEA is based on the *Levin's* method [25] which allows to compute thermal noise in a test mass. The method calculates the noise by considering a hypothetical oscillatory force (with a spatial profile matching the laser beam) applied to the mirror surface.

However, the method is not adapted for temperature distribution, but is still a first approach to describe thermal noise using FEM. This method will not be discussed in this manuscript.

## 1.5 Motivation

Using FEM might be really necessary for the design of the next generation of gravitational wave detectors. Indeed, examining the E-TEST prototype [9] (Figure 1.9), the radiative cryogenic system will introduce thermal noise and is difficult to describe analytically. Moreover, analytical methods are not capable of describing systems as detailed as FEM models. In terms of geometry, but also in terms of temperature distribution (this will be discussed later).

It can be concluded that, at this stage of development, introducing thermal noise into FEM models is essential to design the next generation of gravitational wave detectors such as ET (for Thermal Noise at least).



Figure 1.9: Picture of the E-TEST prototype

## 1.6 Objectives

### 1.6.1 Theoretical review

The first objective of this manuscript is to start from the bottom of knowledge. To do so, a complete review of the Fluctuation and Dissipation Theorem will be performed to clearly identify the assumptions. This theorem is really the basis of Thermal Noise theory, but was written in 1951. Furthermore, a fairly complete description of the phenomena of internal dissipation will be performed to clearly understand, not only the origin but also the impacting parameters that drive the dissipation.

### 1.6.2 Analytical study of basic pendulum system

Next, an analytical study of basic systems will be performed to clearly understand the behaviour of such a noise. The goal will be to identify the main influencing parameters and to understand why they have such an impact on the Thermal Noise.

### 1.6.3 New Finite Element Method approach

After having set the theoretical background, the goal will be to adapt what already exists to another technique. Indeed, it will be seen that the most common description of the TN is based on analytical

models. Thus, the purpose will be to adapt the TN theory to Finite Element (FE) computation. Even if a FEA method already exists (*Levin's* method), this one is different and will allow to compute more detailed systems. Indeed, the method, that will be described later, is based on the rough computation of the state space matrices rather than strain energies computation.

The first objective will be to build the method and adapt it to higher DoF models.

The second objective will be to prove the exactness of this method by making comparisons with basic analytical systems.

The third objective will be to use this method in a more complex model to identify the impact of using radiative cryogenic cooling systems, as there is no existing study on it.

The fourth objective will be to use this method on actual complex models to get an accurate description of their thermal noise estimation.

Last but not least, an important purpose is to identify all the advantages of using this method compared to what already exists, but also to talk about the actual limitations.

#### 1.6.4 E-TEST

Since there is no existing study on the TN impact of a radiative cooling strategy, the purpose will be, with the developed method, to estimate the TN response of the mirror for the E-TEST concept.

It should be noted that the goal is mainly to see the impact of adding such panels. The E-TEST prototype was not designed to minimize this noise and does not have to be compared with the ET requirements [16]. The main objective is also to show that the developed method can be used for such a complex system.

#### 1.6.5 ET-LF

The last objective of this manuscript is to describe the TN response of the mirror from the ET-LF concept.

The purpose will be to compare the impact of both cooling strategies on the mirror TN.

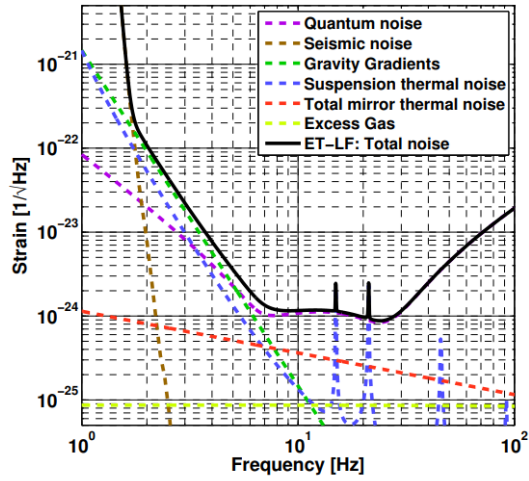
Moreover, some additional study will be performed such as :

- The effect of the temperature of the top wire.
- The increase in the top wire diameter, since introducing an RHE means an important addition of mass. Then the initial top wire might not have been design to support this amount of weight.
- Reinforcements in the RHE that increase the stiffness of the fins.
- The connections between the RHE and the Platform (PF).

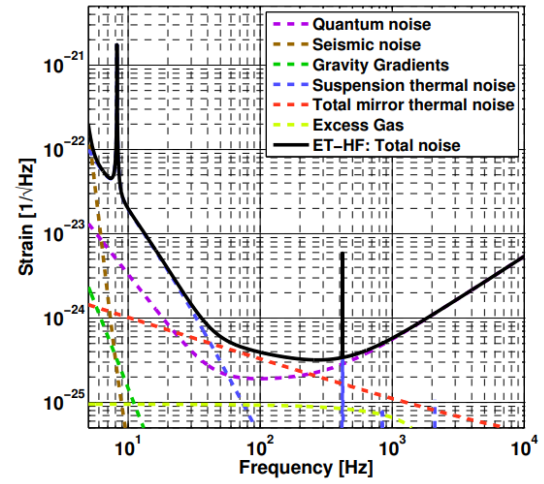
Eventually, a comparison with the model of *Koroveshi et al.* [22] will be made.

All results will be compared with the ET-LF sensitivity requirements [16].

The sensitivity curves are represented in Figure 1.10.



(a) Low Frequency (LF) interferometer



(b) High Frequency (HF) interferometer

Figure 1.10: ET-D sensitivity curve with the different noise contribution (Noise budget) [16]

The "D" refers to a sensitivity based on the inclusion of many noise sources for ET.

## Chapter 2

# Thermal Noise theory : Fluctuation and Dissipation Theorem & Origin of the dissipation

The goal of this section is to follow the development made by *Herbert B. Callen and Theodore A. Welton* [5] for the fluctuation and dissipation theory but with more details (thanks to [35] & [23]) to clearly identify the assumptions made in this method. No new information will be derived than the one in the references. Both dissipation and fluctuation will be considered and then merge into the *Nyquist* relation, which is the expression to reach. Then, the goal will be to have an expression of the power spectral density of the thermal noise excitation and the response of the system (which is not directly expressed in [5]).

### 2.1 Dissipation

A system is considered to be dissipative if it absorbs energy when subjected to a perturbation. In this case, the goal is to determine the response of a quantum system when subjected to a force. The force will be represented, for the system, as a perturbation. Then, the **time-dependent perturbation theory** from quantum mechanics will be used.

In this representation, the system is described by  $\hat{H}(t)$ , where  $\hat{H}$  is the hamiltonian of the system :  $\hat{H} = \hat{H}_0 + \hat{H}_p(t)$ .

- $\hat{H}_0$  is the hamiltonian of the unperturbed system.  $\hat{H}_0$  is a function of the coordinates  $q_1 \cdots q_k$  and momenta  $p_1 \cdots p_k$  of the system (considered to be of order k).
- $\hat{H}_p(t) = V(t) \hat{Q}$  is the perturbation composed of
  - $V(t)$  is a function of time that measures the instantaneous magnitude of the perturbation.
  - $Q$  is a function of the coordinates and momenta. In quantum terms, this is an operator that is an observable of the system on which the perturbation is applied. In other words,  $Q$  describes how the system reacts to the perturbation.

The definitions being made, the system can be solved using the Equation of Motion (EoM), which is in this case the *Schrödinger* equation (Definition 2.1.1).

#### Definition 2.1.1 - Schrödinger Equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle \quad (2.1)$$

where  $\hbar = 1.05 \times 10^{-34}$  J.s is the reduced *Planck* constant.  
Applying the perturbation to the system :

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}_0 |\psi(t)\rangle + V(t) \hat{Q} |\psi(t)\rangle \quad (2.2)$$

It is known that for a conservative system, when the hamiltonian is independent of time, the eigenvalues  $E_n$  and eigenvectors  $|\phi_n\rangle$  of the hamiltonian :  $\hat{H} |\phi_n\rangle = E_n |\phi_n\rangle$  are also independent of time.  $|\phi_n\rangle$  are the eigenstates of the hamiltonian, also called the stationary states which forms an ortho-normed base which allows to express (considering  $t_0 = 0$  s) :

$$|\psi(t)\rangle = \sum_n c_n(t) |\phi_n\rangle = \sum_n c_n(t) e^{-iE_n t/\hbar} |\phi_n\rangle \quad (2.3)$$

where  $c_n(t)$  are the coefficient that should be found.

A parallel can be made with the theory of vibrations, when considering a linear system, the solution to any perturbation can be found using a combination of the eigenstates.

Returning to our case, the hamiltonian of the unperturbed system being independent of time, it leads to :  $\hat{H}_0 |\phi_n\rangle = E_n |\phi_n\rangle$ .

If the perturbation varies sinusoidally with time, such as  $V(t) = V_0 \sin(\omega t)$ , equation (2.2) becomes :

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}_0 |\psi(t)\rangle + V_0 \sin(\omega t) \hat{Q} |\psi(t)\rangle \quad (2.4)$$

Taking into account Equation (2.3), Equation (2.4) can be expressed as a set of first-order equations, which can be integrated to determine the coefficients  $c_n$ .

If the energy levels of the system are **densely distributed**, the total induced transition rate (variation over time of the probability to have a transition) of a system initially in the state  $|\phi_n\rangle$  can be determined using the *Fermi's Golden Rule*.

Injecting Equation (2.3) and  $\hat{H}_0 |\phi_n\rangle = E_n |\phi_n\rangle$  into the *Schrödinger* Equation (2.1.1) it gives :

$$\underbrace{i\hbar \frac{\partial}{\partial t} \left( \sum_n c_n(t) e^{-iE_n t/\hbar} |\phi_n\rangle \right)}_{RT} = \sum_n c_n(t) e^{-iE_n t/\hbar} E_n |\phi_n\rangle + \sum_n c_n(t) e^{-iE_n t/\hbar} \hat{H}_p |\phi_n\rangle \quad (2.5)$$

where the right term ( $RT$ ) can be derived :

$$RT = i\hbar \sum_n \frac{\partial c_n(t)}{\partial t} e^{-iE_n t/\hbar} |\phi_n\rangle + i\hbar \sum_n c_n(t) \left( \frac{-iE_n}{\hbar} \right) e^{-iE_n t/\hbar} |\phi_n\rangle$$

After simplifications and projection to the state  $\langle \phi_k |$  (State reached with the transition due to the perturbation), Equation (2.5) becomes (thanks to orthogonality relationships) :

$$\begin{aligned} i\hbar \frac{\partial c_k(t)}{\partial t} e^{-iE_k t/\hbar} &= \sum_n c_n(t) \langle \phi_k | \hat{H}_p |\phi_n\rangle e^{-iE_n t/\hbar} \\ \iff \boxed{\frac{\partial c_k(t)}{\partial t} &= \frac{1}{i\hbar} \sum_n c_n(t) \langle \phi_k | \hat{H}_p |\phi_n\rangle e^{-i(E_n - E_k)t/\hbar}} \end{aligned} \quad (2.6)$$

Equation (2.6) is the one to solve to find the solution of the system. At this step, no assumptions are made. However, as the expression is quite hard to solve, two **assumptions** will be made :

1. **The system is initially in an eigenstate**  $|\phi_m\rangle$  of  $\hat{H}_0$ , which means that at  $t_0$  :  $c_j(t = t_0) = \delta_{j,m}$
2. **The perturbation is considered weak** and **applied for a short period of time**, such that all coefficients remain nearly unchanged  $\rightarrow c_j(t) \simeq c_j(t_0)$ .

Taking into account these assumptions, Equation (2.6) can be expressed as :

$$\begin{aligned}
\frac{\partial c_k(t)}{\partial t} &= \frac{1}{i\hbar} \overbrace{c_m(t)}^{\simeq 1} \langle \phi_k | \hat{H}_p | \phi_m \rangle e^{-i(E_m - E_k)t/\hbar} \\
\iff c_k(t) &= \frac{1}{i\hbar} \int_0^t \langle \phi_k | \hat{H}_p | \phi_m \rangle e^{-i(E_m - E_k)\tau/\hbar} d\tau \\
\iff c_k(t) &= \frac{1}{i\hbar} \int_0^t \langle \phi_k | \hat{H}_p | \phi_m \rangle e^{i\omega_{m,k}\tau} d\tau
\end{aligned} \tag{2.7}$$

where the energy of transition was replaced by :  $E_k - E_m = \omega_{m,k} \hbar$ .

Equation (2.7) gives an expression of the coefficient. Then the probability of going from a state to the state  $k$  is given by :

$$\begin{aligned}
P_{m \rightarrow k} &= P(E_k) = |\langle \psi_k | \psi \rangle|^2 = |c_k(t)|^2 \\
&= \frac{1}{\hbar^2} \left| \int_0^t \langle \phi_k | \hat{H}_p | \phi_m \rangle e^{i\omega_{m,k}\tau} d\tau \right|^2
\end{aligned} \tag{2.8}$$

Returning to the case considered, the perturbation is considered harmonic (considering a sinusoidal perturbation) and is expressed as :  $\hat{H}_p = V(t)\hat{Q} = V_0 \sin(\omega t)\hat{Q} = \frac{V_0}{2i} (e^{i\omega t} - e^{-i\omega t}) \hat{Q}$ . Then, the coefficient  $c_k$  can be expressed :

$$\begin{aligned}
c_k(t) &= \frac{-V_0}{2\hbar} \langle \phi_k | \hat{Q} | \phi_m \rangle \int_0^t \left( e^{i(\omega_{m,k} + \omega)\tau} - e^{i(\omega_{m,k} - \omega)\tau} \right) d\tau \\
&= \frac{-V_0}{2\hbar} \langle \phi_k | \hat{Q} | \phi_m \rangle \left[ \frac{e^{i(\omega_{m,k} + \omega)t} - 1}{i(\omega_{m,k} + \omega)} - \frac{e^{i(\omega_{m,k} - \omega)t} - 1}{i(\omega_{m,k} - \omega)} \right]
\end{aligned} \tag{2.9}$$

Then Equation (2.9) can be injected into Equation (2.8) which gives :

$$P(E_k) = \frac{V_0^2}{4\hbar^2} \left| \langle \phi_k | \hat{Q} | \phi_m \rangle \right|^2 \left| \frac{e^{i(\omega_{m,k} + \omega)t} - 1}{i(\omega_{m,k} + \omega)} - \frac{e^{i(\omega_{m,k} - \omega)t} - 1}{i(\omega_{m,k} - \omega)} \right|^2 \tag{2.10}$$

At this step, there are two cases to consider when the time  $t \rightarrow \infty$  :

1.  $E_k \simeq E_m - \hbar\omega \Rightarrow \omega \simeq -\omega_{m,k} \equiv \text{Emission}$
2.  $E_k \simeq E_m + \hbar\omega \Rightarrow \omega \simeq \omega_{m,k} \equiv \text{Absorption}$

Considering the emission case (the absorption case is similar and leads to the same results). Then Equation (2.10) becomes (using the relationship developed in Appendix 8.2; Relation 1.) :

$$\begin{aligned}
P(E_k) &= \frac{V_0^2}{4\hbar^2} \left| \langle \phi_k | \hat{Q} | \phi_m \rangle \right|^2 \left| \frac{e^{i(\omega_{m,k} - \omega)t} - 1}{i(\omega_{m,k} - \omega)} \right|^2 \\
&= \frac{V_0^2}{4\hbar^2} \left| \langle \phi_k | \hat{Q} | \phi_m \rangle \right|^2 \left| \frac{2 \sin((\omega_{m,k} - \omega)t/2)}{(\omega_{m,k} - \omega)} \right|^2
\end{aligned} \tag{2.11}$$

To consider a continuous transition between the state  $m$  and  $k$ , all states to which the system can transit should be considered. This is done by integrating the probability multiplied by the *density of states*  $\rho(E)$  over all the accessible states ( $E_{as}$ ). This leads to :

$$\begin{aligned}
P(t) &= \int_{E_{as}} P(E_k) \rho(E) dE \\
&= \frac{V_0^2}{4\hbar^2} \int_{E_{as}} \left| \langle \phi_k | \hat{Q} | \phi_m \rangle \right|^2 \underbrace{\left| \frac{2 \sin((\omega_{m,k} - \omega)t/2)}{(\omega_{m,k} - \omega)} \right|^2}_{F(\omega, t)} \rho(E) dE
\end{aligned}$$

From this expression, the function  $F(\omega, t)$  can be identified. Figure 2.1 shows its evolution for both parameters. For higher duration time, the function is only important in a short frequency range near the resonance (The frequency needed to transit from state  $m$  to  $k$ ).

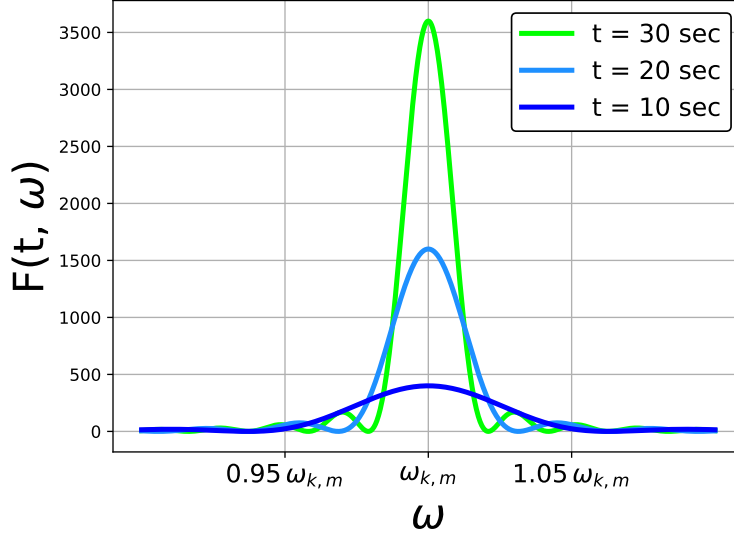


Figure 2.1:  $F(\omega, t)$  Function evolution with frequency of the perturbation  $\omega$  and the time  $t$

This means that  $\left| \langle \phi_k | \hat{Q} | \phi_m \rangle \right|^2$  and  $\rho(E)$  can be considered to be nearly constant (Energy levels are considered to be densely distributed). Then, replacing the expression of the energy as  $E = \hbar\omega$  the expression becomes :

$$\begin{aligned}
 P(t) &\simeq \frac{V_0^2}{4\hbar^2} \left| \langle \phi_k | \hat{Q} | \phi_m \rangle \right|^2 \rho(E_k) \hbar \int_{E_{as}} \left| \frac{2 \sin((\omega_{m,k} - \omega)t/2)}{(\omega_{m,k} - \omega)} \right|^2 d\omega \\
 &\simeq \frac{V_0^2}{4\hbar} \left| \langle \phi_k | \hat{Q} | \phi_m \rangle \right|^2 \rho(E_k) \left( \frac{2}{t} \right) t^2 \underbrace{\int_{-\infty}^{+\infty} \frac{\sin^2(x)}{x^2} dx}_{\pi} \\
 &\simeq \frac{\pi V_0^2}{2\hbar} \left| \langle \phi_k | \hat{Q} | \phi_m \rangle \right|^2 \rho(E_k) t
 \end{aligned} \tag{2.12}$$

Then from this expression, the *Fermi's Golden Rule* can be found, since the transition rate is given by the derivative over time of the probability such as expressed by Definition 2.1.2.

#### Definition 2.1.2 - Fermi's Golden Rule - Transition Rate

The transition rate between the state  $m$  and the state  $k$  due to a harmonic perturbation is given by

$$\Omega = \frac{\partial P(t)}{\partial t} \simeq \frac{\pi V_0^2}{2\hbar} \left| \langle \phi_k | \hat{Q} | \phi_m \rangle \right|^2 \rho(E_k)$$

This expression is not a general one, but an adaptation to the case considered.

In the general case of dissipation, both absorption and emission might occur. Indeed, each transition from the initial state to the higher (resp. lower) energy state is accompanied by the absorption (resp. emission) of energy ( $\hbar\omega$ ). Considering the initial state to be written  $|E_n\rangle$ , the higher and lower energy states will be written respectively  $|E_n + \hbar\omega\rangle$  and  $|E_n - \hbar\omega\rangle$ . Then the total transition rate is given by :

$$\Omega_{Tot} = \frac{\pi V_0^2}{2\hbar} \left[ \left| \langle E_n + \hbar\omega | \hat{Q} | E_n \rangle \right|^2 \rho(E_n + \hbar\omega) + \left| \langle E_n - \hbar\omega | \hat{Q} | E_n \rangle \right|^2 \rho(E_n - \hbar\omega) \right] \tag{2.13}$$

In this case, the rate of **energy absorption** is given by :

$$\frac{\pi V_0^2}{2} \omega \left[ \left| \langle E_n + \hbar\omega | \hat{Q} | E_n \rangle \right|^2 \rho(E_n + \hbar\omega) - \left| \langle E_n - \hbar\omega | \hat{Q} | E_n \rangle \right|^2 \rho(E_n - \hbar\omega) \right]$$

To predict the behaviour of a real thermodynamic system, it must be averaged over-all initial states, weighting each according to the *Boltzmann* factor :  $f(E_n) = e^{-E_n/k_B T}$ . This then leads to the definition of the **Dissipation Power** :

$$\mathcal{P}(\omega) = \frac{1}{2} \pi V_0^2 \omega \sum_n \left[ \left| \langle E_n + \hbar\omega | \hat{Q} | E_n \rangle \right|^2 \rho(E_n + \hbar\omega) - \left| \langle E_n - \hbar\omega | \hat{Q} | E_n \rangle \right|^2 \rho(E_n - \hbar\omega) \right] f(E_n) \quad (2.14)$$

In this expression, the sum over the  $n$  initial states can be replaced by an integration over all the energy (considering the density of states  $\rho(E)$ ):

$$\mathcal{P}(\omega) = \frac{1}{2} \pi V_0^2 \omega \int_0^\infty \left[ \left| \langle E + \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E + \hbar\omega) - \left| \langle E - \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E - \hbar\omega) \right] \rho(E) f(E) dE \quad (2.15)$$

This expression means that the small and periodic perturbation applied to a system in which the eigenstates are densely distributed in energy leads to a power dissipation that is quadratic with the perturbation.

In linear systems, the impedance  $Z(\omega)$  is defined as :

$$V = Z(\omega) \dot{Q} \quad (2.16)$$

where  $V$  is the excitation force and  $\dot{Q}$  the velocity response of the system. The average Power is also defined as :

$$\mathcal{P}(\omega) = \frac{1}{2} \frac{\text{Re}\{Z(\omega)\}}{|Z(\omega)|^2} V_0^2 \quad (2.17)$$

Then injecting Equation (2.15) into Equation (2.17) leads to the following expression :

$$\frac{\text{Re}\{Z(\omega)\}}{|Z(\omega)|^2} = \pi \omega \int_0^\infty \left[ \left| \langle E + \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E + \hbar\omega) - \left| \langle E - \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E - \hbar\omega) \right] \rho(E) f(E) dE \quad (2.18)$$

## 2.2 Fluctuation

The system is now considered to be in **thermal equilibrium**, with no applied force. The system is expected to exhibit a spontaneous fluctuating  $\dot{Q}$ , which might be associated with a spontaneously fluctuating force.

Consider  $\langle V^2 \rangle$  as the mean square value of the spontaneously fluctuating force and then  $\langle \dot{Q}^2 \rangle$  as the mean square value of the fluctuating velocity response  $\dot{Q}$ .

In a same way, considering that the system is in the eigenstate  $|E_n\rangle$  such that :

$$\begin{aligned} \langle \dot{Q}^2 \rangle_n &= \langle E_n | \dot{Q}^2 | E_n \rangle = \sum_k \langle E_n | \dot{Q} | E_k \rangle \langle E_k | \dot{Q} | E_n \rangle \\ &= \sum_k \langle E_n | \frac{i}{\hbar} [\hat{H}_0, \hat{Q}] | E_k \rangle \langle E_k | \frac{i}{\hbar} [\hat{H}_0, \hat{Q}] | E_n \rangle \end{aligned} \quad (2.19)$$

$$\begin{aligned} &= -\frac{1}{\hbar^2} \sum_k \langle E_n | \hat{H}_0 \hat{Q} - \hat{Q} \hat{H}_0 | E_k \rangle \langle E_k | \hat{H}_0 \hat{Q} - \hat{Q} \hat{H}_0 | E_n \rangle \\ &= -\frac{1}{\hbar^2} \sum_k \left[ (E_n - E_k) \langle E_n | \hat{Q} | E_k \rangle (E_k - E_n) \langle E_k | \hat{Q} | E_n \rangle \right] \\ &= \frac{1}{\hbar^2} \sum_k (E_n - E_k)^2 \left[ \langle E_k | \hat{Q}^\dagger | E_n \rangle^* \langle E_k | \hat{Q} | E_n \rangle \right] \\ &= \frac{1}{\hbar^2} \sum_k (E_n - E_k)^2 \left| \langle E_k | \hat{Q} | E_n \rangle \right|^2 \end{aligned} \quad (2.20)$$

Equation (2.19) was obtained using the *Heisenberg* representation of the *Schrödinger* Equation, where the observable varies over time. Using the formula 2. in Appendix 8.2, it can be obtained that :  $\dot{Q} = \frac{i}{\hbar} [\hat{H}_0, \hat{Q}]$  (Since now the equilibrium system is considered :  $\hat{H} = \hat{H}_0$ ).

Equation (2.20) is finally obtained using the fact that  $\hat{Q}$  is an *Observable* and then an Hermitian operator :  $\hat{Q}^\dagger = \hat{Q}$ .

The frequency  $\omega$  which corresponds to :  $\hbar\omega = |E_n - E_k|$ , can be introduced. Furthermore, the sum can be replaced by two integrals over  $\omega$ , one when  $E_n < E_k$  and the other when  $E_n > E_k$ . Using integrals means that the density of states  $\rho(E)$  should be introduced :

$$\begin{aligned} \langle \dot{Q}^2 \rangle_n &= \frac{1}{\hbar^2} \int_0^\infty (\hbar\omega)^2 \left| \langle E_n + \hbar\omega | \hat{Q} | E_n \rangle \right|^2 \rho(E_n + \hbar\omega) \hbar d\omega \\ &\quad + \frac{1}{\hbar^2} \int_0^\infty (\hbar\omega)^2 \left| \langle E_n - \hbar\omega | \hat{Q} | E_n \rangle \right|^2 \rho(E_n - \hbar\omega) \hbar d\omega \\ &= \int_0^\infty \hbar\omega^2 \left[ \left| \langle E_n + \hbar\omega | \hat{Q} | E_n \rangle \right|^2 \rho(E_n + \hbar\omega) + \left| \langle E_n - \hbar\omega | \hat{Q} | E_n \rangle \right|^2 \rho(E_n - \hbar\omega) \right] d\omega \end{aligned} \quad (2.21)$$

In a same way as in the previous Section, to describe the fluctuation of a real system, Equation (2.21) should be multiplied by the weighting factor  $f(E_n)$  and a sum should be done over all the states :

$$\begin{aligned} \langle \dot{Q}^2 \rangle &= \sum_n \langle \dot{Q}^2 \rangle_n f(E_n) \\ &= \sum_n f(E_n) \int_0^\infty \hbar\omega^2 \left[ \left| \langle E_n + \hbar\omega | \hat{Q} | E_n \rangle \right|^2 \rho(E_n + \hbar\omega) + \left| \langle E_n - \hbar\omega | \hat{Q} | E_n \rangle \right|^2 \rho(E_n - \hbar\omega) \right] d\omega \end{aligned}$$

Again, in a same way as for the *dissipation* the sum might be replaced by an integral over the energy, taking into account the density of states  $\rho(E)$  :

$$\langle \dot{Q}^2 \rangle = \int_0^\infty \hbar\omega^2 \left\{ \int_0^\infty \rho(E) f(E) \left[ \left| \langle E + \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E + \hbar\omega) + \left| \langle E - \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E - \hbar\omega) \right] dE \right\} d\omega \quad (2.22)$$

Using the impedance relation (2.16); it leads to :

$$\langle V^2 \rangle = \int_0^\infty |Z(\omega)|^2 \hbar\omega^2 \left\{ \int_0^\infty \rho(E) f(E) \left[ \left| \langle E + \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E + \hbar\omega) + \left| \langle E - \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E - \hbar\omega) \right] dE \right\} d\omega \quad (2.23)$$

## 2.3 Nyquist Relation

Now that the expressions of  $\langle V^2 \rangle$  and  $\text{Re}\{Z(\omega)\}/|Z(\omega)|^2$  are known, the goal is to relate them. The integral over the energy is nearly similar and differ by one sign. For  $\langle V^2 \rangle$  there is a + and for  $\text{Re}\{Z(\omega)\}/|Z(\omega)|^2$  a - sign. The goal of this section is to link the two expressions in order to find the *Nyquist* Relation.

Calling the "minus" expression  $C_-$  and the "plus" expression  $C_+$  :

$$\begin{aligned} C_- &= \int_0^\infty \left[ \left| \langle E + \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E + \hbar\omega) - \left| \langle E - \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E - \hbar\omega) \right] \rho(E) f(E) dE \\ &= \int_0^\infty \left| \langle E + \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E + \hbar\omega) f(E) \rho(E) dE \\ &\quad - \underbrace{\int_0^\infty \left| \langle E - \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E - \hbar\omega) \rho(E) f(E) dE}_{SI} \end{aligned}$$

In the second integral (SI) the transformation  $E \rightarrow E + \hbar\omega$  in the integration variable can be made, in addition to using the hermitian property of  $\hat{Q}$  (since it is an observable). The second integral (SI) can then be written as :

$$\begin{aligned} SI &= \int_0^\infty \left| \langle E | \hat{Q} | E + \hbar\omega \rangle \right|^2 \rho(E) \rho(E + \hbar\omega) f(E + \hbar\omega) d(E + \hbar\omega) \\ &= \int_0^\infty \left| \langle E + \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E) \rho(E + \hbar\omega) f(E + \hbar\omega) dE \end{aligned}$$

Then injecting this expression in  $C_-$  leads to :

$$\begin{aligned} C_- &= \int_0^\infty \left| \langle E + \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E + \hbar\omega) f(E) \rho(E) \left( 1 - \frac{f(E + \hbar\omega)}{f(E)} \right) dE \\ &= \left( 1 - e^{-\hbar\omega/k_B T} \right) \int_0^\infty \left| \langle E + \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E + \hbar\omega) f(E) \rho(E) dE \end{aligned} \quad (2.24)$$

$C_+$  can be obtained in a similar way :

$$C_+ = \left( 1 + e^{-\hbar\omega/k_B T} \right) \int_0^\infty \left| \langle E + \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E + \hbar\omega) f(E) \rho(E) dE \quad (2.25)$$

From this the following relations can be obtained :

$$\begin{cases} \frac{\text{Re}\{Z(\omega)\}}{|Z(\omega)|^2} = \pi\omega \left( 1 - e^{-\hbar\omega/k_B T} \right) \int_0^\infty \left| \langle E + \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E + \hbar\omega) f(E) \rho(E) dE \\ \langle V^2 \rangle = \int_0^\infty |Z(\omega)|^2 \hbar\omega^2 \left( 1 + e^{-\hbar\omega/k_B T} \right) \left\{ \int_0^\infty \left| \langle E + \hbar\omega | \hat{Q} | E \rangle \right|^2 \rho(E + \hbar\omega) f(E) \rho(E) dE \right\} d\omega \end{cases}$$

Then by comparing these expressions :

$$\begin{aligned} \langle V^2 \rangle &= \int_0^\infty |Z(\omega)|^2 \hbar\omega^2 \left( \frac{1 + e^{-\hbar\omega/k_B T}}{1 - e^{-\hbar\omega/k_B T}} \right) \frac{\text{Re}\{Z(\omega)\}}{|Z(\omega)|^2} \frac{1}{\pi\omega} d\omega \\ &= \frac{1}{\pi} \int_0^\infty \text{Re}\{Z(\omega)\} \hbar\omega \left( \frac{1 + e^{-\hbar\omega/k_B T}}{1 - e^{-\hbar\omega/k_B T}} \right) d\omega \\ &= \frac{2}{\pi} \int_0^\infty \text{Re}\{Z(\omega)\} \underbrace{\hbar\omega \left( \frac{1}{2} + \frac{1}{e^{\hbar\omega/k_B T} - 1} \right)}_{E(\omega, T)} d\omega \end{aligned} \quad (2.26)$$

where  $E(\omega, T)$  is the mean energy of an oscillator of a natural frequency  $\omega$  which is a the temperature  $T$ .

Equation (2.26) can be obtained using the relation 3. in Appendix 8.2. The *Bose-Einstein Distribution* can be seen, which is directly linked to the number of phonons.

Then the *Nyquist Relation* is obtained.

### Definition 2.3.3 - Nyquist Relation

$$\langle V^2 \rangle = \frac{2}{\pi} \int_0^\infty \text{Re}\{Z(\omega)\} E(\omega, T) d\omega \quad (2.27)$$

It can be noted that when  $k_B T \gg \hbar\omega \rightarrow e^{\hbar\omega/k_B T} = 1 + \frac{\hbar\omega}{k_B T} + \underbrace{\dots}_{\simeq 0}$ . Then the mean energy can be approximated by :

$$E(\omega, T) \simeq \frac{1}{2} \hbar\omega + k_B T \simeq k_B T$$

which leads to the expression :

$$\langle V^2 \rangle = \frac{2}{\pi} k_B T \int_0^\infty \text{Re}\{Z(\omega)\} d\omega \quad (2.28)$$

The **Power Spectral Density** being defined by :

**Definition 2.3.4 - Power Spectral Density (PSD)**

The PSD  $S_x(\omega)$  is defined as :

$$\langle x^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_x(\omega) d\omega$$

where  $\langle x^2 \rangle$  is the mean square value.

The PSD of the excitation force in this case can be expressed by :

$$S_f(\omega, T) = 4k_B T \text{Re}\{Z(\omega)\} \quad (2.29)$$

From this expression, the PSD of the response can be found using the impedance relation (2.16) and the fact that  $\dot{Q}(\omega) = i\omega Q(\omega)$  :

$$\begin{aligned} S_x(\omega, T) &= \frac{S_f(\omega)}{\omega^2 |Z(\omega)|^2} \\ &= \frac{4k_B T}{\omega^2} \frac{\text{Re}\{Z(\omega)\}}{|Z(\omega)|^2} \\ &= \frac{4k_B T}{\omega^2} \text{Re}\left\{\frac{1}{Z(\omega)}\right\} \end{aligned}$$

The inverse of the impedance is called the *admittance*  $Y(\omega) = Z^{-1}(\omega) = \dot{Q}/V$ . Then the final expression for the PSD response is given by :

$$S_x(\omega, T) = \frac{4k_B T}{\omega^2} \text{Re}\{Y(\omega)\} \quad (2.30)$$

The real part of the impedance and the admittance are called, respectively, the *mechanical resistance*  $R(\omega)$  and the *mechanical conductance*  $\sigma(\omega)$ .

In a more general case (when  $k_B T$  is not much higher than  $\hbar\omega$ ), the general expression for the PSD response is given by :

$$S_x(\omega, T) = \frac{4\hbar}{\omega} \left( \frac{1}{2} + \frac{1}{e^{\hbar\omega/k_B T} - 1} \right) \text{Re}\{Y(\omega)\} \quad (2.31)$$

The PSD of the response and the excitation are directly related to the temperature  $T$  and increase with higher temperature.

Eventually, both the thermal noise excitation and response depend on the imaginary part of the transfer function of the system. In other words, here, the link with the dissipation is straightforward, since without any dissipation the excitation and then the response tend to zero. However, it will be seen in the next sections that the dependence is not that easy for the response of a system to the thermal noise.

## 2.4 Assumption Validation

In this Section, the assumption of :  $k_B T \gg \hbar\omega$  will be verified.

For a given system, the following results can be obtained :

| Temperature    | 1e-2 K     | 1e-9 K   | 1e-10 K |
|----------------|------------|----------|---------|
| Max Rel. Diff. | 1.6e-7 [%] | 1.87 [%] | 59 [%]  |

Table 2.1: Maximum Relative difference between the simplified ( $k_B T \gg \hbar \omega$ ) and the full PSD expression for different temperatures in the frequency range :  $[0, 10]$  Hz

At  $T = 1\text{e-}2\text{K}$  the maximum relative difference increase of 1 order of magnitude (so  $1\text{e-}6\%$ ) at a frequency of  $1\text{e}5$  Hz.

Whatever the system, the same order of magnitude will be observed. Indeed, to validate the assumption the ratio  $T/\omega$  should be much higher than  $k_B/\hbar \approx 7.6\text{e} - 12$  which is only false in extreme cases (really high frequency), this will never be observed in such a study.

## 2.5 Origin of the thermal noise : Energy Dissipation

As previously seen, thermal noise only occurs when the system dissipate energy. Indeed, without dissipation, the real part of the admittance (the imaginary part of the transfer function) is equal to zero.

There are two main kinds of dissipation : the viscous and the internal dissipations.

### 2.5.1 Viscous Dissipation

Viscous dissipation is the common source of dissipation in systems. This dissipation is directly related to the velocity of the DoF. Consider a basic system with a dashpot to represent the dissipation (Figure 2.2).

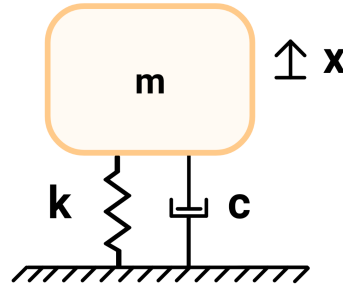


Figure 2.2: 1-degree-of-freedom system : Viscous damping

The Equation of Motion (EoM) of such a system is given by :

$$m \ddot{x}(t) + c \dot{x}(t) + k x(t) = f(t)$$

This leads to the following transfer function :

$$H(\omega) = \frac{1}{k - \omega^2 m + i\omega c}$$

From this expression the real part of the impedance ( $Z(\omega) = (i\omega H(\omega))^{-1}$ ) and admittance ( $Y(\omega) = Z(\omega)^{-1}$ ) can be found :

$$\text{Re}\{Z(\omega)\} = c \quad \& \quad \text{Re}\{Y(\omega)\} = \frac{\omega^2 c}{(k - \omega^2 m)^2 + \omega^2 c^2}$$

The PSD of the thermal noise excitation is then given by :

$$S_f(\omega, T) = 4k_B T c$$

As expected, the excitation has a linear dependence on  $c$  and then increases with dissipation. The thermal noise response of the system to this dissipation is given by :

$$S_x(\omega, T) = \frac{4k_B T c}{(k - \omega^2 m)^2 + \omega^2 c^2}$$

In this case, the response does not necessarily increase with dissipation. Indeed, when the dissipation ( $c$ ) tends to infinite, the response tends to zero. In contrast, when the dissipation tends to zero, the response also tends to zero.

These results can be understood as following :

- When there is no dissipation, there is no force, so no response can occur.
- When the dissipation is infinite, the force increases, but the system fully dissipates this excitation, and no response result.

Then it means that, for each system, a maximum can be found for the response at a given frequency. The dissipation ( $c$ ) that leads to a maximum response of the system to thermal noise excitation is given by :

$$c_{max} = \frac{|k - \omega^2 m|}{\omega}$$

However, such dissipations should not be dominant in the case of interest. Indeed, since the mirror is in vacuum, no viscous damping should result from air. However, a study of the damping from the residual gas might be interesting to perform.

In this manuscript, no viscous damping will be considered.

### 2.5.2 Internal Dissipation

Another source of dissipation is the internal dissipation of the structures. For a one-degree-of-freedom system without viscous damping, the Equation of Motion is expressed as :

$$(k_{eff} - \omega^2 m) X = F \quad (2.32)$$

where  $k_{eff} = k(1 + i\phi(\omega))$ ,  $\phi(\omega)$  being the loss angle function. This function describes the internal dissipation which depends on the frequency. This function is well described for fluids since it is the viscous damping ( $\phi(\omega) = c\omega/k$ ). Nevertheless, for solids, the loss angle can have a completely different expression. The determination of such an expression is still difficult to determine [36] [13]. Some contributions to this dissipation have a description, while others are determined approximatively by experimentation [13].

From the EoM, the *admittance* can be found :

$$Y(\omega) = \frac{i\omega X(\omega)}{F(\omega)} = \frac{i\omega(k - \omega^2 m - i\phi(\omega)k)}{(k - \omega^2 m)^2 + \phi(\omega)^2 k^2} \quad (2.33)$$

This then allows to find the PSD excitation and response to the thermal noise excitation such as :

$$\begin{aligned} S_x(\omega, T) &= \frac{4k_B T}{\omega^2} \text{Re} \{Y(\omega)\} \\ &= \frac{4k_B T}{\omega^2} \left( \frac{\omega \phi(\omega) k}{(k - \omega^2 m)^2 + \phi(\omega)^2 k^2} \right) \\ &= \frac{4k_B T \phi(\omega) k}{\omega [(k - \omega^2 m)^2 + \phi(\omega)^2 k^2]} \end{aligned} \quad (2.34)$$

$$(2.35)$$

$$\begin{aligned} S_f(\omega, T) &= 4k_B T \text{Re} \{Y^{-1}(\omega)\} \\ &= 4k_B T \text{Re} \left\{ -i \left( \frac{k - \omega^2 m + i\phi(\omega)k}{\omega} \right) \right\} \\ &= 4k_B T \frac{k\phi(\omega)}{\omega} \end{aligned} \quad (2.36)$$

As mentioned previously, one of the most complicated tasks to perform when studying thermal noise is to have an expression for the loss angle for solids. In many studies, this function is considered constant over frequency.

Although this approach is the only one to determine the dissipation, it should be noted that there is a limitation. Indeed, at low frequency, the PSDs tend to infinite when considering a constant loss angle (see Appendix 8.3). This shows an important limitation, since the thermal noise is always overestimated at low frequency (especially for high  $\phi$  values). In most cases, this overestimation is not that important, but can start being significant for study of TN at really low frequency.

A way to determine the loss angle function is to rely on the *anelasticity* phenomenon described by *Zener* [32]. The internal dissipation measured by *Wert* and *Zener* (see example in Figure 2.3) exhibited a maximum value at a certain frequency, which is called a *Debye's peak*.

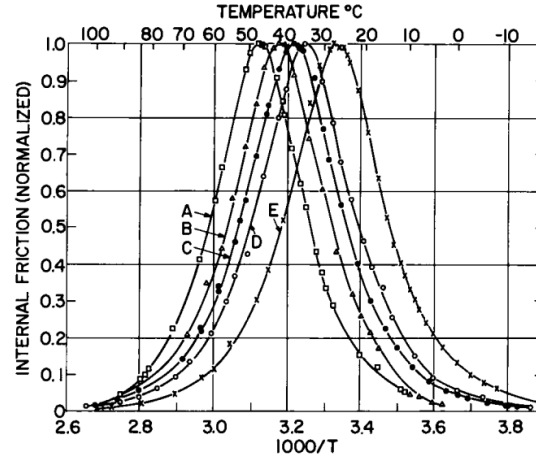


Figure 2.3: Example of the experimental measurements from *Wert* and *Zener*, 1949 [32]

*Zener* interpreted this phenomenon as resulting from the coupling between applied stress and internal defect-related degrees of freedom [32].

The *Zener* modelled the internal dissipation following the representation in Figure 2.4.

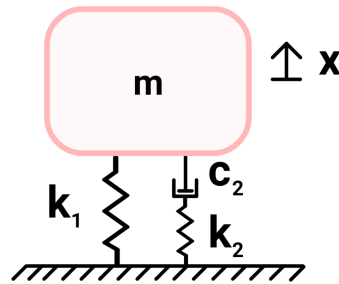


Figure 2.4: 1-degree-of-freedom system : Internal damping

where the loss angle function can be expressed as :

$$\phi(\omega) = \Delta \frac{\omega\tau}{1 + \omega^2\tau^2} \quad (2.37)$$

where  $\Delta = k_2/k \equiv$  Relaxation strength and  $\tau = c_2/k_2 \equiv$  Relaxation Time.

### 2.5.2.1 Thermoelastic Damping

For many dissipation origins, the loss angle is considered to be a constant of the frequency. However, one which is not is the thermoelastic dissipation. This dissipation is due to the fact that any material

will dissipate energy due to strain. Indeed, when considering a beam in flexion, as the Coefficient of Thermal Expansion (CTE) is non zero, a small temperature gradient will be created as one part of the beam will be in compression and the other in tension. Then a heat flux will result to restore the equilibrium. This leads to a decrease in the restoring force and then to a smaller energy equilibrium. Even if the CTE is equal to zero, a dissipation will still result since the Young Modulus varies with the temperature (this is the nonlinear thermoelastic damping contribution).

Following *Zener's* anelastic model [32], the loss angle due to the thermoelastic damping is given by :

$$\phi_{TE}(\omega, T) = \Delta(T) \frac{\tau\omega}{1 + (\tau\omega)^2} \quad (2.38)$$

where in the case of a circular cross section wire in flexion [32] :

- $\Delta(T) = \frac{ET}{c} \left( \alpha - \frac{P}{E^2} \frac{\partial E}{\partial T} \right)^2$  is the **relaxation strength**<sup>1</sup>.
- $\tau = \frac{d^2}{4.32\pi D}$  is the **relaxation time**.

$\alpha$  being the Coefficient of Thermal Expansion ,  $P$  the stress in the wires in [Pa] (tension  $\sigma$  in [N] per unit of cross-sectional area),  $T$  the temperature,  $c = c_p \rho$  the specific heat per unit of volume ( $c_p \equiv$  specific heat,  $\rho \equiv$  density) of the wire material and  $D = \lambda/c$  the thermal diffusion coefficient ( $\lambda \equiv$  thermal conductivity). The thermoelastic coefficient can be defined as follows :  $\beta(T) = \frac{1}{E} \frac{\partial E}{\partial T}$ , which comes from the nonlinear contribution of the Young Modulus temperature dependence.

In the case of a rectangular cross section, a ribbon that has a thickness<sup>2</sup> $t$ , the relaxation time is expressed as follows [32] :

$$\tau = \frac{t^2}{\pi^2 D}$$

It can be noted that, in both cases, the dissipation :

- increases with the Young modulus.
- decreases with the density and specific heat.
- increases with the CTE (origin of the thermal gradient).

In the same way as in Section 2.5.1, the loss angle has a maximum value depending on the relaxation time (at a given frequency).

Indeed :

- if linear TE damping is considered<sup>3</sup>: When the diameter (or thickness) tends to zero, the loss angle tends to zero. When there is no distance, there is no thermal gradient, and there is no thermal heat flux. In contrast, when the diameter (or thickness) tends to infinite, the distance being so high that the equilibrium cannot be restored, leading to zero dissipation.
- when the thermal conductivity tends to zero, no heat flux can be generated. When it tends to infinite, no thermal gradient can be generated. Both situations lead to zero dissipation.

Then the value that generates the maximum dissipation can be computed :

$$\tau_{max} = \frac{1}{\omega}$$

The frequency distribution of the TE dissipation can be plotted. The function is represented in Figure 2.5.

<sup>1</sup>This expression is not the one of *Zener* but comes from a more complex analysis of nonlinear thermoelastic damping [4]

<sup>2</sup>Note that neither the length nor the width impact the TE dissipation.

<sup>3</sup>Indeed, when considering the nonlinear contribution, if  $d \rightarrow 0$  mm makes  $P$  tends to infinite and then the dissipation tends to infinite. The limit here is structural, it is impossible to have a high tension with a low diameter without braking.

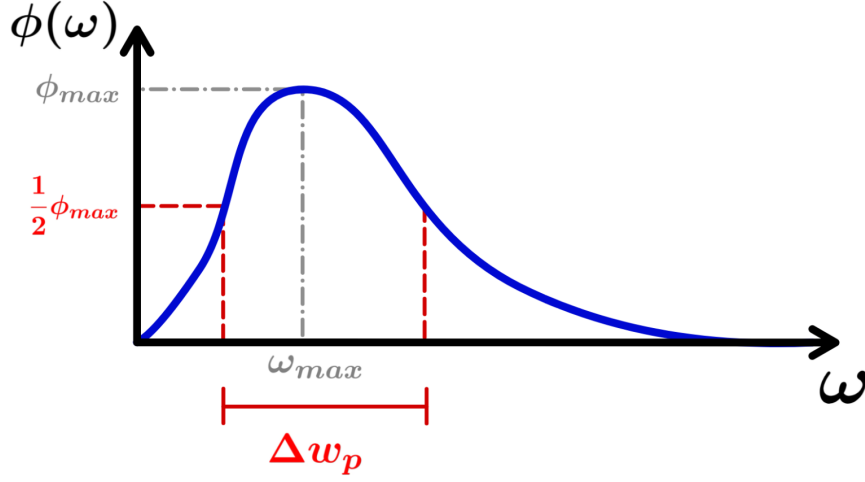


Figure 2.5: Thermoelastic (TE) damping function over the frequency

$\omega_{max}$ ,  $\phi_{max}$  and  $\Delta\omega_p$  can be computed :

$$\begin{aligned}\frac{\partial\phi(\omega)}{\partial\omega} &= \frac{\partial}{\partial\omega} \left( \Delta \frac{\tau\omega}{1 + \tau^2\omega^2} \right) \\ &= \frac{\Delta\tau (1 - \tau^2\omega^2)}{(1 + \tau^2\omega^2)^2}\end{aligned}$$

From this expression, considering  $\omega \geq 0$  rad/s, the following parameter can be obtained :

$$\omega_{max} = \frac{1}{\tau} \quad \& \quad \phi_{max} = \frac{\Delta}{2} \quad (2.39)$$

This result is quite interesting, as it shows that the maximum amplitude of the TE dissipation is fully independent of the relaxation time and then of the geometry (diameter or thickness).

The width of the peak  $\Delta\omega_p$  that is considered to be at  $\phi(\omega) = \phi_{max}/2$ , can be computed :

$$\begin{aligned}\frac{\phi_{max}}{2} = \frac{\Delta}{4} &= \Delta \frac{\tau\omega_{1,2}}{1 + \tau^2\omega_{1,2}^2} \\ \iff \omega_{1,2} &= \frac{2 \pm \sqrt{3}}{\tau}\end{aligned}$$

This leads to :

$$\Delta\omega_p = \frac{2\sqrt{3}}{\tau} \quad (2.40)$$

These results clearly identify the impact of both relaxation strength and time :

- **Relaxation strength** ( $\Delta$ ) linearly influences the TE dissipation. It plays a role on the global amplitude (for every frequency).
- **Relaxation time** ( $\tau$ ) does not modify the maximum amplitude but defines the location of the maximum in frequency and its width. It plays a role in the frequency distribution.

The impact of  $\tau$  can be seen in Figure 2.6.

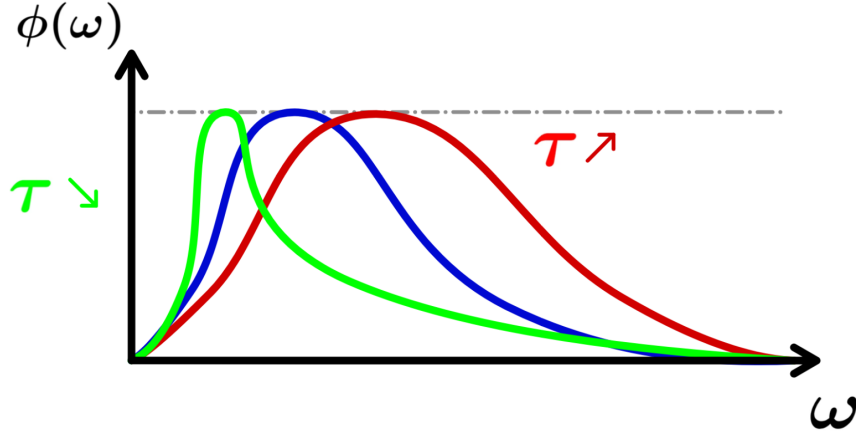


Figure 2.6: Thermoelastic (TE) damping function over the frequency - Relaxation time  $\tau$  impact

The relaxation time being linked to the geometry (diameter or thickness), it can be concluded that the geometry does not affect the maximum amplitude of the TE dissipation but its distribution over the frequency.

The maximum amplitude of the TE dissipation is fully driven by the material properties. The relaxation strength also depends on the tension  $\sigma$ . However, it will be seen later that this contribution in most cases is negligible.

### 2.5.2.2 Bulk dissipation

The bulk dissipation comes from the intrinsic losses in the bulk material. The origin of such a dissipation can be explained by the structural composition of the material and especially by all the defects it can have. These losses depend mainly on the temperature and frequency. However, in most cases, the frequency dependence is neglected, and the loss angle is considered constant. The bulk losses are mainly determined experimentally for different temperatures (for example, in [3] and [20]).

### 2.5.2.3 Surface dissipation

Surface losses differ from bulk losses because they occur in a thin surface layer.

This dissipation depends mainly on the quality of the surface and on its treatment. The loss angle is experimentally determined with the surface loss parameter  $\alpha_{surf}$  and can be expressed as [22] :

$$\phi_{surf} = \alpha_{surf} \frac{\mu A_{surf}}{V} \quad (2.41)$$

Where  $\mu$  is the geometry factor (equal to 2 for thin circular cross section wires [15]),  $A_{surf}$  the surface area of the wire ( $\pi d L$  for a circular cross section wire of diameter  $d$  and length  $L$ ) and  $V$  the volume of the wire ( $\pi d^2 L/4$  for a circular cross section wire of diameter  $d$  and length  $L$ ).

In this expression, the loss angle is considered to be a constant of the frequency.

### 2.5.2.4 Other dissipation sources

Many other sources of dissipation exist as :

- The joining losses that might occur due to bolt connection, weld connection, etc. Although these losses are not characterized theoretically or experimentally, they might be an important source of internal dissipation.
- The friction losses between parts.
- Viscous damping due to the residual gas.

These losses are not covered and taken into account in this study as their descriptions are not yet well defined.

## Chapter 3

# Study of basic pendulum systems

Now that the theoretical background has been defined, the computation of the PSD of the Thermal Noise can be done, in a same way as in [36].

The developments will be performed for the PSD, but the Amplitude Spectral Density (ASD) will be plotted in all the manuscript (ASD = square root of the PSD).

Note that all systems in this manuscript are considered linear, as the displacements of the masses are small.

### 3.1 One Degree Of Freedom Pendulum

#### 3.1.1 Analytical derivation

Consider the pendulum system in Figure 3.1. Where a mass  $m$  is suspended by a number  $n$  of wires of length  $L$  (connected to the CoM of  $m$ ) and with an effective stiffness  $k_{eff}$ . The suspended mass is considered to be a concentrated mass with no inertia.

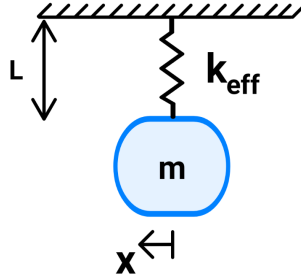


Figure 3.1: 1-degree-of-freedom Pendulum system

For pendulum systems, a natural stiffness is provided by gravity. The other is just the stiffness in flexion of the wire under tension. In case of high tension in the wire, the stiffness can be approximated by :  $k_{f,wire} = \frac{n\sqrt{EI}\sigma}{2L^2}$ . Then the effective stiffness can be expressed as follows, considering internal damping :

$$\begin{aligned} k_{eff} &= k_g + k_{f,wire}(1 + i\phi_{sus}(\omega)) \\ &= \frac{mg}{L} + \frac{n\sqrt{EI}\sigma}{2L^2}(1 + i\phi_{sus}(\omega)) \end{aligned} \quad (3.1)$$

where  $E$  is the young modulus,  $g$  the gravity constant, The second moment of area  $I = \pi d^4/64$  (for a circular cross section wire),  $d$  the diameter of one wire,  $\sigma = mg/n$  the tension in each wire in [N] and  $\phi_{sus}(\omega)$  the total loss angle of one wire.

The loss angle is the main concern when thermal noise is studied. Indeed, the expression is hard to determine for solids because there are many sources of dissipation. For many dissipation phenomena, the loss angle is considered to be independent of the frequency (constant value). However, one which

is not is the thermoelastic (TE) dissipation. Then in this basic study, only this TE damping will be considered. Following the nonlinear thermoelastic damping model of [4], the loss angle due to thermoelastic damping is given by :

$$\phi_{TE}(\omega, T) = \Delta(T) \frac{\tau\omega}{1 + (\tau\omega)^2} \quad (3.2)$$

where in the case of a circular cross section wire (diameter  $d$ ) in flexion :

- $\Delta(T) = \frac{ET}{c} \left( \alpha - \frac{P}{E}\beta \right)^2$  is the relaxation strength
- $\tau = \frac{d^2}{4.32\pi D}$  is the relaxation time

The PSD of the thermal noise excitation and the response of the mass can be known. Using the EoM of the system, the real part of the impedance can be found :

$$\begin{aligned} (k_{eff} - m\omega^2)X &= F \\ \iff Z(\omega) &= \frac{F}{i\omega X} = -\frac{i}{\omega}(k_g + k_{f,wire} + i\phi_{sus}(\omega, T)k_{f,wire} - m\omega^2) \\ \iff \text{Re}\{Z(\omega)\} &= \frac{\phi_{sus}(\omega, T)k_{f,wire}}{\omega} \end{aligned} \quad (3.3)$$

This leads to the PSD of the thermal noise excitation :

$$S_f(\omega, T) = 4k_B T \frac{\phi_{sus}(\omega, T)k_{f,wire}}{\omega} \quad (3.4)$$

In a same way, from the EoM, the real part of the admittance can be found :

$$\begin{aligned} Y(\omega) &= \frac{i\omega X}{F} = i\omega \left( \frac{1}{k_g + k_{f,wire} + i\phi_{sus}(\omega, T)k_{f,wire} - m\omega^2} \right) \\ \iff \text{Re}\{Y(\omega)\} &= \frac{\omega\phi_{sus}(\omega, T)k_{f,wire}}{(k_g + k_{f,wire} - m\omega^2)^2 + \phi_{sus}^2(\omega, T)k_{f,wire}^2} \end{aligned} \quad (3.5)$$

This leads to the PSD of the thermal noise response of the mass :

$$S_x(\omega, T) = 4k_B T \frac{\phi_{sus}(\omega, T)k_{f,wire}}{\omega \left[ (k_g + k_{f,wire} - m\omega^2)^2 + \phi_{sus}^2(\omega, T)k_{f,wire}^2 \right]} \quad (3.6)$$

Equations (3.4) and (3.6) are quite interesting. Indeed, it can be directly seen that when the dissipation tends to 0 for a given frequency, the excitation tends to zero, and no response results. However, when the dissipation tends to infinite at a given frequency, the excitation tends also to infinite but the response tends to 0. This result is also quite logical, since when the dissipation is infinite, no response can result from any excitation.

This means that between low and high dissipation, there is a maximum for the response (at a given frequency). This observation means that, at some point, increasing the dissipation will be preferable than decreasing it. Moreover, this conclusion can also be drawn for the elastic stiffness of the wire. This will be discussed in the sensitivity analysis.

Eventually, it is important to recall that these expressions of the PSDs are only for the flexion of the wire (1 DoF considered).

### 3.1.2 Sensitivity Analysis

The influence of the different parameters of the system can be studied. For this study, n wires will be considered at 20K and to be made of Ti6Al4V (properties assumed in Table 3.1)

| $\rho$ [kg/m <sup>3</sup> ] | E [GPa] | $c_p$ [J/kgK] | $\lambda$ [W/mK] | $\alpha$ [1/K] | $\beta$ [1/K] |
|-----------------------------|---------|---------------|------------------|----------------|---------------|
| 4430                        | 125     | 30            | 2                | 1e-6           | -4.6e-6       |

Table 3.1: Ti6Al4V assumed properties at 20 K

These properties are totally assumed (in the order of magnitude of Ti6Al4V) and do not represent reality.

At low temperature, the parameter  $\beta$  is generally small enough to have a negligible impact on dissipation. To verify it, a quick computation of order of magnitude can be made.

Recall that the Relaxation strength is given by :  $\frac{ET}{c} (\alpha - \frac{P}{E}\beta)^2$ , where  $\frac{P}{E}\beta$  is the nonlinear contribution. This contribution should be of the order of magnitude of the CTE ( $\alpha$ ) to provide a significant impact. Taking into account the Ti6Al4V, the orders of magnitude are :

- $\alpha \sim 10^{-6}$  [1/K]
- $\beta \sim -10^{-6}$  [1/K]
- $E \sim 10^{11}$  [Pa]

Considering a diameter which is in mm, it can be obtained that :

$$\begin{aligned}
 P &\sim \frac{mg}{d^2} \sim \frac{\alpha E}{\beta} \\
 \Leftrightarrow m &\sim \frac{10^{-6} \cdot 10^{11} \cdot (10^{-3})^2}{10 \cdot 10^{-6}} \\
 m &\sim 10^4 \text{ [kg]}
 \end{aligned}$$

Having a mass of 10 tons is not expected in the context of this manuscript.

This result depends a lot on the parameters used. Generally, the nonlinear contribution of TE damping can become important for low diameter, high mass, low Young Modulus, and high absolute  $\beta$ . For the next discussions in the sensitivity analysis, the impact of the nonlinear TE will be considered negligible, such as the Relaxation strength is only :  $\Delta(T) = \frac{ET}{c}\alpha^2$

It should be noted that for a positive value of  $\beta$ , the TE dissipation can be equal to zero.

### 3.1.2.1 Mass

First, a study of the effect of the suspended mass can be done. All other parameters are considered constant and can be found in Table 3.2.

| T [K] | n [-] | d [m] | L [m] |
|-------|-------|-------|-------|
| 20    | 1     | 4e-3  | 1.2   |

Table 3.2: Constant parameters for the mass variation for the 1 Degree Of Freedom pendulum system

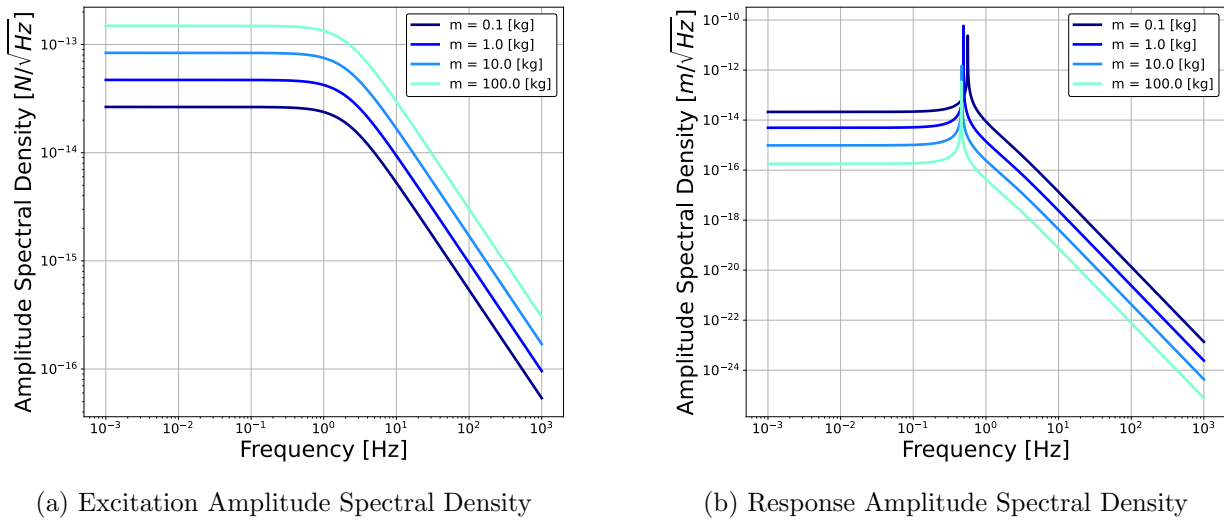


Figure 3.2: Thermal noise for mass variation comparison for the 1-degree-of-freedom pendulum system

Figure 3.2 depicts that the thermal noise excitation increases with the mass, while it is the opposite for the response. Looking at Equation (3.4), it can be seen that the dependence of the force on the mass is due to the tension in the wire  $\sigma = mg/n$ . Indeed, having a higher mass leads to a higher tension in the wire and then a higher stiffness in flexion for the wire. All of this leads to a higher thermal noise excitation.

However, the thermal noise response of the system decreases with increasing mass. This result was expected, but at some point, if the mass decreases, the tension in the wire decreases, then the thermal noise excitation will also decrease, and the response is expected to decrease. This is not the case. Indeed, when looking at Equation (3.6) the entire denominator is linked to the suspended mass, then the maximum predicted in the beginning of the section will not appear when the mass is changed. This is the only particular case of this sensitivity analysis in which no maximum TN response can be found for a given mass.

Nevertheless, this result is quite strange and seems to be non physical. Indeed, how a system which is not excited can have a response that tends to infinite ?

The problem comes from the fact that the expression of the stiffness of the wires in flexion is an approximation solution for high tension in the wire. Indeed, the expression is such that there is no stiffness of the wire when there is no suspended mass.

This is a limitation of the approximation; this is why later in this manuscript the full solution of the stiffness will be computed. The exact stiffness being non zero when there is no tension.

Eventually, the resonance that can be observed at 0.4-0.5 Hz, is the pendulum mode (see Figure 8.1). Increasing the mass makes it appear at lower frequency but not linearly like  $\omega_0^2 \sim k/m$ . Indeed, in this pendulum case, the stiffness also increases with the mass but with  $k \sim \sqrt{m}$ . Later, with the exact value of the stiffness, it will be seen that increasing the mass can increase the frequency of a pendulum mode (for a higher number of DoF).

### 3.1.2.2 Wires dimensions

#### 3.1.2.2.a) Number of wires

The effect of the number of wires, linked to the suspended mass, on the system behaviour, can be analysed. All other parameters are considered constant and can be found in Table 3.3.

| T [K] | m [kg] | d [m] | L [m] |
|-------|--------|-------|-------|
| 20    | 1      | 4e-3  | 1.2   |

Table 3.3: Constant parameters for the number of wires variation for the 1 Degree Of Freedom pendulum system

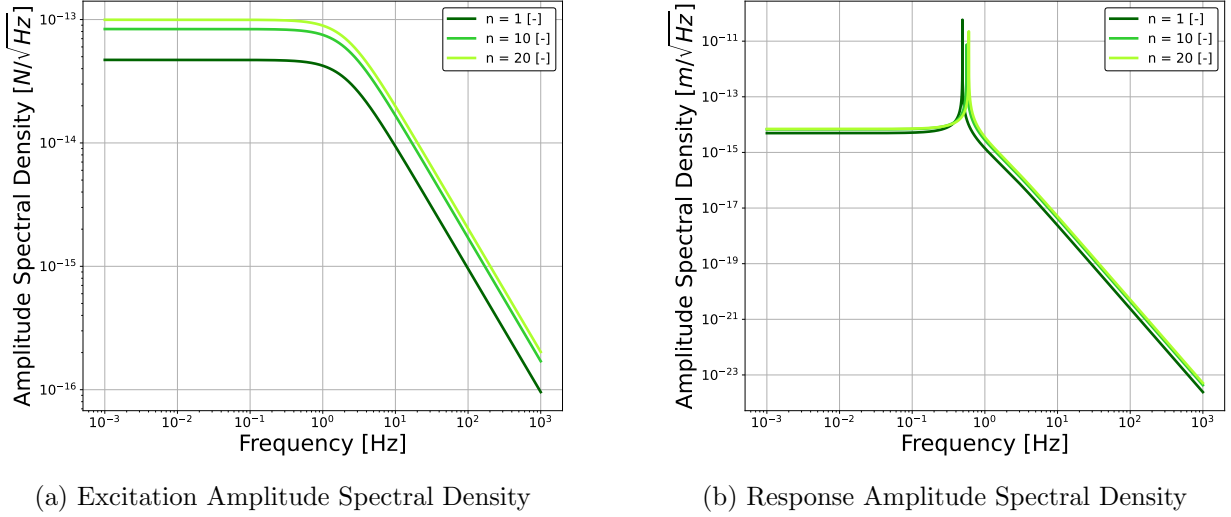


Figure 3.3: Thermal noise for Number of wires variation comparison for the 1-degree-of-freedom pendulum system

As apparent in Figure 3.3, increasing the number of cables increases the thermal noise excitation. Indeed, more wires leads to more sources of noise by logic. In equation (3.4) this is due to the total stiffness in flexion that increases with the number of wires (in parallel). Indeed, even if the tension in the wire is reduced, the stiffness depends on the square root of the number of wires.

Looking at the response, it can be seen that increasing the number of wires globally increases the TN response. However, at a certain point, for a given frequency, increasing the number of wires increases the stiffness and leads to a decrease in the TN response. This means that a maximum can be found for a given frequency :

Consider that the frequency is fixed and that the loss angle does not depend on the tension in the wires ( $-\frac{P}{E}\beta < 0$ ). Equation (3.6) can be seen as :

$$f(x) = \frac{x}{\alpha x^2 + \beta x + \gamma} \quad (3.7)$$

Where  $\alpha = \frac{\omega}{4k_B T \phi} + \frac{\phi \omega}{4k_B T}$  ;  $\beta = 2 \frac{\omega}{4k_B T \phi} (k_g - m\omega^2)$  ;  $\gamma = \frac{\omega}{4k_B T \phi} (k_g - m\omega^2)^2$  &  $x = k_{wire}$   
The maximum value that the response Amplitude Spectral Density can reach at a given frequency, depending on the elastic stiffness of the wires is given by :

$$\frac{\partial f(x)}{\partial x} = \frac{\alpha x^2 + \beta x + \gamma - 2\alpha x^2 - \beta x}{(\alpha x^2 + \beta x + \gamma)^2} = 0$$

The solution being :  $x = \sqrt{\gamma/\alpha}$ . After simplifications, it means that the thermal noise response is maximum at a given frequency for the following elastic stiffness :

$$k_{wire,max} = \sqrt{\frac{1}{1 + \phi(\omega)}} |k_g - m\omega^2| \quad (3.8)$$

Using the expression of  $k_{wire}$  leads to the following expression :

$$\begin{aligned} n_{max} &= \frac{4k_{wire}^2 L^4}{EI\omega g} \\ &= \left( \frac{1}{1 + \phi(\omega)} \right) \frac{4(k_g - \omega^2 m)^2 L^4}{EI\omega g} \end{aligned} \quad (3.9)$$

When considering only thermoelastic damping, the maximum response to thermal noise can be easily computed at the frequency = 0 Hz. Indeed, the loss angle of this dissipation tends to 0 when  $\omega = 0$  rad/s but this corresponds also to the plateau of the response at low frequency. The expression (3.9) becomes :

$$n_{max} = \frac{4mgL^2}{EI} \quad (3.10)$$

For our special case of study, the number of wires that leads to a maximum response of the system to thermal noise at low frequency is equal to 36 [-]. This value is high and unreachable, especially when the mass increases.

To have a lower response than with one wire, the number of wires should be greater than 5000 [-]. This is clearly unrealistic but gives the idea that in this case only one way to reduce the TN response should be chosen.

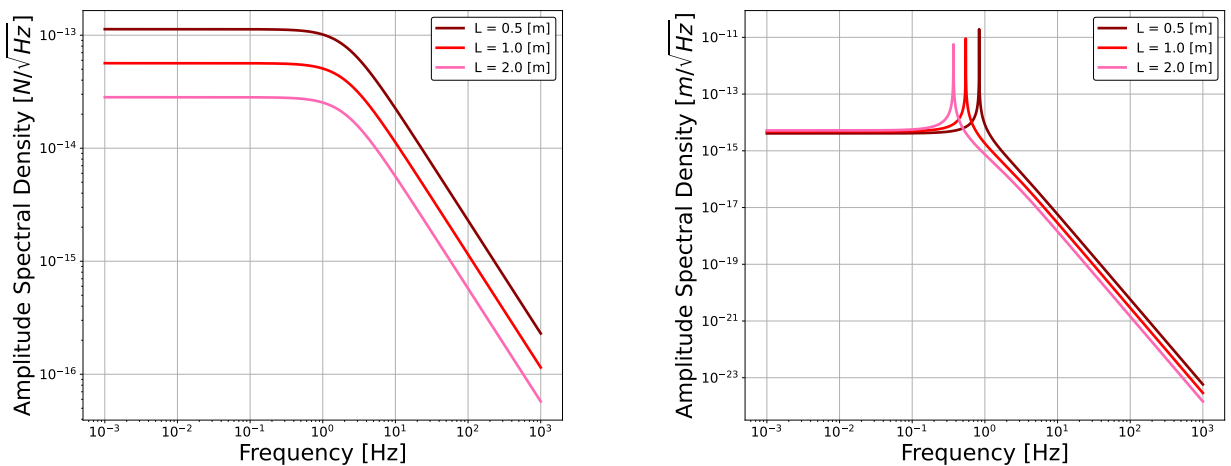
Eventually, it is important to keep in mind that the number of wires does not change the response much. Indeed, in the example, the difference between 1 and 20 wires is not that important. In this way, it can be concluded that the number of wires does not play a huge role.

### 3.1.2.2.b) Length of the wires

The length of the wires has a similar (but opposite) effect to the number of wires as they both impact the stiffness, even if the number of wires does not appear in the gravitational stiffness. Other parameters are considered constant and can be found in Table 3.4.

| T [K] | m [kg] | d [m] | n [-] |
|-------|--------|-------|-------|
| 20    | 1      | 4e-3  | 1     |

Table 3.4: Constant parameters for the length of wires variation for the 1 Degree Of Freedom pendulum system



(a) Excitation Amplitude Spectral Density

(b) Response Amplitude Spectral Density

Figure 3.4: Thermal noise for Length of wires variation comparison for the 1-degree-of-freedom pendulum system

In Figure 3.4, as expected, the behaviour is similar; in this case, the longer the wires, the lower will be the thermal noise excitation as the stiffness decreases.

The thermal noise response is higher or lower, depending on the frequency. Indeed, in Equation (3.6) the length plays a role in the gravitational stiffness. Then the maximum response at a given frequency is directly driven by the resonance frequency of the system. Indeed, changing the length leads to a change in the stiffness that moves the resonance frequency. Then the maximum response at a given frequency is directly due to this 'motion' of the resonance. Figure 3.4 shows that, at more or less 0.5 Hz, the response is maximum for  $L = 1$  m as the resonance passes through it.

Then unlike for the number of wire, no maximum can be found at  $\omega = 0$  rad/s but an infinite length. Note that the length does not influence the dissipation at all.

### 3.1.2.2.c) Diameter of the wires

The diameter is a more impactful parameter as it acts on the stiffness through the second moment of area and the cross section area in the tension expression, but it also has an impact on the thermoelastic losses from the relaxation time.

All other parameters are considered constant and can be found in Table 3.5.

| T [K] | m [kg] | L [m] | n [-] |
|-------|--------|-------|-------|
| 20    | 1      | 1.2   | 1     |

Table 3.5: Constant parameters for the diameter of wires variation for the 1 Degree Of Freedom pendulum system

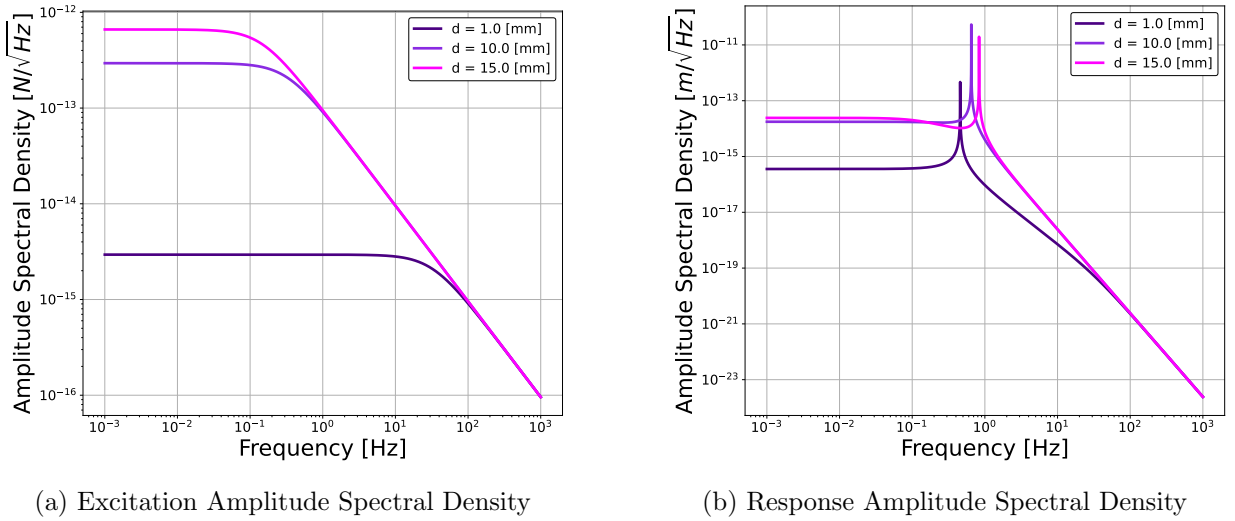


Figure 3.5: Thermal noise for Diameter of wires variation comparison for the 1-degree-of-freedom pendulum system

Figure 3.5a displays that, as expected, the higher the diameter, the higher the stiffness, and then the higher the thermal noise excitation. However, at high frequency, it seems to change. Indeed, this is due to the frequency and diameter dependence of the thermoelastic dissipation.

Looking at the expression of the thermoelastic loss angle :

$$\phi_{TE}(\omega, T) = \frac{ET}{c_p \rho} \left( \alpha - \underbrace{\frac{P\beta}{E}}_{\approx 0} \right)^2 \frac{(\omega d^2)}{4.32\pi\lambda + \frac{\rho^2 c_p^2}{4.32\pi\lambda} (\omega d^2)^2} \quad (3.11)$$

The function shows a clear dependence on the diameter. It can be referred to the study performed for the relaxation time. Indeed, in Section 2.5.2.1, it was found that changing it was only playing a role

on the location and width of the dissipation peak in frequency.

Recall that, when the diameter increases, the relaxation time increases (longer to return to equilibrium) and therefore smaller (in width) and at lower frequency is the TE dissipation peak (see Section 2.5.2.1 and Figure 2.6).

In other words, when the diameter increases, the dissipation is higher at low frequencies but drops sooner at high frequencies, which can be observed in the behaviour of the curves of Figures 3.5a. The TN excitation, directly related to the loss angle, drops sooner in frequency for larger diameters. At low frequency, the amplitude of the plateau is mainly determined by the stiffness.

For the response, the dissipation distribution is less easy to link with the behaviour of the curves from Figure 3.5b. However, it seems to show the same kind of behaviour. Indeed, when the diameter increases, the ASD seems to decrease sooner and sharper. 'Sooner' is relative since the increase in diameter makes the wire stiffer and then the resonance appears at a higher frequency. This might be more clear when looking in Figure 8.5, provided in Appendix. It is important to note that the diameter does not have a direct influence on the overall maximal amplitude of the dissipation, but rather on the amplitude at a certain frequency.

As for the other parameters, a maximum can be found for the TN response at a certain frequency for a given diameter. In this case, it is quite hard to determine it analytically. However, it is known that increasing the diameter will, at some point, decrease the TN response. In order to get an idea, in this case, at a frequency of 1e-3 Hz, the TN response with a diameter of 10 mm and 25 cm have nearly the same order of magnitude. Having a large diameter is an idea to reduce the TN response. However, the isolation is then completely lost. The results can be seen in Figure 8.5, provided in Appendix 8.5.

### 3.1.2.3 Wires Material Properties

Now the effect of the material properties on thermal noise can be determined. Changing one parameter without modifying the other is not completely physical, but it gives insight into what properties should be considered when choosing a material.

#### 3.1.2.3.a) Density and Specific Heat

The density and the specific heat have the same effect on thermal noise. When one of them tends to infinite, the loss angle tends to 0 (Equation (3.11)). Indeed, even at  $\omega \rightarrow 0$  Hz as the denominator being a sum with a constant, in all cases the loss angle tends to 0. If dissipation tends to zero, the thermal noise excitation also tends to 0 (Equation (3.4)).

If the specific heat is infinite, it means that the energy required to increase the temperature to 1 K of a 1 kg mass of this material is infinite. Then increasing the temperature is impossible; if there are no local changes in temperature, there is no heat flux, and then there is no dissipation.

This is directly related to the thermal diffusion coefficient :  $D = \lambda/(\rho c_p)$ .

The thermal noise response, looking at Equation (3.6), tends to zero when the loss angle tends to zero. However, this is not straightforward at  $\omega \rightarrow 0$  Hz. From the expression of the PSD response when  $\omega \rightarrow 0$  Hz with the entire expression of the loss angle :

$$S_x(\omega, T) \sim 4k_B T \frac{\phi_{sus}(\omega, T) k_{f,wire}}{\omega (k_g + k_{f,wire})^2} \quad (3.12)$$

$$\sim 4k_B T \frac{k_{f,wire}}{(k_g + k_{f,wire})^2} E T \left( \alpha - \frac{P\beta}{E} \right)^2 \frac{d^2}{4.32\pi\lambda + \frac{\rho^2 c_p^2 d^4 \omega^2}{4.32\pi\lambda}} \quad (3.13)$$

Then in this case, it is hard to say what is the behaviour when  $\omega \rightarrow 0$  and  $\rho$  or  $c_p \rightarrow \infty$ . However, there are two cases :

1.  $\rho c_p \omega \rightarrow \infty$  : the response tends to 0 at low frequency.
2.  $\rho c_p \omega \rightarrow 0$  or *cst* : the response tends to a constant at low frequencies.

This is a limit study; the second case should be considered.

The only concern when the loss angle tends to zero is at the resonance. Indeed, no dissipation means that the resonance peak tends to infinite. Then reducing the loss will reduce the thermal noise excitation for all frequencies. This is also the case for the response, but not at the resonance where the response will increase when the dissipation is reduced. This is expected.

Eventually, when  $\rho$  or  $c_p$  tends to zero, the thermal diffusivity tends to infinite. Then the expression (3.11) cannot be used since it was simplified. Here, the relaxation time should be checked. When the thermal diffusivity tends to infinite, the loss angle tends to zero in a same way as before.

In this case, this is quite simple to understand; when the thermal conductivity is infinite, the relaxation time is equal to zero, so the equilibrium is instantaneously reached and there is no dissipation.

Then it means that, in order to reduce the dissipation and then the thermal noise, the density and heat capacity should be low or high. In most practical cases, the easier way to reach a minimum will be to have a high value of them.

### 3.1.2.3.b) Thermal conductivity

The thermal conductivity is directly related to the previous section. Indeed, the thermal diffusivity being defined has :  $D = \lambda/(\rho c_p)$ . When  $\lambda \rightarrow 0$ , the interpretation is the same as before. Furthermore, it can be seen from Equation (3.11) that the loss angle also tends to 0.

Again, when  $\lambda \rightarrow \infty$  the loss angle tends to zero.

Then it means that, in order to reduce the dissipation and then thermal noise, the thermal diffusivity should be either low or high. In most practical cases, the easier way to reach a minimum will be to have a high value of it.

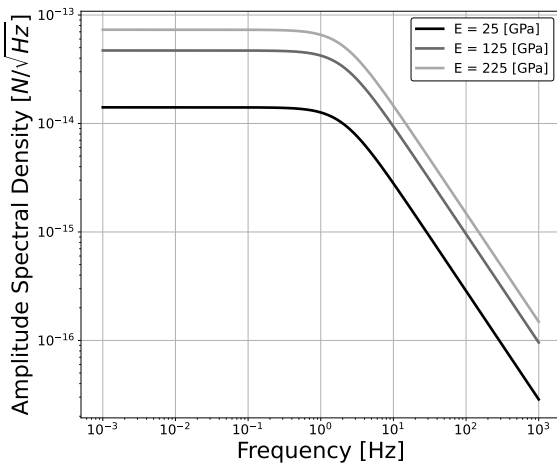
### 3.1.2.3.c) Young Modulus

As opposed to the previous material properties, the Young Modulus has an impact in both dissipation (Relaxation strength) and stiffness of the wire.

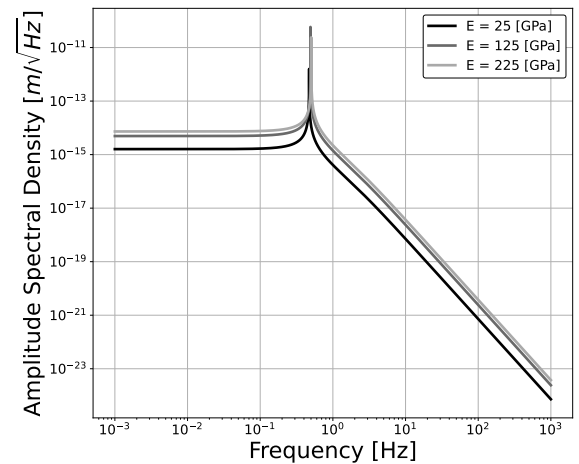
The other parameters remain constant and are given in Table 3.6.

| T [K] | m [kg] | L [m] | n [-] | d [m] |
|-------|--------|-------|-------|-------|
| 20    | 1      | 1.2   | 1     | 4e-3  |

Table 3.6: Constant parameters for Young Modulus variation for the 1 Degree Of Freedom pendulum system



(a) Excitation Amplitude Spectral Density



(b) Response Amplitude Spectral Density

Figure 3.6: Thermal noise for Young Modulus variation comparison for the 1-degree-of-freedom pendulum system

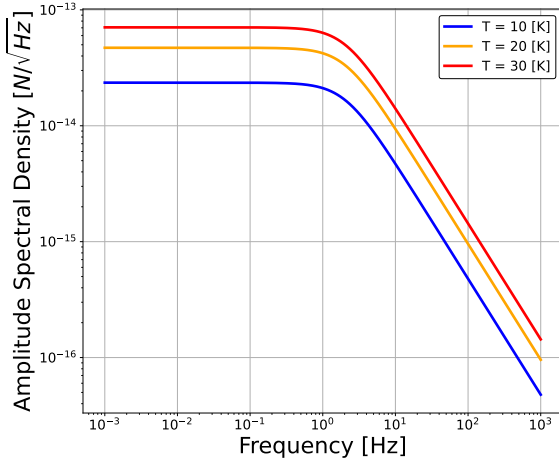
In this case, the Young Modulus is directly linked to the loss angle. Indeed, the higher the Young Modulus, the higher the dissipation, regardless of the frequency. Only modifying it will not change the frequency shape of the loss angle but the overall amplitude.

Figure 3.6 shows that a higher  $E$  leads to a higher dissipation (for all frequencies) and a higher stiffness. Then both thermal noise excitation and response increase with increasing Young Modulus.

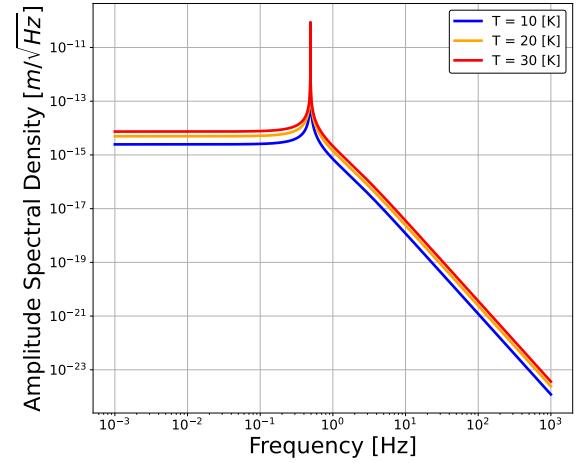
Physically, for a given strain, a higher  $E$  will result in a larger stress (Hooke's Law) and a higher energy dissipation (higher relaxation strength).

### 3.1.2.4 Temperature

The temperature parameter is quite difficult to study, as the material properties change with it. However, in a small range of temperatures they can be considered constant.



(a) Excitation Amplitude Spectral Density



(b) Response Amplitude Spectral Density

Figure 3.7: Thermal noise for Temperature variation comparison for the 1-degree-of-freedom pendulum system

Figure 3.7 shows that, as expected, the lower the temperature, the lower the thermal noise excitation and response. This is a basic result of thermal noise that is quite logical when looking at Equations (3.4) & (3.6). The results will be the same when considering temperature-dependent properties, since the link with the temperature in Equations (3.4) & (3.6) is dominant. Some shape changes are expected, but not high enough to globally change the behaviour.

### 3.1.3 Elastic stiffness of the wires

The expression of elastic stiffness :  $k_{wire} = \frac{n\sqrt{EI\sigma}}{2L^2}$  is an approximate solution of the *Euler-Bernoulli* equation at low frequency and when the tension in the wire is high. Indeed, this expression tends to zero when the tension is low, stating that only the tension gives a stiffness to the wire. Moreover, it does not describe the internal wire modes that will dissipate energy.

The exact solution is needed thus. Moreover, a comparison with the simplified expression will be made.

The Euler-Bernoulli Equation for a wire under tension is given by :

**Definition 3.1.5 - Euler-Bernoulli Equation**

$$-EI \frac{\partial^4 x(y, t)}{\partial y^4} + \sigma \frac{\partial^2 x(y, t)}{\partial y^2} = \mu \frac{\partial^2 x(y, t)}{\partial t^2} \quad (3.14)$$

$$\stackrel{\mathcal{L}}{\Rightarrow} EI \frac{\partial^4 X(y, \omega)}{\partial y^4} - \sigma \frac{\partial^2 X(y, \omega)}{\partial y^2} - \mu \omega^2 X(y, \omega) = 0 \quad (3.15)$$

where E is the Young modulus, I is the area moment of inertia,  $\sigma$  is the tension force in the wire,  $\mu$  is the linear mass density of the wire and  $X(y)$  is the continuous displacement of the wire along y (y being the longitudinal direction of the wire).

using the following boundary conditions (BC) [12] :

1.  $X(0, \omega) = 0$  (wire clamped at the top)
2.  $\left. \frac{\partial X(y, \omega)}{\partial y} \right|_{y=0} = 0$  (wire clamped at the top)
3.  $X(L, \omega) = X_1(\omega)$  (wire connected to the centre of mass (CoM) of the suspended mass, the DoF of this mass being  $X_1(\omega)$ )
4.  $\left. \frac{\partial^2 X(y, \omega)}{\partial y^2} \right|_{y=L} = 0$  (Point mass, attached to the wire at the CoM)

the EoM of the suspended mass is given by :

$$-EI \left. \frac{\partial^3 X(y, \omega)}{\partial y^3} \right|_{y=L} + \sigma \left. \frac{\partial X(y, \omega)}{\partial y} \right|_{y=L} - m\omega^2 X(L, \omega) = F(\omega) \quad (3.16)$$

Equation (3.15) has a specific solution, which is expressed as : ([12])

$$X(y, \omega) = C_1(\omega) \sin(k_s y) + C_2(\omega) \cos(k_s y) + C_3(\omega) e^{k_e y} + C_4(\omega) e^{-k_e y}$$

where  $k_s = \sqrt{\frac{\sigma + \sqrt{\sigma^2 + 4EI\rho A\omega^2}}{2EI}}$  &  $k_e = \sqrt{\frac{-\sigma + \sqrt{\sigma^2 + 4EI\rho A\omega^2}}{2EI}}$  [12]

Using the boundary conditions, the coefficients  $C_{1 \rightarrow 4}$  can be determined. These computations might be performed by hand, but a methodology is used to facilitate the determination (the methodology is easily adaptable). Indeed, the system generated by the boundary conditions can be seen as a matrix problem :

$$\underbrace{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} C_1(\omega) \\ C_2(\omega) \\ C_3(\omega) \\ C_4(\omega) \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ X_1(\omega) \\ 0 \end{pmatrix}}_{\mathbf{b}} \quad (3.17)$$

Then all the coefficient can be expressed by linear functions of  $X_1(\omega)$ .

This leads to the following solution :  $X(y, \omega) = f(y, \omega) X_1(\omega)$

$f(y, \omega)$  being easily computable with the expression of  $C_{1 \rightarrow 4}(\omega)$ .

Using this in Equation of Motion (3.16), it becomes :

$$F(\omega) = \left[ -EI \left. \frac{\partial^3 f(y, \omega)}{\partial y^3} \right|_{y=L} + \sigma \left. \frac{\partial f(y, \omega)}{\partial y} \right|_{y=L} - m\omega^2 f(L, \omega) \right] X_1(\omega) \quad (3.18)$$

$$\Leftrightarrow H(\omega) = \frac{X_1(\omega)}{F(\omega)} = \left[ -EI \left. \frac{\partial^3 f(y, \omega)}{\partial y^3} \right|_{y=L} + \sigma \left. \frac{\partial f(y, \omega)}{\partial y} \right|_{y=L} - m\omega^2 f(L, \omega) \right] \quad (3.19)$$

The transfer function being known, the PSD of the thermal noise excitation and response can be computed.

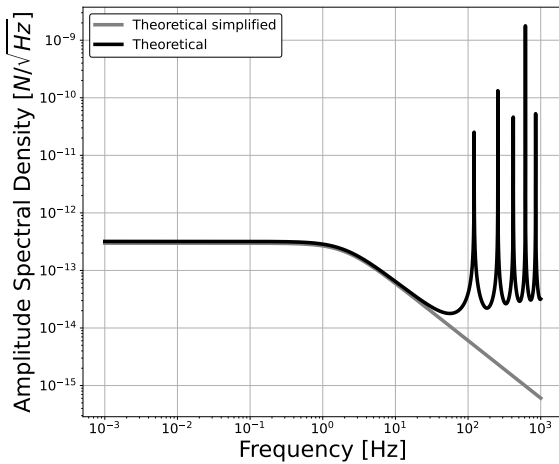
In this case, the dissipation is described by the Young Modulus such as :  $E_{eff} = E(1 + i\phi(\omega))$ .

Furthermore, only the thermoelastic dissipation is considered here.

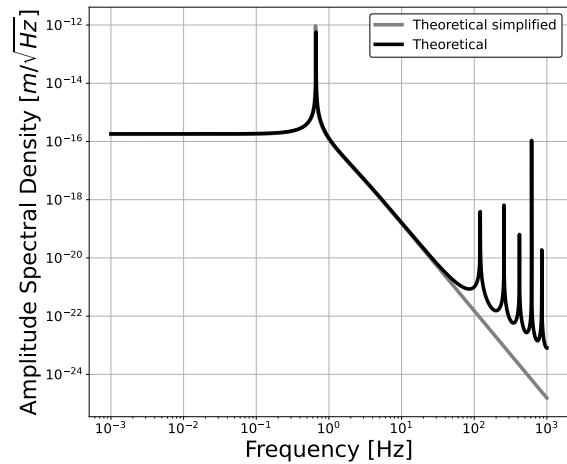
For the following parameters :

| T      | m        | L       | n     | d      | Material                   |
|--------|----------|---------|-------|--------|----------------------------|
| 20 [K] | 100 [kg] | 0.6 [m] | 1 [-] | 4 [mm] | Ti6Al4V (at 20K → Tab.3.1) |

Table 3.7: Parameters used for the comparison between the simplified and exact elastic stiffness of a wire



(a) Excitation Amplitude Spectral Density

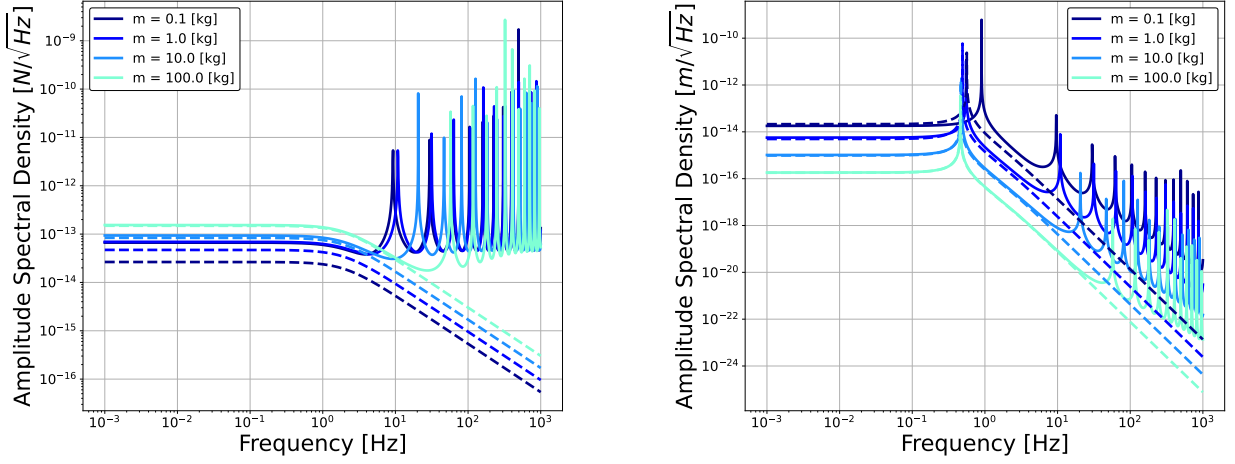


(b) Response Amplitude Spectral Density

Figure 3.8: Thermal noise comparison between the simplified and exact elastic stiffness of a wire for the 1-degree-of-freedom pendulum system

In Figures 3.8, the wire modes appear clearly and lead to a larger thermal noise response at high frequency. These modes are called the 'violin modes' (Figure 8.2) and cannot be neglected. The first peak is the pendulum mode described previously.

Eventually, these values were obtained for a mass of 100 kg. The differences for lower  $\sigma$  can be computed (the wire length is now  $L = 1.2$  m to see more violin modes in the same frequency range).



(a) Excitation Amplitude Spectral Density

(b) Response Amplitude Spectral Density

Figure 3.9: Thermal noise for Mass variation and comparison between the simplified (dashed lines) and exact (full lines) elastic stiffness of a wire for the 1-degree-of-freedom pendulum system

Figure 3.9 displays that, as expected, the difference between the simplified and exact stiffness is higher when the mass and tension decrease, especially for the pendulum mode.

### 3.1.4 Finite Element Method

The Euler-Bernoulli equation (3.15) can be hard to determine for a higher order system. In this context, the Finite Element Method can be a solution to such a complexity. However, in many FEM software, the internal damping as described in this manuscript cannot be applied. Then the goal is to simulate the system using FEM without dissipation, this step will be realised using **Nastran** solver [29]. The state space model of the system can then be extracted from the software from two simulations :

1. A modal simulation where gravity is applied to take into account the added stiffness due to gravity. This will serve to get the real part of the global stiffness matrix (as gravity does not dissipate any energy) :  $\mathbf{K}_g + \mathbf{K}_e$
2. A modal simulation without gravity to get the stiffness matrix  $\mathbf{K}_e$  that will serve to apply the dissipation and get the imaginary part of the global stiffness.

The goal of these two simulations is to obtain the following Equation of Motion :

$$(\mathbf{K}_{eff}(\omega) - \omega^2 \mathbf{M}) \mathbf{X}(\omega) = \mathbf{F}(\omega) \quad (3.20)$$

where  $\mathbf{K}_{eff}(\omega) = (\mathbf{K}_g + \mathbf{K}_e) + i (\mathbf{K}_e \odot \Phi(\omega))$ . The dissipation matrix  $\Phi(\omega)$  is therefore applied manually. (Symbol :  $\odot$  being defined as the multiplication element per element of the matrices)

The system can then be solved using an LU factorization to inverse the impedance matrix  $\mathbf{Z}(\omega)$ , which will be a sparse matrix. Eventually, the thermal noise excitation and response ASD can be estimated. However, these steps must be done for each frequency, since the stiffness matrix depends on the frequency through the dissipation matrix.

This approach is summarized in Figure 3.10.

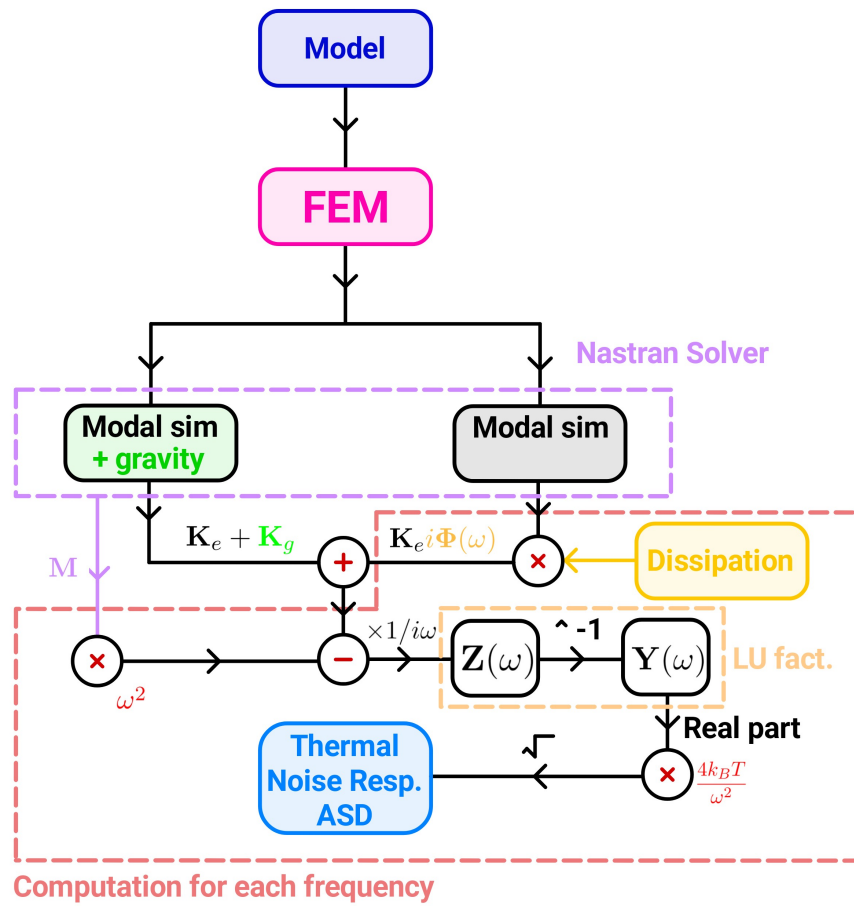
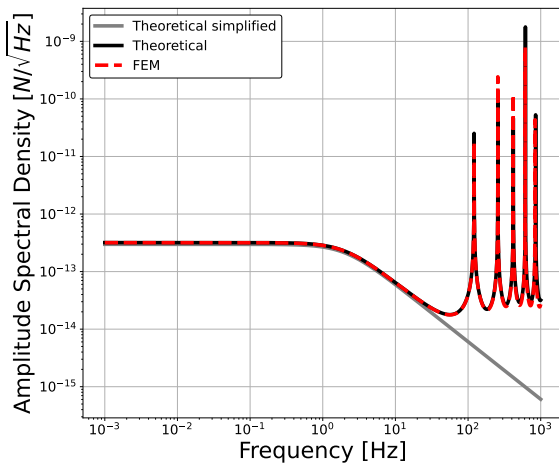


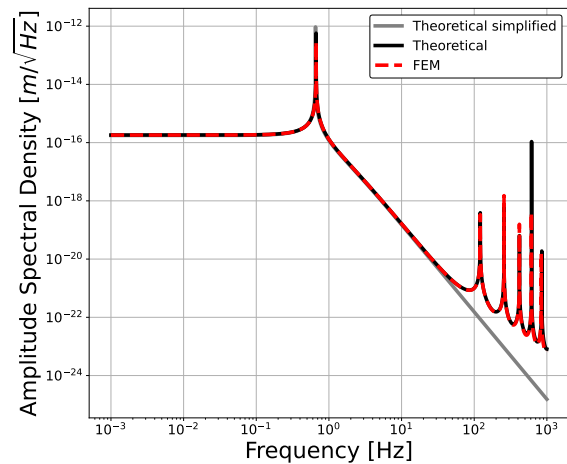
Figure 3.10: Finite Element Method : Thermal noise estimation scheme

A way to see whether this method can be applied is to compare the FEM results with those derived analytically. If the prediction matches, then it means that more complex system would be able to be represented.

The scheme in Figure 3.10 is applied to the 1-DoF pendulum considering 10 beam elements for the wire (same geometrical and material properties as before). For the dissipation, only the TE dissipation is applied and for the full stiffness matrix. In other words, every term of  $K_e$  is multiplied by the same value of  $\phi(\omega)$ .



(a) Excitation Amplitude Spectral Density



(b) Response Amplitude Spectral Density

Figure 3.11: Thermal noise comparison between the simplified, exact elastic stiffness of a wire and the Finite Element Method for the 1-degree-of-freedom pendulum system

Figure 3.11 shows that the results fit quite well, even if the high frequency violin modes start shifting. This should be improved by increasing the number of elements. A way to verify it is to perform a convergence study of the mesh.

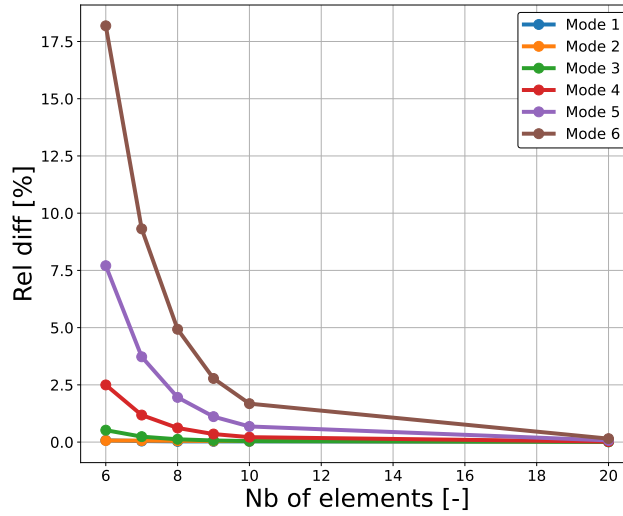


Figure 3.12: Finite Element Method resonance frequencies convergence for the 1-degree-of-freedom pendulum system

Figure 3.12 highlights that using 10 beam elements is a good choice, since the relative difference<sup>1</sup> is lower than 2% for the 6 first resonance frequencies.

### 3.1.5 Extended Suspended Mass

Rather than considering a concentrated mass, a geometry and thus inertia  $J$  can be applied to the suspended mass.

Analytically, the only change lies in the fourth Boundary Condition of the system [12] :

$$-EI \frac{\partial^2 X(y, \omega)}{\partial y^2} \Big|_{y=L} = J\omega^2 \frac{\partial X(y, \omega)}{\partial y} \Big|_{y=L}$$

The analytical results can be, again, compared with the FEM ones (still using 10 elements for the wire).

<sup>1</sup>The relative difference was computed for the resonance frequencies considering the analytical results to be exact.

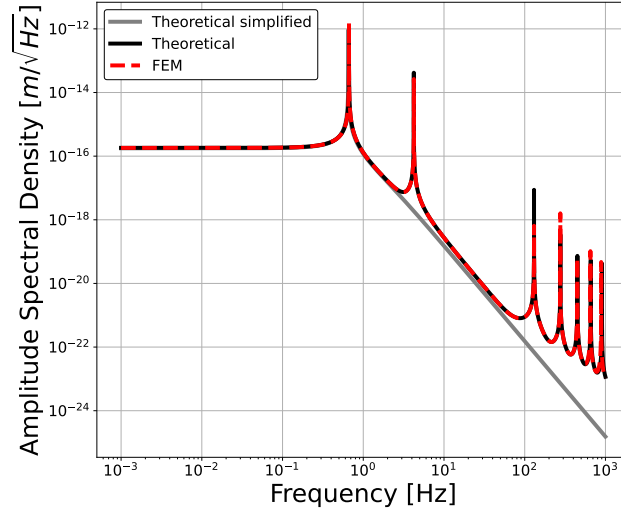


Figure 3.13: Thermal noise Response Amplitude Spectral Density - comparison between the simplified, exact elastic stiffness of a wire and the Finite Element Method for the 1-degree-of-freedom pendulum system (Extended Mass)

Figure 3.13 again displays the good agreement between the FEM and the exact analytical solution. Moreover, another resonance appears at nearly 4 Hz, which is the rotational mode of the mass around its CoM. This is called the *Rocking Mode* (Figure 8.3).

### 3.2 Two Degree Of Freedom Pendulum

Now that the methodology has been validated on a single pendulum system, it will be applied to the double pendulum (without inertia to the masses) described in Figure 3.14.

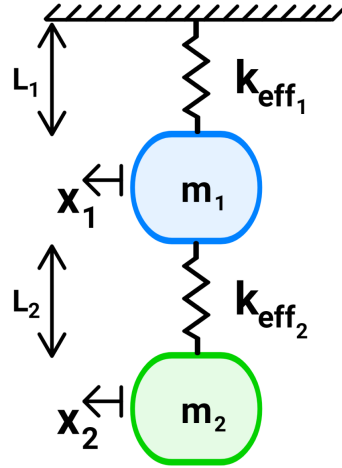


Figure 3.14: Two-degree-of-freedom Pendulum system

Koroveshi *et al.* [22] developed a description of a double pendulum. In their study, they considered the simplified expression of the stiffness for the top wire, as the tension is high, and solved the Euler-Bernoulli equation for the second wire. In this manuscript, the full resolution of the Euler-Bernoulli equations for the two wires will be performed. A comparison will be made later in this section.

The same methodology can be applied to solve these equations.

Here there are 2 equations, so 2 solutions, as there are two wires, so 8 BCs that are :

1.  $X_1(0, \omega) = 0$  (top wire clamped at the top)

2.  $\frac{\partial X_1(y, \omega)}{\partial y} \Big|_{y=0} = 0$  (top wire clamped at the top)
3.  $X_1(L_1, \omega) = X_1(\omega)$  (wire connected to the centre of mass (CoM) of the first suspended mass, the DoF of this mass being  $X_1(\omega)$ )
4.  $X_2(0, \omega) = X_1(\omega)$  (second wire connected to the centre of mass (CoM) of the first suspended mass)
5.  $\frac{\partial X_1(y, \omega)}{\partial y} \Big|_{y=L_1} = \frac{\partial X_2(y, \omega)}{\partial y} \Big|_{y=0}$  (continuity)
6.  $E_1 I_1 \frac{\partial^2 X_1(y, \omega)}{\partial y^2} \Big|_{y=L_1} = E_2 I_2 \frac{\partial^2 X_2(y, \omega)}{\partial y^2} \Big|_{y=0}$  (continuity of internal bending moment)
7.  $X_2(L_2, \omega) = X_2(\omega)$  (second wire connected to the centre of mass (CoM) of the second suspended mass, the DoF of this mass being  $X_2(\omega)$ )
8.  $\frac{\partial^2 X_2(y, \omega)}{\partial y^2} \Big|_{y=L_2} = 0$  (Point mass, attached to the second wire at the CoM)

The system to solve is then :

$$\underbrace{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} C_1(\omega) \\ C_2(\omega) \\ C_3(\omega) \\ C_4(\omega) \\ C_5(\omega) \\ C_6(\omega) \\ C_7(\omega) \\ C_8(\omega) \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ X_1(\omega) \\ X_1(\omega) \\ 0 \\ 0 \\ X_2(\omega) \\ 0 \end{pmatrix}}_{\mathbf{b}} \quad (3.21)$$

Each  $C_i(\omega)$  being linearly dependent of  $X_1(\omega)$  &  $X_2(\omega)$ , allowing to express the solutions of the Euler-Bernoulli equations as :

$$\begin{cases} X_1(y_1, \omega) = f(y_1)X_1(\omega) + g(y_1)X_2(\omega) \\ X_2(y_2, \omega) = h(y_2)X_1(\omega) + i(y_2)X_2(\omega) \end{cases}$$

This leads to the EoM system (annotating  $\partial/\partial y$  by the symbol : ' ):

$$\underbrace{\left( -\mathbf{EI} \begin{bmatrix} f'''(L_1) & g'''(L_1) \\ h'''(L_1) & i'''(L_2) \end{bmatrix} + \boldsymbol{\sigma} \begin{bmatrix} f'(L_1) & g'(L_1) \\ h'(L_1) & i'(L_2) \end{bmatrix} - \omega^2 \mathbf{M} \begin{bmatrix} f(L_1) & g(L_1) \\ h(L_1) & i(L_2) \end{bmatrix} \right)}_{\mathbf{H}^{-1}(\omega)} \begin{bmatrix} X_1(\omega) \\ X_2(\omega) \end{bmatrix} = \begin{bmatrix} F_{Th,1}(\omega) \\ F_{Th,2}(\omega) \end{bmatrix} \quad (3.22)$$

The system can then be solved analytically and compared with the FEM model (considering only thermoelastic dissipation). Figure 3.15 shows the comparison, when considering the following parameters (which are the same for the two wires) :

| T      | L       | n     | d      | Material                   |
|--------|---------|-------|--------|----------------------------|
| 20 [K] | 0.6 [m] | 1 [-] | 4 [mm] | Ti6Al4V (at 20K → Tab.3.1) |

Table 3.8: Parameters used for the comparison between the analytical and FEM description of a double pendulum

The first suspended mass was set at 150 [kg] and the second one at 100 [kg].

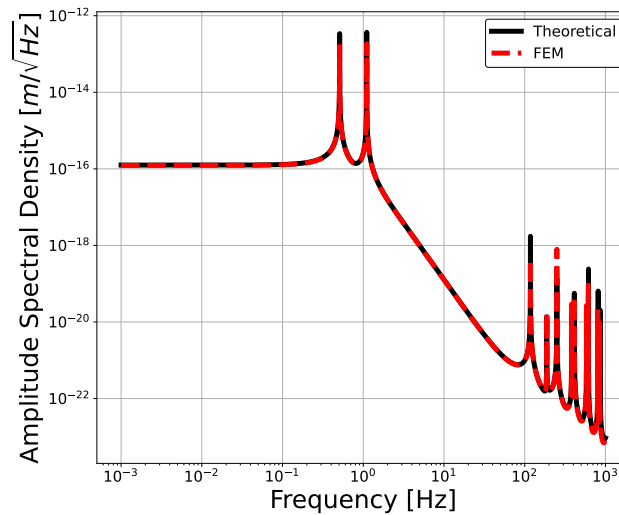


Figure 3.15: Response Amplitude Spectral Density Thermal noise - comparison between the analytical and FEM solutions for the 2-degree-of-freedom pendulum system

Figure 3.15 displays that two pendulum modes appear, these are the in-phase and the out-of-phase modes, which are represented in Figures 8.4. Moreover, more violin modes are present since there are two wires.

Eventually, the results fit quite well, which is a good sign that the FEM method can be applied to describe the thermal noise of complex systems.

The approximation made by *Koroveshi et al.* [22] can be compared to the exact solution. To do so, 3 different cases will be considered :

1. **Simplified** : The approximate solution of the elastic stiffness of a wire under tension is applied to both wires of the double pendulum.
2. **Intermediate** : The approximated solution of *Koroveshi et al.* [22] where the approached solution of the elastic stiffness of a wire under tension is applied to the top wire and for the second, the Euler-Bernoulli equation is solved.

This approach differs slightly from the one of [22]. Indeed, in their study, they consider the following boundary condition :  $\frac{\partial X_2(y, \omega)}{\partial y} \Big|_{y=L_2} = 0$  at the bottom of the second wire. This assumption will be discussed later (Section 6.4.6) but is not the one used here.

3. **Exact** : Euler-Bernoulli equations are solved for the complete system.

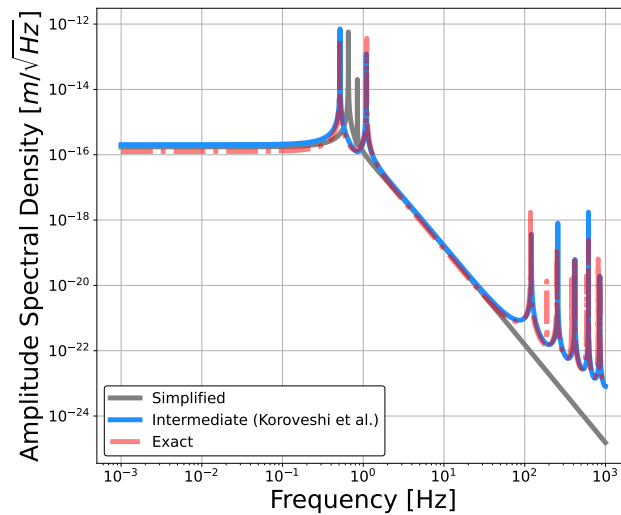


Figure 3.16: Response Amplitude Spectral Density Thermal noise - comparison between the 3 approaches of solving analytically the 2-degree-of-freedom pendulum system

Figure 3.16 highlights that the simplified approach is very bad and shows its limit. For the intermediate approach, it can be seen that this is quite good in comparison to the exact one. Some violin modes are missing, since only one wire is fully described. Even if resonances are quite well predicted, the amplitude seems to be overestimated for the intermediate approach.

### 3.3 Conclusion

This Section enables to describe and understand the thermal noise on basic pendulum systems. The impacts of different parameters were determined and physically described.

Next, a new FEM method, based on the state space matrices, has been developed and shown to be accurate in comparison with analytical results.

Eventually, this method enabled to compare the approaches from the literature, especially from [22].

## Chapter 4

# First Radiative Heat Exchanger (RHE) Model

The purpose of this section is to study the effect of placing an RHE on a two Degree Of Freedom pendulum. The added fins follow the radiative cooling system of E-TEST [9]. In this first model, geometries and materials will be chosen arbitrarily (still close to realistic values). However, in the next sections, an approach of the E-TEST prototype and ET-LF model will be performed with more accurate values.

### 4.1 Model Description

The model is a two Degree Of Freedom pendulum in which a certain number of plates are added to the top mass (Cold Platform). Figure 4.1 represents the scheme of the model. Even if the RHE system of E-TEST has fins all around the cold platform, here only two sides will be considered.

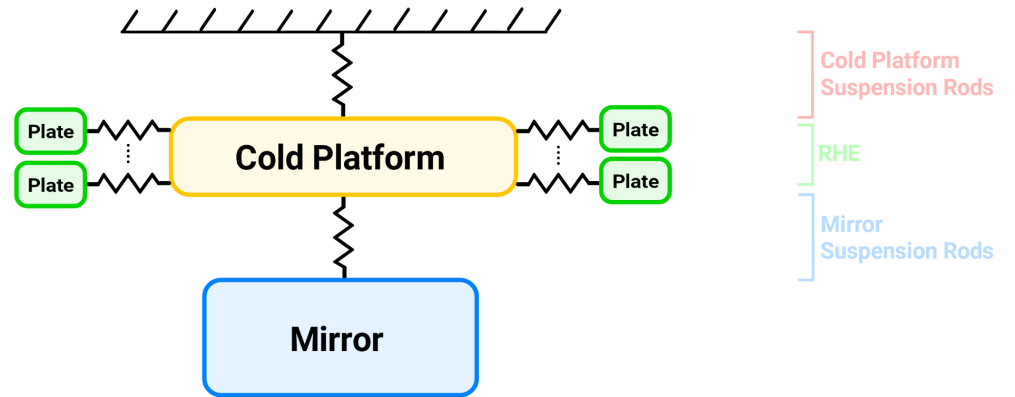


Figure 4.1: First Cryostat Model - Scheme

Figure 4.2 presents the resulting FE model considering 50 fins.

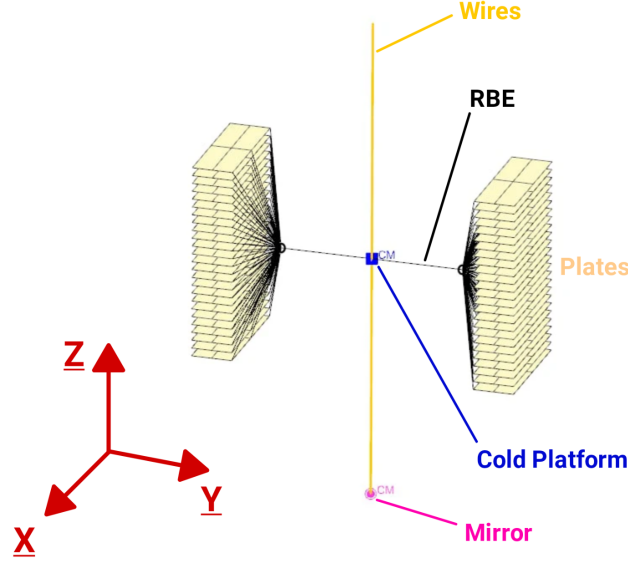


Figure 4.2: First Cryostat Model - FEM model with 50 plates

The Mirror and Cold Platform are considered as 100 and 70 kg point masses (without inertia), respectively. The two wires (1D beam elements) are identical with a length of 0.6 m and a diameter of 4 mm. The plates (2D CQUAD4 elements) are rigidly connected to the cold platform thanks to Rigid Body Element (RBE) by one edge (see Figure 4.2). The plates are attached at 300 mm (in the y axis) of the cold platform point mass.

The dimensions of the plates are : Length = 300 mm, Width = 100 mm, and Thickness = 0.5 mm. Furthermore, there is a 20 mm distance between each panel.

The wires are considered to be made of Ti6Al4V with the properties (assumed) in Table 4.1 :

| $\rho$ [kg/m <sup>3</sup> ] | $E$ [GPa] | $\nu$ [-] | $\lambda$ [W/m/K] | $c_p$ [J/kg/K] | $\alpha$ [K <sup>-1</sup> ] | $\beta$ [K <sup>-1</sup> ] | $\phi_{bulk}$ [-] | $\alpha_{surf}$ [m] |
|-----------------------------|-----------|-----------|-------------------|----------------|-----------------------------|----------------------------|-------------------|---------------------|
| 4430                        | 125       | 0.3       | 2                 | 30             | 1e-6                        | -4.6e-6                    | 0                 | 0                   |

Table 4.1: Assumed properties of the Ti6Al4V for the First Model @ 20 K

For the material of the fins, almost the same properties as the wires were chosen, only the density is different and equal to 33 333.33 kg/m<sup>3</sup> (this value was chosen to have plates of 0.5 kg each).

For the mesh, the wires are made up of 10 beam elements each. For the plates, CQUAD4 shell elements, from NASTRAN, with a number of two nodes per edge, were used.

The axes that will be used in the study are represented in Figure 4.2.

## 4.2 Dissipation Model

This section is one of the most important, as it aims to define a dissipation model to apply on such systems. Previously in this master's thesis, the dissipation was applied everywhere on the stiffness matrix because the wires were identical (dimensions and material). Moreover, only one axis was studied.

However, in more complex systems, the dimensions and materials are not the same, there are 2D elements, and responses along different axes are needed. This means that the dissipations are not the same in the stiffness matrix.

First, different materials and dimensions mean different values of the loss angle. Secondly, different axes mean different dissipation contributions. The torsional dissipation is not the same as the flexural one, since the Thermoelastic damping does not occur in torsion.

All these reasons show that a dissipation model is needed. To do so, since the mass and stiffness

matrices are exported from the software, the stiffness matrix has to be modified manually.

As illustrated on the right of Figure 3.10 the stiffness matrix is multiplied by the dissipation. A dissipation matrix  $\Phi(\omega)$  should be created and then multiplied by the stiffness matrix, element per element.

#### 4.2.1 1D Beam elements

Applying such a method to 1D Beam elements is straightforward, since the matrix of an element is decomposed into axial, flexural, and torsional stiffness. Then, the axial  $\phi_\eta(\omega)$ , flexural  $\phi_\gamma(\omega)$  and torsional  $\phi_\tau(\omega)$  dissipations can be applied to their respective stiffness (see Figure 4.3).

$$K_{el} = \begin{bmatrix} \frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{l^3} & \frac{12EI_y}{l^3} & 0 & 0 & 0 & 0 & \frac{12EI_z}{l^3} & \frac{12EI_y}{l^3} & 0 & 0 & 0 \\ 0 & 0 & \frac{12EI_y}{l^3} & 0 & 0 & 0 & 0 & 0 & \frac{12EI_x}{l^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{GJ_x}{l} & 0 & 0 & 0 & 0 & 0 & \frac{GJ_x}{l} & 0 & 0 \\ 0 & 0 & -\frac{6EI_y}{l^2} & 0 & \frac{4EI_y}{l} & 0 & 0 & 0 & 0 & 0 & \frac{4EI_z}{l} & 0 \\ 0 & \frac{6EI_z}{l^2} & 0 & 0 & 0 & \frac{4EI_z}{l} & 0 & 0 & 0 & 0 & 0 & \frac{4EI_x}{l} \\ -\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{l} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI_z}{l^3} & 0 & 0 & 0 & 0 & 0 & -\frac{12EI_z}{l^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{12EI_y}{l^3} & 0 & 0 & 0 & 0 & 0 & -\frac{12EI_y}{l^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{GJ_x}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ_x}{l} & 0 & 0 \\ 0 & 0 & \frac{6EI_y}{l^2} & 0 & \frac{2EI_y}{l} & 0 & 0 & 0 & 0 & \frac{2EI_y}{l} & 0 & 0 \\ 0 & 0 & -\frac{6EI_z}{l^2} & 0 & 0 & \frac{2EI_z}{l} & 0 & 0 & 0 & 0 & \frac{2EI_z}{l} & 0 \\ 0 & \frac{6EI_z}{l^2} & 0 & 0 & 0 & 0 & 0 & -\frac{6EI_z}{l^2} & 0 & 0 & 0 & \frac{4EI_z}{l} \end{bmatrix}$$

× $\phi_\eta(\omega)$ 
× $\phi_\gamma(\omega)$ 
× $\phi_\tau(\omega)$

Figure 4.3: Dissipation Scheme applied on a 1D Beam element stiffness matrix

The expressions of the different loss angles are not the purpose of this section. Following this method, the loss angle matrix that is defined for a 1D beam element is :

$$\Phi_{el} = \begin{pmatrix} \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau \\ \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau \\ \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau \\ \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau \\ \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau \\ \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau \\ \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau \\ \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau \\ \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau \\ \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau \\ \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau & \phi_\eta & \phi_\gamma & \phi_\tau \end{pmatrix} \quad (4.1)$$

which is multiplied element per element with the stiffness matrix described in Figure 4.3. Many terms will be equal to zero. Multiplication also includes the imaginary number and leads to the yellow part in Figure 3.10.

#### 4.2.2 2D elements

The same kind of methodology can be applied to 2D elements. However, the dissipation is not directly applied in the software but by post-processing the stiffness matrix. For 1D beam elements, this task

was quite direct, since the element stiffness matrices are on the main diagonal of the global stiffness matrix. However, for 2D elements, the nodes are not necessarily well determined (software defined), and it can be hard to play with it. Particularly for a high number of 2D elements. An approximation is made therefore by applying the same loss angle  $\phi_p(\omega)$  to all elements of the fins. To be conservative, the maximum value of the loss angle will be applied (all dissipation contributions considered for all the DoFs).

### 4.2.3 First model dissipation

For this simple model, only the thermoelastic damping was kept for its frequency dependence.

Then, for simplicity, the TE damping will be applied for the flexural stiffness but also for the axial and torsional one.

For the wires, the TE damping of a circular cross section wire will be used, while for the fins, the ribbon formula was chosen (Equation (2.38)).

## 4.3 Sensitivity analysis

The goal of performing such a simplified model is to play with the parameters and identify their impacts on the system. The objective is then to describe the behaviour of each modification in order to facilitate future design of such radiative heat exchangers.

### 4.3.1 Number of plates

The first parameter to study is the number of fins added to the basic double pendulum case. Three cases will be considered for this comparison :

- a model without fins;
- a model with 6 fins;
- a model with 50 fins;

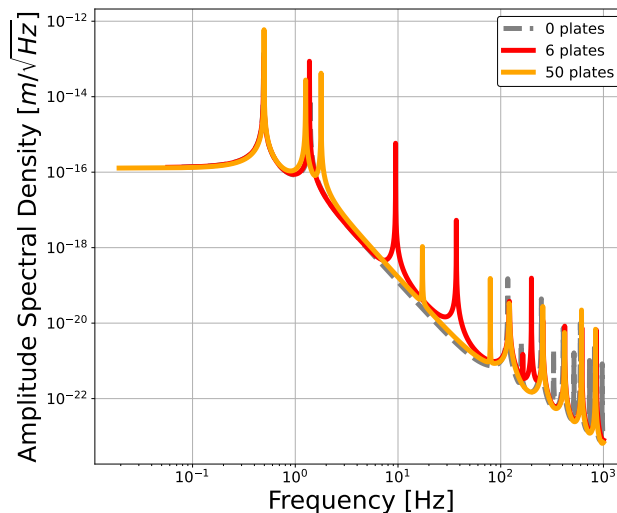


Figure 4.4: Thermal noise Response ASD - comparison for different number of plates for the First Model system (in the x axis)

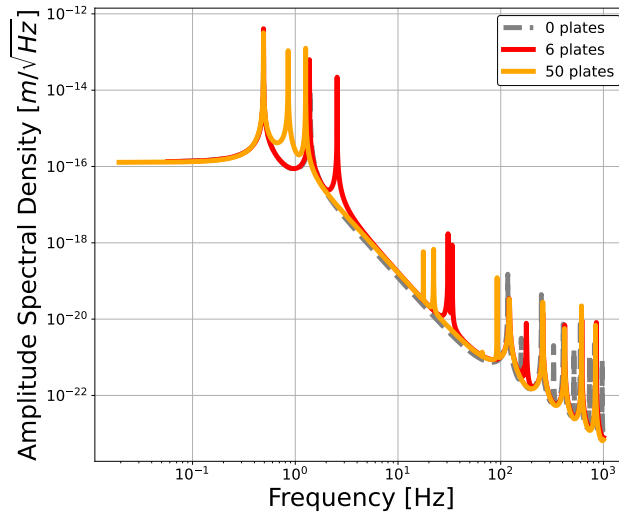


Figure 4.5: Thermal noise Response ASD - comparison for different number of plates for the First Model system (in the y axis)

In the x and y axes, Figures 4.4 & 4.5 show that the addition of plates has an impact on the mass (and then tension) and on the inertia. Pendulum modes are slightly modified and a new mode appears due to inertia<sup>1</sup> (see Figure 4.4 for the 6 plates case at  $\approx 9$  Hz and  $\approx 2.5$  Hz in Figure 4.5). Furthermore, the fins are flexible; and their modes appear. There are new resonances that locally increase the thermal noise. As the plates deform, they dissipate energy, leading to an increase in TN as well. The peaks amplitudes depend only on the dissipation of the plates (this will be verified later).

For the x axis, the contributing plate modes are the torsional ones which create a rotation around the y axis in the same way as a rocking mode. The rocking mode is coupled with the torsion of the fins, as shown in Figure 4.6.

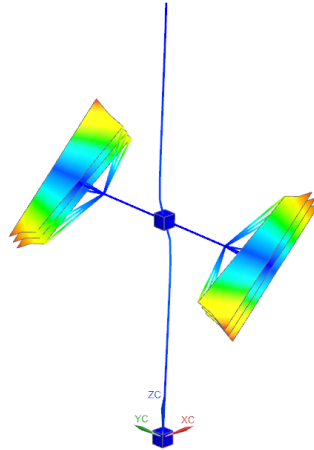


Figure 4.6: First Rocking Mode of the First model with 6 plates

This coupling depends on the geometry and stiffness of the plates (this will be discussed later). The first torsion mode of the plates is shown in Figure 4.7.

<sup>1</sup>It is a new mode since the masses do not have any inertia.

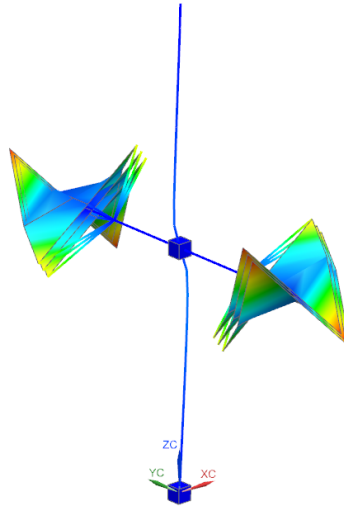


Figure 4.7: First Torsion Mode of the plates from the First model with 6 plates

The coupling is more identifiable in Figure 4.4 as the number of plates should not modify their own resonances. However, the fin modes are not the same for 6 as for 50 panels. In this case this is mainly the inertia that is changed, 50 panels in torsion will make the resonance appear sooner in frequency.

For the y axis, the Out-Of-Phase flexion modes of the fins create a rotation around the x axis. Again, this motion is coupled with the inertia and the same conclusions can be drawn. The first Out-Of-Phase mode of the plates is displayed in Figure 4.8.

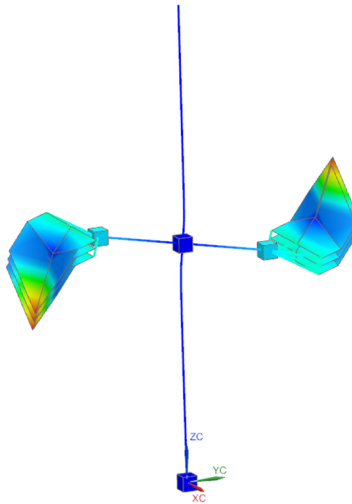


Figure 4.8: First Out-Of-Phase flexion Mode of the plates from the First model with 6 plates

At low frequencies, the plates seem to have no influence. This is because they move rigidly (see Figure 4.12) at low frequency, so no deformation can occur and be the cause of dissipation. Nevertheless, the addition of fins increases the mass and then the tension in the top wire. Since this added mass is not significant, no important changes can be seen. Eventually, as expected, the violin modes are nearly not impacted at all.

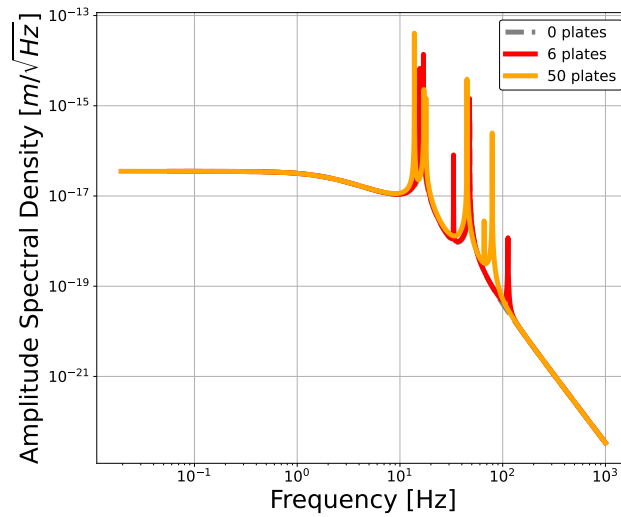


Figure 4.9: Thermal noise Response ASD - comparison for different number of plates for the First Model system (in the z axis)

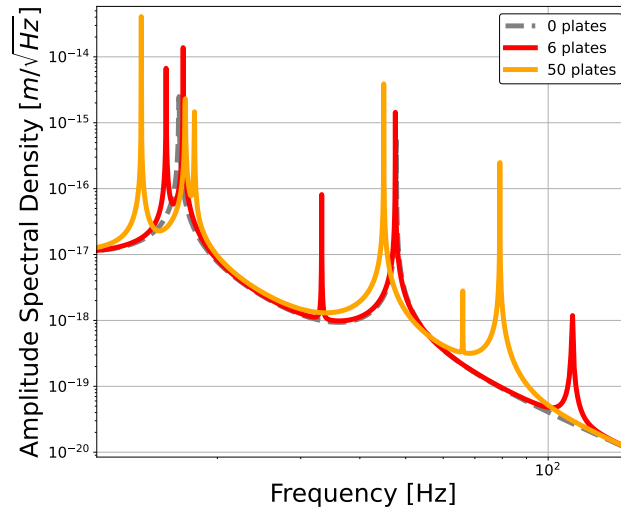


Figure 4.10: Thermal noise Response ASD - comparison for different number of plates for the First Model system (in the z axis) - Zoom

For the z axis, Figures 4.9 & 4.10 display that the In-Phase flexion modes of the fins create a direct motion in the z axis.

In this case, at low frequency there is no difference. Increasing the mass means increasing the tension in the wires, but the axial stiffness does not depend on this tension.

It can be noted that the rotational DoF of the mirror could have been shown, since the method can compute them. However, these curves will not provide new information.

### 4.3.2 Internal Dissipation of the RHE

An interesting parameter to play with is the internal dissipation of the plates. The thermoelastic damping is still considered.

Increasing the dissipation should increase the thermal noise, as previously seen. However, the purpose is to identify the impact of the dissipation of the fins.

Five cases will be considered for the comparison :

- a model without fins.

- a model with 6 fins with a dissipation  $\phi_p$  (the dissipation computed at the previous section).
- a model with 6 fins with twice the dissipation  $2 \times \phi_p$
- a model with 6 fins with ten times the dissipation  $10 \times \phi_p$
- a model with 6 fins with fifty times the dissipation  $50 \times \phi_p$

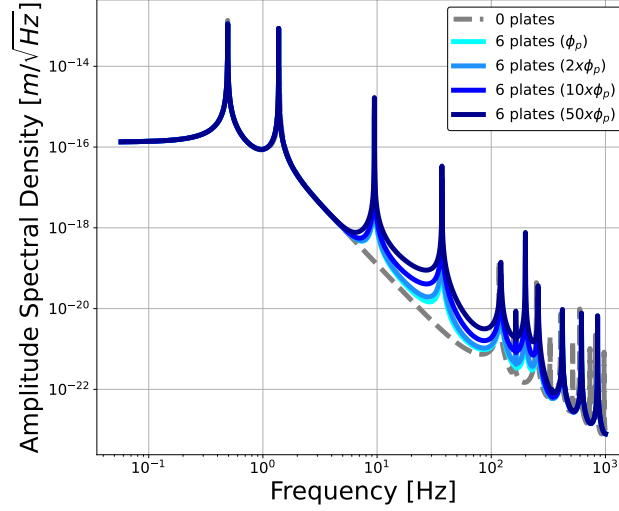


Figure 4.11: Thermal noise Response ASD - comparison for different plate dissipations (loss angles) for the First Model system (in the x axis)

Figure 4.11 highlight that the amplitude of the two first peaks did not change. Since these resonances correspond to the in-phase and out-of-phase pendulum modes. For these modes, the plates are not deformed at all and the dissipation only comes from the wires (see Figure 4.12).

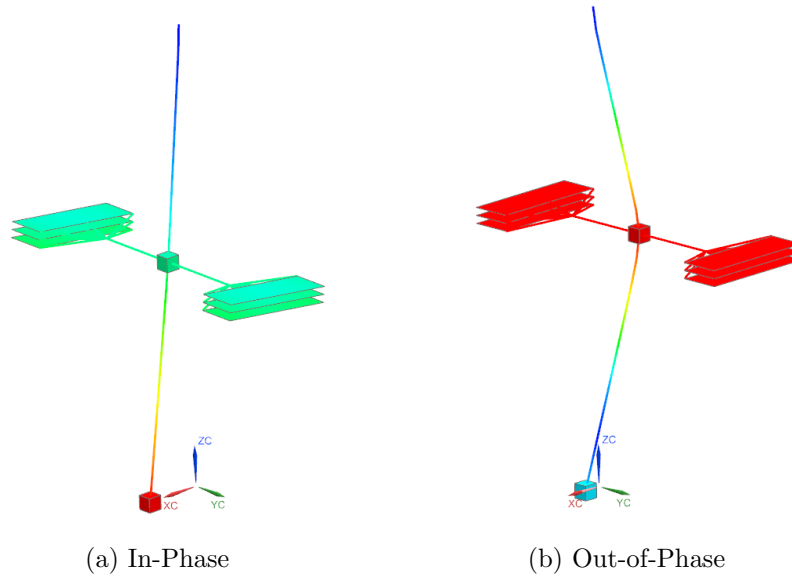


Figure 4.12: Pendulum modes of the First model with 6 plates (in the x axis)

As the plates do not dissipate any energy in this frequency range, their loss angle does not impact the response. However, between 10 and 100 Hz, the dissipation of the plates seems to have an important impact. As expected, the thermal noise increases with the dissipation but not everywhere. The mode at  $\approx 10$  Hz corresponds to the rocking mode, but still seems to be impacted. The coupling with the

fins deformation is clearly identified. If the plates were stiffer, this peak would have been independent of the dissipation of the plates.

The mode at  $\approx 35$  Hz comes completely from the torsional mode of the plates. The fins being deformed, they dissipate energy. Then, increasing the dissipation of the panels only impacts the frequency at which they do not act as rigid bodies.

Moreover, as expected, the violin modes are not at all influenced as they do not deform the plates.

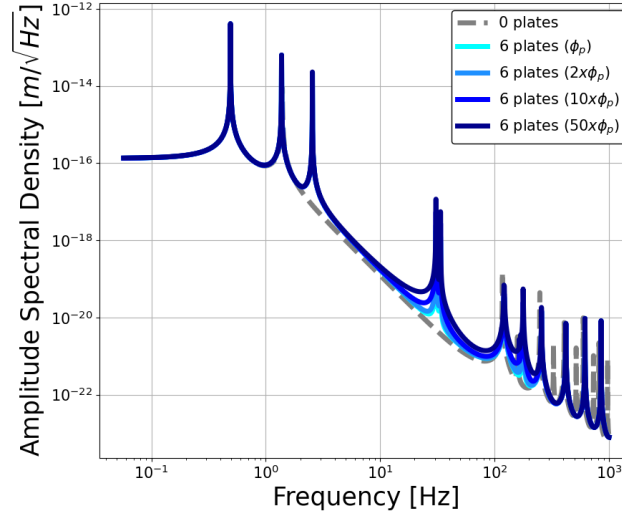


Figure 4.13: Thermal noise Response ASD - comparison for different plate dissipations (loss angles) for the First Model system (in the y axis)

From Figure 4.13, the same conclusions can be drawn for Out-Of-Phase flexion modes of the fins. For the rocking mode at  $\approx 2.5$  Hz, the coupling with the plates is extremely small. Indeed, looking at the configuration of the fins (Figure 4.2) and how they are attached to the Cold Platform, they exhibit higher stiffness in flexion than in torsion.

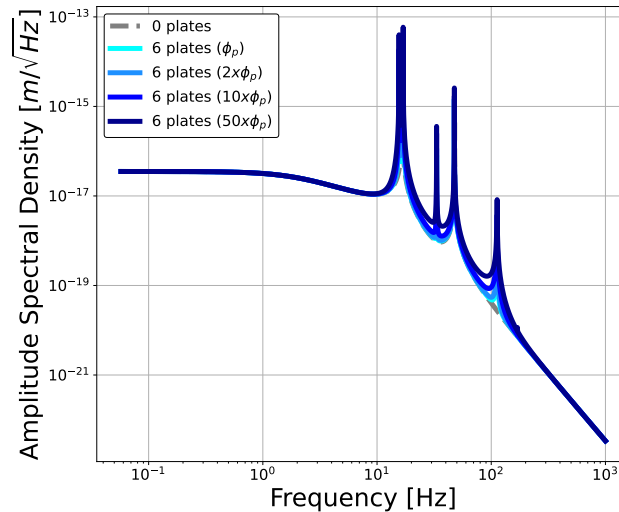


Figure 4.14: Thermal noise Response ASD - comparison for different plate dissipations (loss angles) for the First Model system (in the z axis)

Again, from Figure 4.14, the same conclusions can be made for the z-axis. The In-Phase flexion modes are highly coupled with the axial modes of the two masses. If the plates were stiffer, these axial modes might become totally independent of the dissipation of the fins.

Note that the results shown in Figure 4.14 are more readable in Figure 4.15

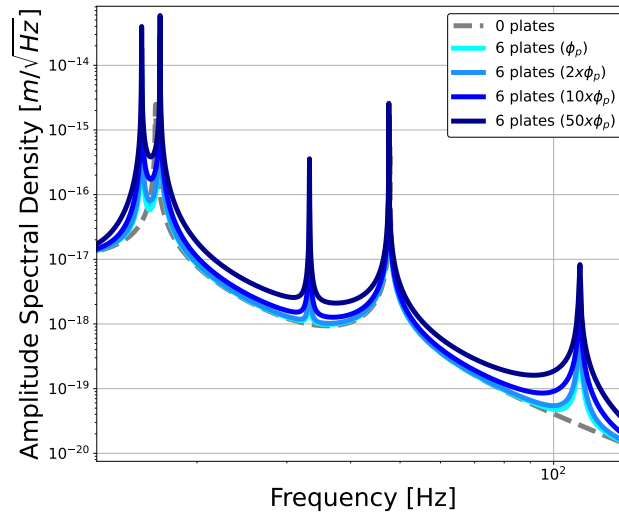


Figure 4.15: Thermal noise Response ASD - comparison for different plate dissipations (loss angles) for the First Model system (in the z axis) - Zoom

#### 4.3.2.1 Conclusion

As physically expected, the impact of the plates dissipation mainly occurs for their own mode. The pendulum, purely inertial, and the violin modes are not impacted, with the condition to be far from the modes of the plates. If there is a coupling, these modes will be impacted.

### 4.3.3 Stiffness of the plates

As seen previously, the stiffness of the plates defines their modes. These modes are the main origin of the TN response increase when introducing an RHE. The impact of the stiffness of the fins will be compared distinguishing two cases :

1. The global stiffness without changing the dissipation of the plates.
2. The Young Modulus, that is the origin of the stiffness, but also plays a role in the dissipation (see Equation (2.38)).

#### 4.3.3.1 Global Stiffness

The stiffness of the fins is increased without changing the dissipation (computed with  $E = 125$  GPa).

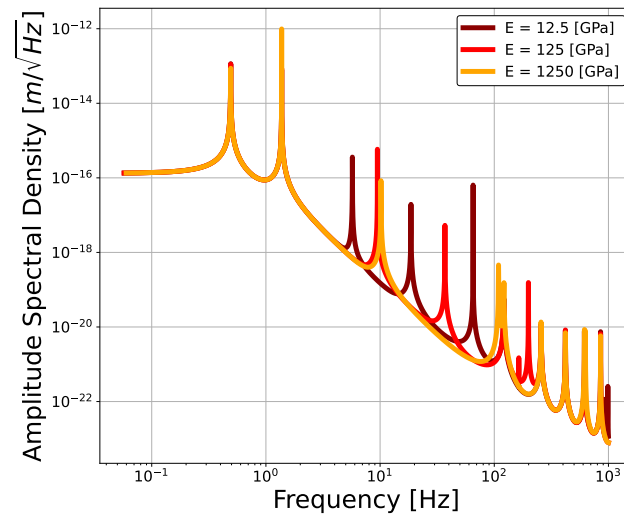


Figure 4.16: Thermal noise Response ASD - comparison for different plate stiffness for the First Model system (in the x axis)

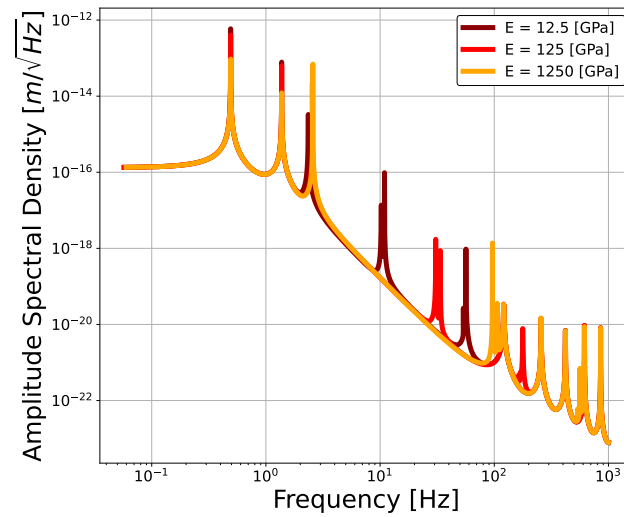


Figure 4.17: Thermal noise Response ASD - comparison for different plate stiffness for the First Model system (in the y axis)

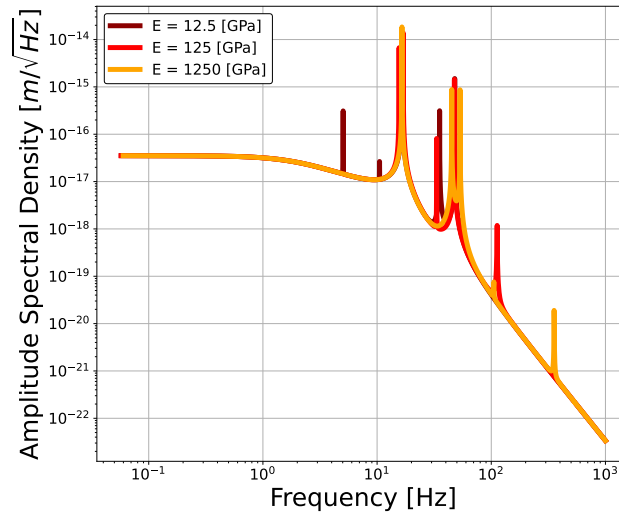


Figure 4.18: Thermal noise Response ASD - comparison for different plate stiffness for the First Model system (in the z axis)

As expected, Figures 4.16, 4.17 and 4.18 shows that the stiffness only affects the resonances of the plates. The stiffer are the fins, at higher frequencies their modes will appear. Moreover, for the x and y axes, the coupling with the Rocking modes can be clearly identified. Indeed, when the stiffness increases, the resonance tends to a single value ( $\approx 10$  Hz for the x axis and  $\approx 2.5$  Hz for the y axis). In contrast, when the stiffness decreases, the coupling is more important, resulting in a decrease of the resonance frequency.

#### 4.3.3.2 Young Modulus

The previous case can be compared with the one where the Young Modulus is also modified for the dissipation computation.

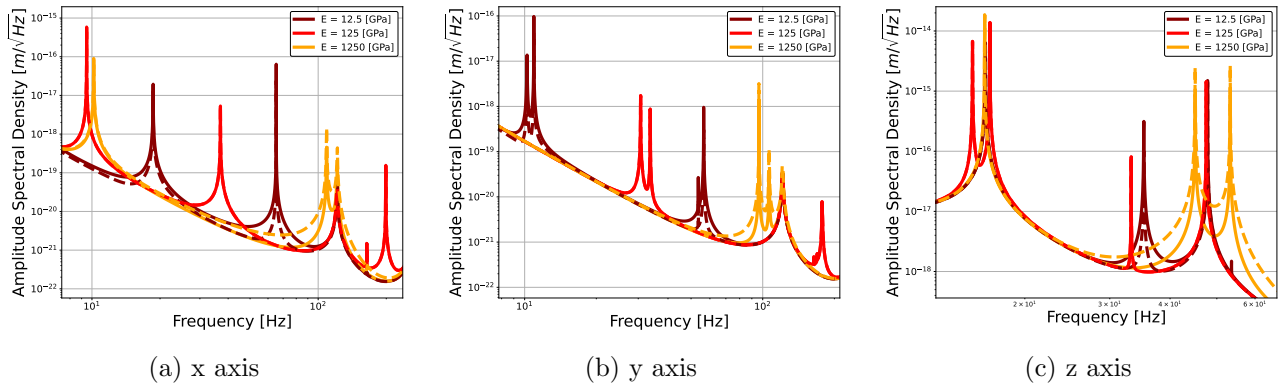


Figure 4.19: Thermal noise Response ASD - comparison between modifying the Global Stiffness (full line) and the Young modulus (dashed line) for the First Model system

Figure 4.19 displays that, as previously determined, the stiffness and dissipation of the plates do not modify the behaviour of the system at low frequency. That is why Figure 4.19 is focused on the frequency range of the fin modes. The only modification follows the conclusion of the previous section. When the Young Modulus increases, the dissipation increases as well. These increments lead to a local rise of the TN response (at the resonances of the plate modes). Eventually, this is the opposite when the Young Modulus decreases.

### 4.3.4 Geometry of the fins

The geometry of the plates is a crucial parameter, as it is directly related to the cooling capacity of the RHE (same as the number of plates). Two cases are distinguished :

1. 2D size : the width and length of the plates. These parameters do not impact the TE dissipation (see Equation (2.38)).
2. The thickness that impact the TE dissipation of the plates (see Equation (2.38)).

#### 4.3.4.1 2D size

In this section, the sizes of the plates will be modified through the width and length. Therefore, two cases will be compared to the initial one :

- Extended in the x direction, the length is doubled (300 mm  $\rightarrow$  600 mm)
- Extended in the y direction, the width is doubled (100 mm  $\rightarrow$  200 mm)

From this, the parameter  $L_w$  (Length-to-width ratio) can be defined and leads to :

- Ext. y :  $L_w = 1.5$  [-]
- Initial case :  $L_w = 3$  [-]
- Ext. x :  $L_w = 6$  [-]

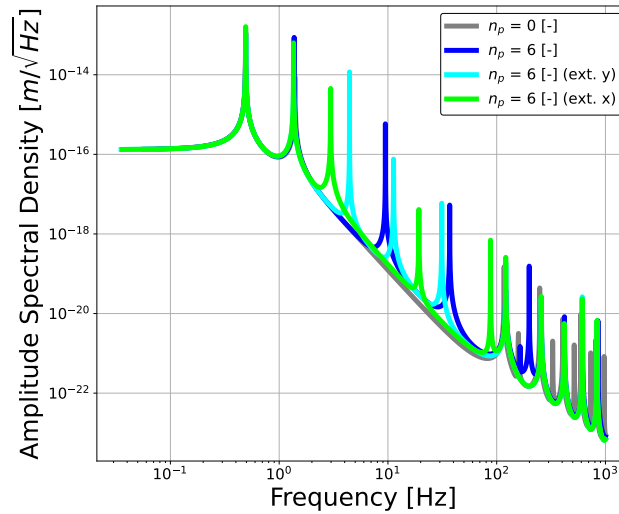


Figure 4.20: Thermal noise Response ASD - comparison for different plate size for the First Model system (in the x axis)

Figure 4.20 shows that these dimensions do not lead to a direct increase in the thermal noise response. Indeed, the thermoelastic dissipation depends only on the thickness (considering the ribbon formula). However, some resonances seem to be modified in frequency. This is explained by the change in inertia. Moreover, changing the dimensions also alters the modes of the plates. Eventually, the pendulum and violin modes are not highly modified. The small changes are due to an increase in the mass that increases the tension in the top wire, making it stiffer in flexion. Nevertheless, the change in mass is too small to exhibit important modifications.

To clearly understand the differences, the modes of the resonances are represented in Figure 4.21 & 4.22.

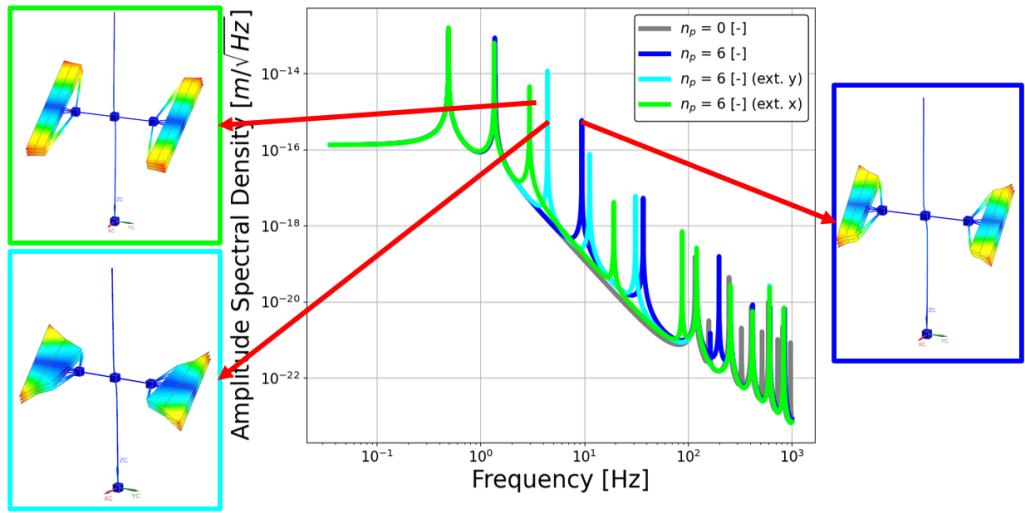


Figure 4.21: Thermal noise Response ASD - comparison for different plate size for the First Model system (in the x axis) - Corresponding First Rocking modes

Figure 4.21 highlights that, in both extended cases, the first Rocking mode appears at a lower frequency. This comes from two different reasons :

1. In the case of the extension in the y direction, the motion in the x axis is due to the rotation around the y axis. However, modifying the width does not modify the inertia around y. Increasing the width reduces the stiffness in torsion of the plates. Therefore, the first plate torsional mode is at a lower frequency (Figure 4.22), making the coupling with inertia more important. This is the same effect as reducing the stiffness of the plates (see previous section).
2. In the case of the extension in the x direction, this is the increase in inertia around the y axis that makes the resonance appear at lower frequency. Even if the stiffness of the plates in torsion is higher.

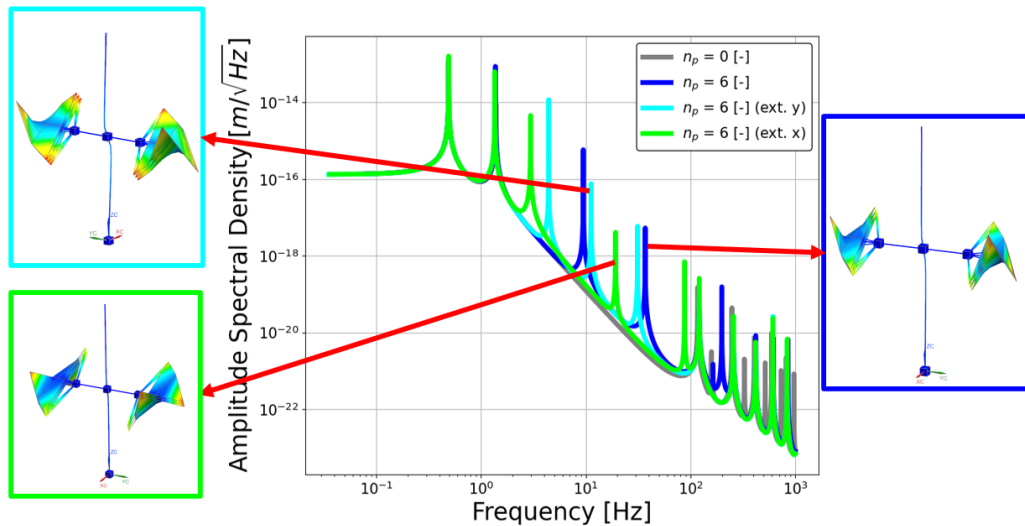


Figure 4.22: Thermal noise Response ASD - comparison for different plate size for the First Model system (in the x axis) - Corresponding First torsion modes of the plates

Figure 4.22 draws the same conclusions, as in this configuration, the torsion modes of the plates are coupled with the Rocking modes of the cold platform mass.

Eventually, there are two types of mode that are affected :

1. The Rocking mode of the cold platform mass :

When the plates are extended in the y direction, the mode is affected by the torsion of the plates which are coupled with the inertia motion. Because this torsion deformation is not the driving motion of the mode, it has less impact than just increasing the inertia. That is why the mode is even at lower frequency for the case of the x extension of the plates. Inertia is the driving parameter.

2. The torsion modes of the plates :

For these modes this is the opposite, the torsion of the plates drives the motion. Thus, in the case of the y extension, the resonance appears at a lower frequency than in the case of the extension in the x axis. The torsional stiffness is the driving parameter.

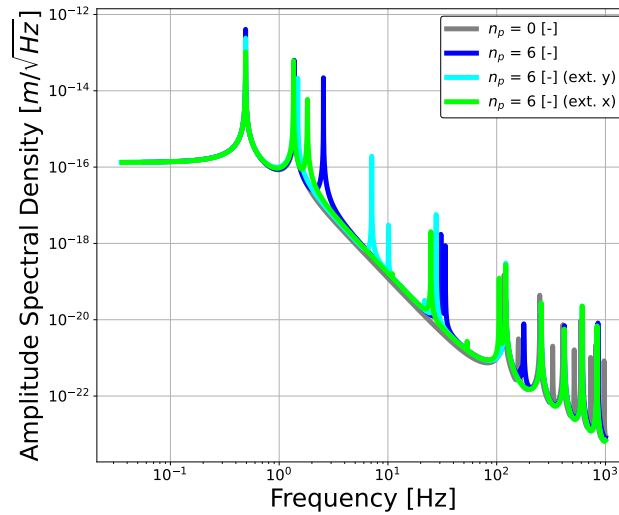


Figure 4.23: Thermal noise Response ASD - comparison for different plate size for the First Model system (in the y axis)

In the y axis, from Figure 4.23, the same conclusions can be drawn. Here, this is the rotation around the x axis that leads to a motion in the y axis. Moreover, this is now the Out-Of-Phase flexion modes of the plates (see Figure 4.8) rather than the torsion modes. Anyway, the same coupling with the inertia is present and leads to the same discussion.

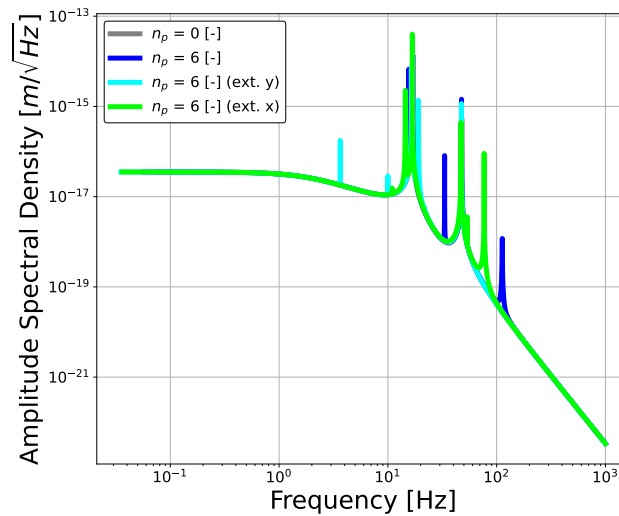


Figure 4.24: Thermal noise Response ASD - comparison for different plate size for the First Model system (in the z axis)

Figure 4.24 displays that, in the z axis, only the In-phase flexion modes of the plates are added. Thus, there is no inertia to consider here.

The only effect of modifying the width or length is to modify the resonances.

The Higher is the  $L_w$ , at lower frequencies the In-phase flexion modes of the plates will appear. This is the same observation as for increasing or decreasing the stiffness of the plates.

Eventually, the In-Phase flexion modes are coupled with the axial modes of the two masses. The result is that it increases the amplitude of the axial modes. If the plates were infinitely stiff, their resonances would not be coupled. The results can be seen more clearly in Figure 4.25.

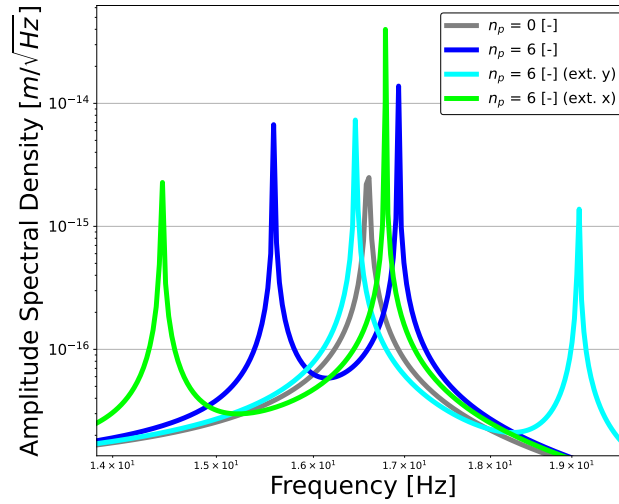


Figure 4.25: Thermal noise Response ASD - comparison for different plate size for the First Model system (in the z axis) - Zoom

#### 4.3.4.2 Thickness

The thickness plays a role in the stiffness and mass of the plates, such as width and length. However, it also impacts the Thermoelastic dissipation of the plates. The thickness, similar to the diameter of the wire, affects the distribution of the dissipation along the frequency (see Section 2.5.2.1).

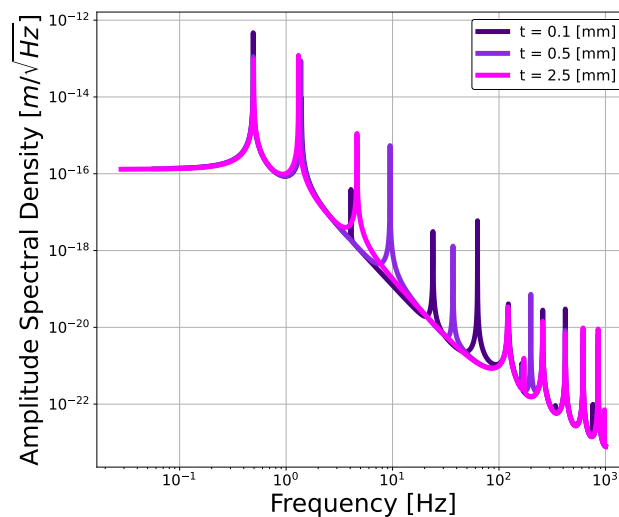


Figure 4.26: Thermal noise Response ASD - comparison for different plate thickness for the First Model system (in the x axis)

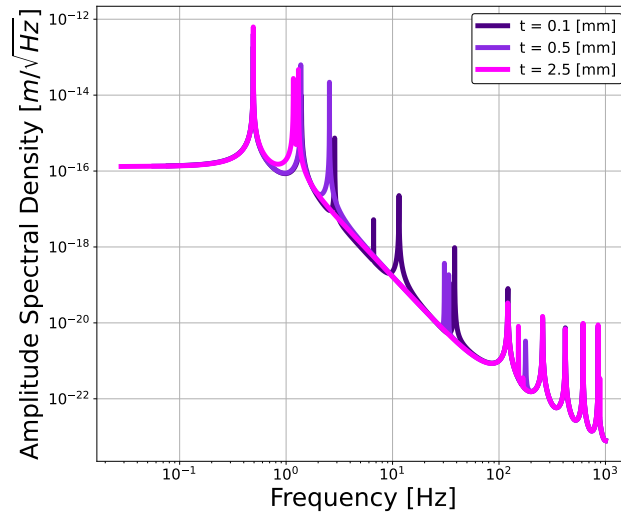


Figure 4.27: Thermal noise Response ASD - comparison for different plate thickness for the First Model system (in the y axis)

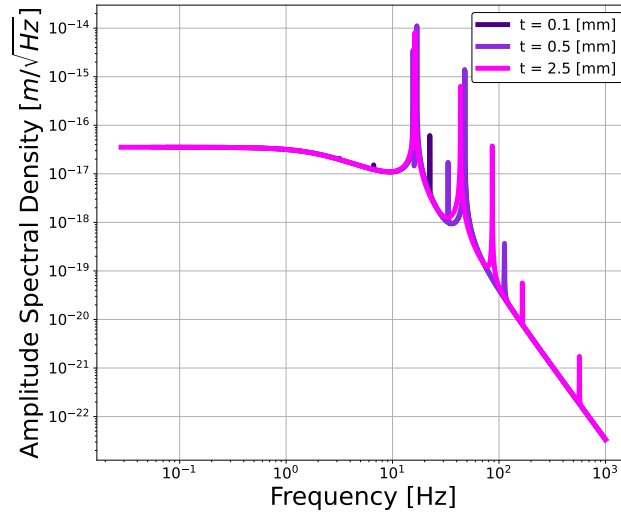


Figure 4.28: Thermal noise Response ASD - comparison for different plate thickness for the First Model system (in the z axis)

Generally, Figures 4.26, 4.27 & 4.28 highlight that the effect of changing the thickness is mainly the same as the stiffness. However, for a thickness of 0.1 mm, the mode at  $\approx 4$  Hz is no longer a rocking mode (since the added inertia is negligible in this case) but the first torsion mode of the plates. The effect of adding masses can be slightly seen at low frequency but is not significant at all.

#### 4.3.5 Conclusion

The main impacts on the TN of adding a Radiative Heat Exchanger can be summarized in three points :

1. Such an RHE is mainly adding resonances from the plates that can contribute to a higher TN depending on their internal dissipation. Moreover, the resonances can be coupled, increasing the amplitude of the TN from the existing resonances (resonances of the system without RHE).
2. Adding an RHE also adds mass to the system, which can change the pendulum and violin modes. However, depending on the added mass, the impact might be negligible.
3. Adding an RHE increases the inertia of the system, which modifies the rocking mode.

## 4.4 RHE Configuration investigation

The Radiative Heat Exchanger was considered to be horizontal in the First Model. However, many configurations can be implemented. In this section, the previous horizontal configuration will be compared to the three vertical ones. To simplify computations, only 6 plates will be considered.

### 4.4.1 Horizontal configuration

The horizontal configuration with 6 plates is represented in Figure 4.29.

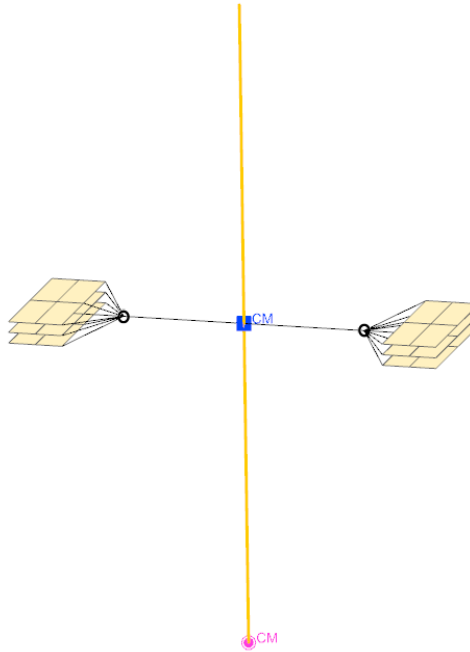


Figure 4.29: First model with an horizontal configuration of the plates ( $n_p = 6$  [-])

This configuration was chosen in the context of the E-TEST RHE design [9]. Although this configuration has been shown to be efficient for cryogenic cooling (see Figure 1.6), the question is to know if this configuration is the most adapted one (for the TN).

### 4.4.2 Vertical configuration

The horizontal configuration is perhaps not the best in terms of assembly [24]. This is the reason why vertical RHE can be a solution used in future RHE designs.

To perform an important characterization, the following three configurations will be studied (see Figure 4.30).

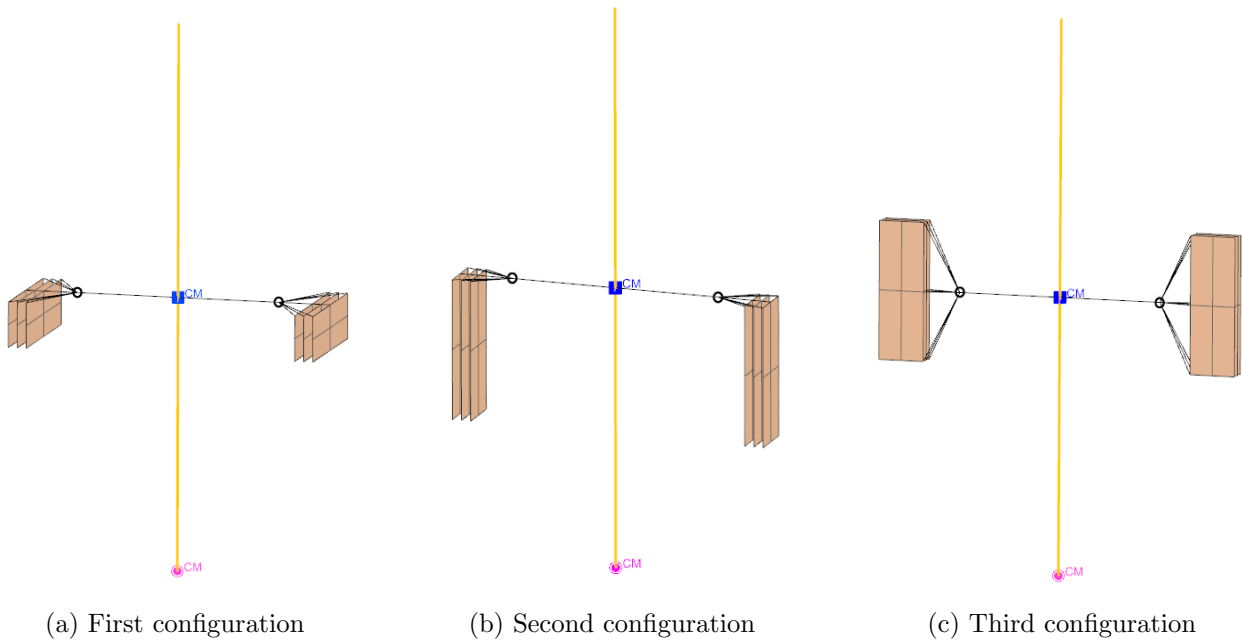


Figure 4.30: First model with vertical configurations of the plates ( $n_p = 6$  [-])

The plate sizes are the same but not oriented and fixed in the same way (the black links are the RBE connections). Many other configurations can be studied to find the optimal design. However, this is not the purpose of this manuscript, as it aims to identify and understand the effects of different configurations.

#### 4.4.3 Thermal noise comparison

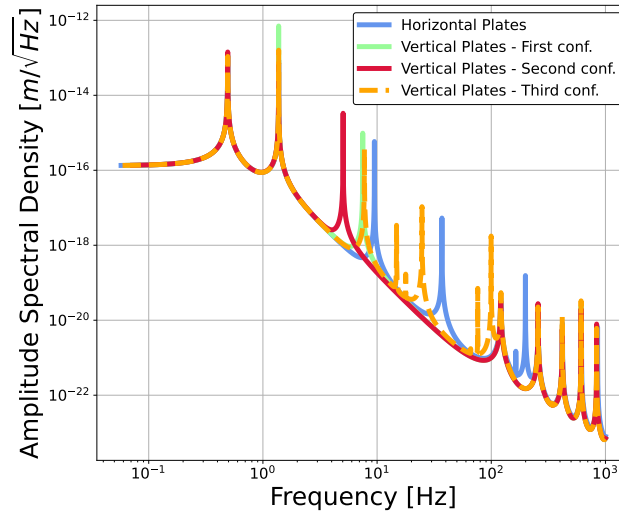


Figure 4.31: Thermal noise Response ASD - comparison for different plate configurations for the First Model system (in the x axis)

In the x axis, Figure 4.31 shows that the rocking mode is, as expected, slightly modified for the different configurations. The torsional mode of the Horizontal plates ( $\approx 3.5$  Hz) seems to disappear for the two first vertical configurations. The torsion of vertical plates will produce a rotation around the z axis rather than around the y axis for the horizontal plates. Then, all the torsional modes that were present in the horizontal configuration are transmitted to another DoF. Moreover, it is important to remember that a rotation around the y axis was producing a motion in the x direction. Rotating around z will no longer lead to a motion in translation axis.

For the third configuration, torsional modes are still present but in addition to the In-Phase Flexion modes of the plates (an example of the mode is represented in Figure 4.32).

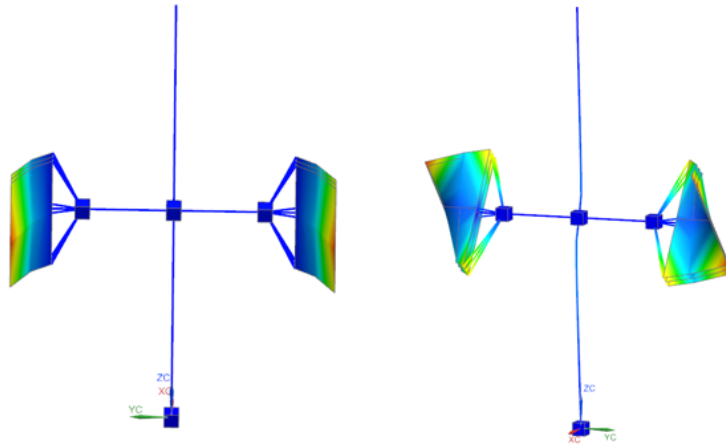


Figure 4.32: First (resp.) In-phase Flexion and Torsion modes of the First Model in the third vertical configuration

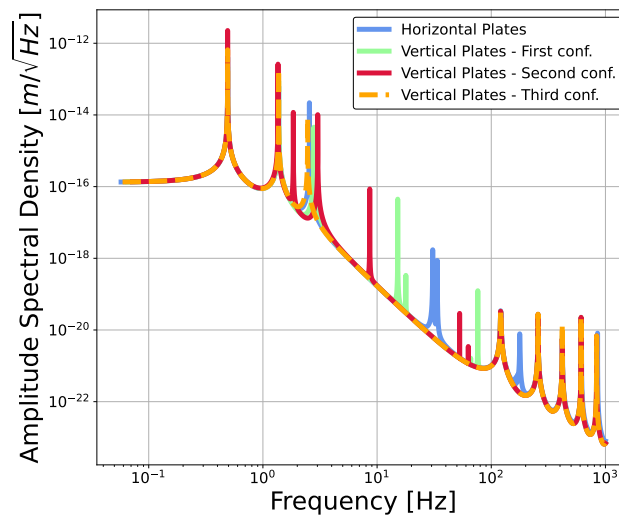


Figure 4.33: Thermal noise Response ASD - comparison for different plate configurations for the First Model system (in the y axis)

From Figure 4.33, the same conclusion can be reached for the y axis. However, this time, for the two first vertical configurations, this is the In-Phase flexion modes that produce a motion in the y axis (Figure 4.34). This motion is directly in the axis, resulting in higher amplitudes. For the horizontal configuration, it was the Out-of-Phase flexion modes that were producing a motion around the x axis resulting in a motion in the y axis.

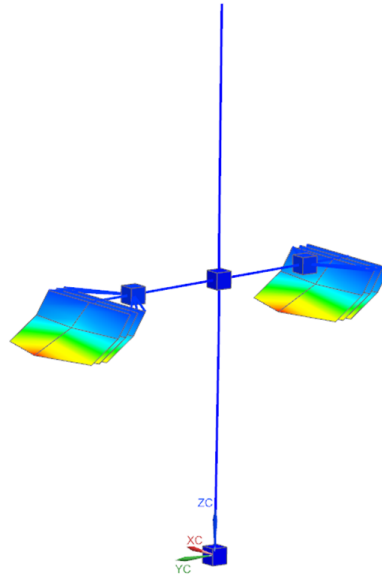


Figure 4.34: First In-phase Flexion mode of the First Model in the first vertical configuration

For the third configuration, as expected, no plate modes are contributing in the y axis (in this frequency range).

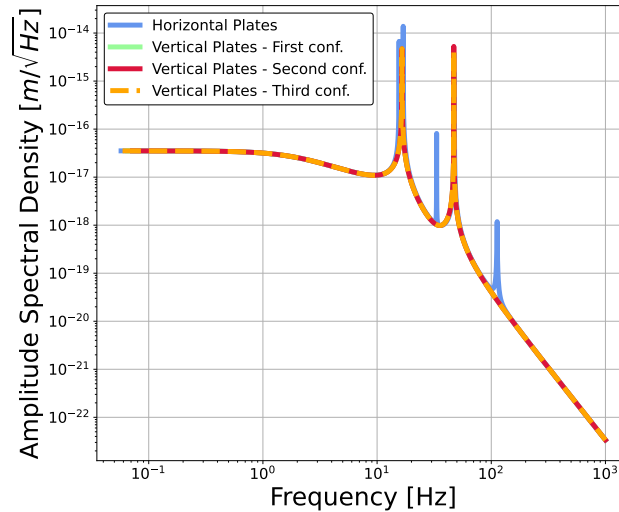


Figure 4.35: Thermal noise Response ASD - comparison for different plate configurations for the First Model system (in the z axis)

Eventually, for the z axis, Figure 4.35 displays that, for vertical configurations, the plate modes do not contribute to the z motion (in this frequency range). This result is quite logical and was expected.

#### 4.4.4 Conclusion

Configurations can highly solve some TN concerns in designs. As seen, changing the configuration might increase or decrease the stiffness of some modes.

If the fins are attached by their big edge, the torsional stiffness is low while the flexural stiffness is high. This is the opposite if the fins are attached by their small edge.

Eventually, vertical configurations shown to be efficient to minimize the impact of some modes from the panels. Changing the orientation of the plates can transfer the effect of a fin mode to another axis. Some modes might also be cancelled by the disposition. The in-phase flexion mode of the plates that occur for the horizontal configuration will not induce any displacement in two first vertical

configurations. For these configurations it is the out-of-phase version of the mode in Figure 4.34.

## 4.5 FDT adapted to Temperature Distribution

One limitation of the Fluctuation and Dissipation Theorem is to consider that the full system is in thermal equilibrium. However, this is not necessarily true, especially dealing with thermal gradients such as for ET-LF concept [22].

To consider it, the computations can be modified using the Adapted Fluctuation Dissipation Theorem (AFDT, called in this manuscript) presented in [21]. In their paper, the PSD of the TN response is computed using this following formula :

$$\mathbf{S}_x(\omega) = \frac{2k_B}{\omega^2} \sum_l T_l \mathbf{Z}^{-1}(\omega) \cdot \left( \mathbf{Z}_l(\omega) + \mathbf{Z}_l^\dagger(\omega) \right) \cdot \mathbf{Z}^{-1\dagger}(\omega) \quad (4.2)$$

Where  $T_l$  is the temperature of the  $l^{th}$  element. The *dagger* sign  $^\dagger$  refers to the complex conjugate of the transposed matrix.

This formulation of the FDT allows to consider, in the computation of the TN, the distribution of the temperature in the system, which was the main limitation of the theorem.

Rewriting Equation (4.2) from [21], the following expression can be found :

$$\mathbf{S}_x(\omega) = \frac{4k_B}{\omega^2} \mathbf{Z}^{-1}(\omega) \cdot \text{Re} \{ \mathbf{Z}^*(\omega) \} \cdot \mathbf{Z}^{-1\dagger}(\omega) \quad (4.3)$$

Where  $\mathbf{Z}^*(\omega) = \mathcal{T} \odot \mathbf{Z}(\omega)$  (Symbol " $\odot$ " corresponding to a multiplication element by element of the matrices).

$\mathcal{T}$  is defined in this master's thesis as the Global Temperature Matrix (GTM) such that :

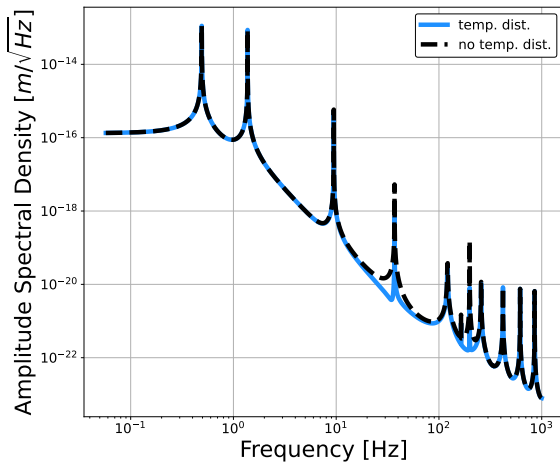
$$\mathcal{T} = \begin{pmatrix} \mathbf{T}_1 + \mathbf{T}_2 & \mathbf{T}_2 & 0 & \cdots \\ \mathbf{T}_2 & \mathbf{T}_2 + \mathbf{T}_3 & \mathbf{T}_3 & \ddots \\ 0 & \mathbf{T}_3 & \ddots & \\ \vdots & \ddots & & \end{pmatrix}$$

Where  $\mathbf{T}_i$  is the matrix temperature of the  $i^{th}$  element.

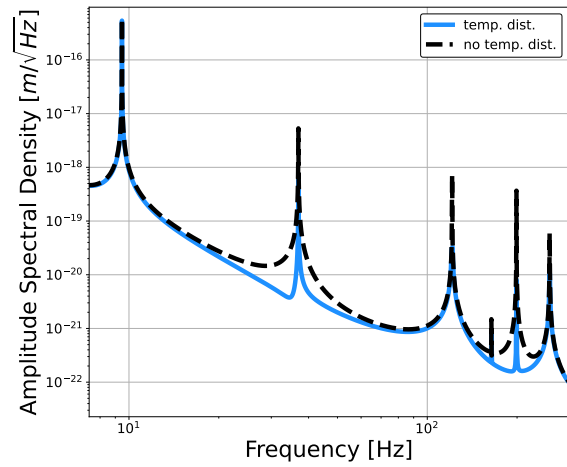
Since the integration by post-processing the matrix is quite hard, no temperature distribution will be considered. Thus, considering the wires and the plates, there are 3 different temperatures that can be defined.

The First model with 6 panels can be computed, considering that the two wires are at 20 K and, in order to see the impact, the plates are at 1 K. Following the sensitivity analysis, the expected results are to see a diminution of the TN where the plates are dissipating energy, i.e at their resonances.

Note that the material properties did not change with the temperature for the dissipation of the plates (as they were assumed).

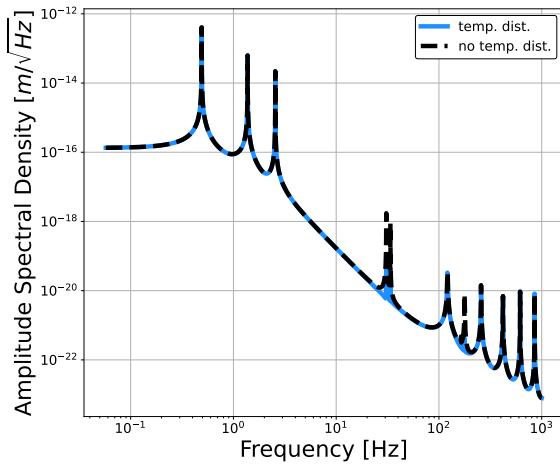


(a) Global ASD

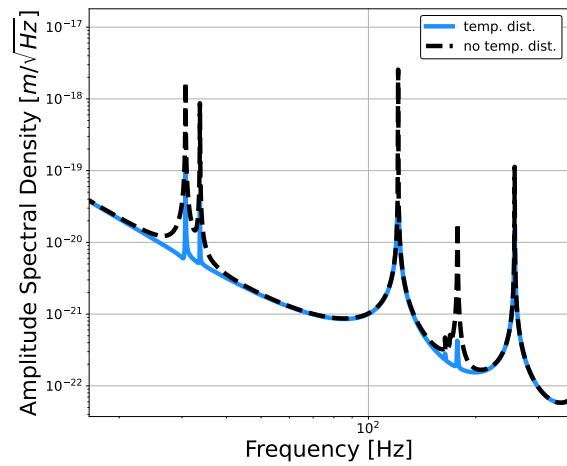


(b) Zoom on the resonances

Figure 4.36: Thermal noise Response ASD - comparison between the FDT (every element at 20 K) and ADFT (wires at 20 K and panels at 1 K) for the First Model system (in the x axis)



(a) Global ASD



(b) Zoom on the resonances

Figure 4.37: Thermal noise Response ASD - comparison between the FDT (every element at 20 K) and ADFT (wires at 20 K and panels at 1 K) for the First Model system (in the y axis)

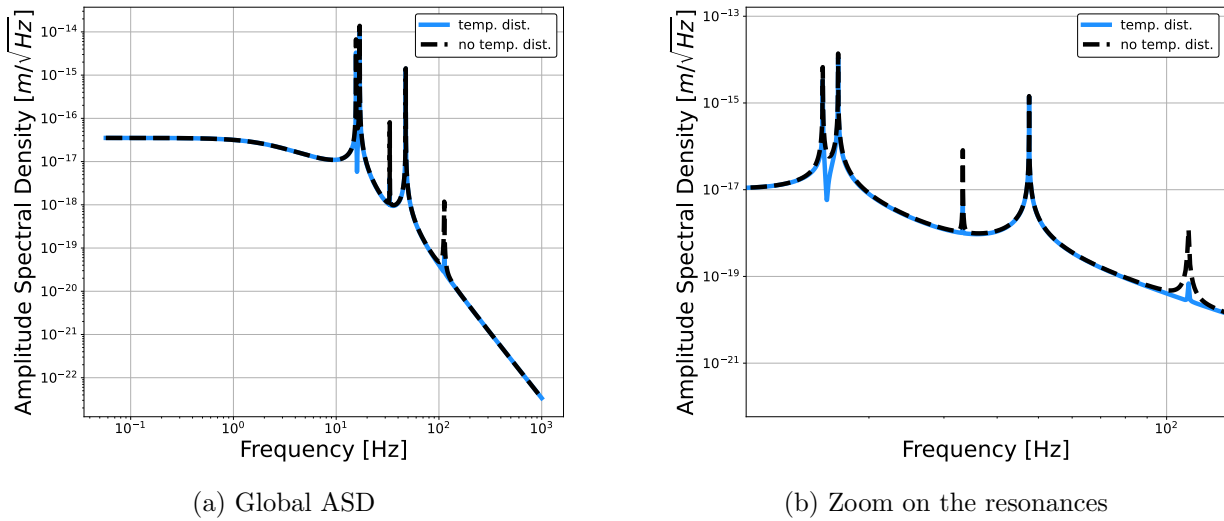


Figure 4.38: Thermal noise Response ASD - comparison between the FDT (every element at 20 K) and ADFT (wires at 20 K and panels at 1 K) for the First Model system (in the z axis)

Figures 4.36, 4.37, 4.38 highlight that, as expected, the temperature only impacts the resonances of the panels.

Moreover, to prove that the method is correct, it can be compared to the basic case of a uniform temperature of the system. The method gives the same results (see Appendix 8.5.3).

Although this method enables to compute more complex cases in terms of temperature distribution, it has to be noted that it requires more computation. Indeed, to get the transfer matrix, an inversion with an LU factorization has to be performed. For the basic FDT, generally, only few elements of the matrix is needed (the desired DoFs), and thus computation can be simplified. Nevertheless, in Equation (4.3), the full inverse matrix should be known, leading to more computation.

An interesting point of using FEM and the ADFT is to perform a temperature distribution in the wires. Indeed, when considering analytical formulas, the output from the computation of the stiffness of the wires was a single value. The wires were considered as unique elements. Using FEM, the wires can be discretized, leading to the application of a temperature distribution to the discretization. The higher will be the number of element, the finer will be the description of the temperature distribution.

# Chapter 5

## E-TEST model

### 5.1 Model description

The E-TEST concept realised in this manuscript is a simplified version based on actual measurements of the prototype [9] that is represented in Figure 5.1.

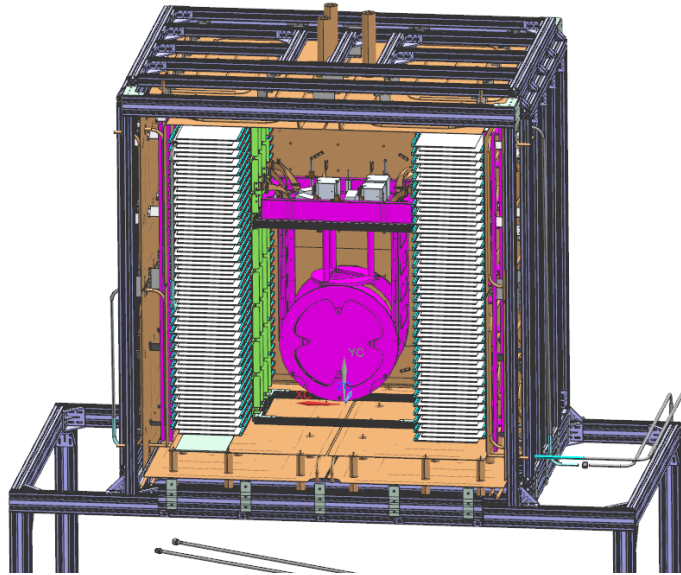


Figure 5.1: E-TEST prototype - Radiative Heat Exchanger CAD Model

To reproduce this model in a simplified way, a double pendulum system is considered with the inner RHE that is attached to the Cold Platform (CP). The FEM model is represented in Figure 5.2.

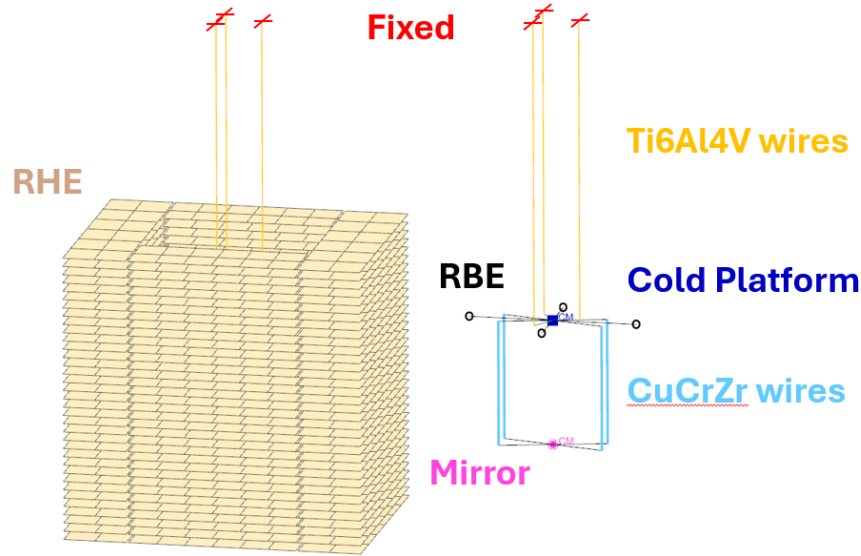


Figure 5.2: E-TEST simplified FEM model [9]

In this simplified model, the RHE is assumed to be rigidly fixed to the Cold Platform (thanks to Rigid Body Elements). The masses and wires properties can be found in Tables 5.1 & 5.2

|                        | Mirror | Cold Platform |
|------------------------|--------|---------------|
| $m$ [kg]               | 100    | 79            |
| $I_{xx}$ [ $kg\ m^2$ ] | 2.531  | 2.029         |
| $I_{yy}$ [ $kg\ m^2$ ] | 1.734  | 0.706         |
| $I_{zz}$ [ $kg\ m^2$ ] | 1.734  | 2.689         |

Table 5.1: Masses properties of the E-TEST model [9]

| Wires | $L$ [m] | $d$ [mm] | Materials |
|-------|---------|----------|-----------|
| F-CP  | 1.46    | 2        | Ti6Al4V   |
| CP-MI | 0.6     | 9        | CuCrZr    |

Table 5.2: Wires properties of the E-TEST model (F : Fixed - CP : Cold Platform - MI : Mirror) [9]

This model is quite the same as the First Model presented previously but with more details. The same kind of behaviour is expected thus.

The wires are composed of 8 beam elements each and the plates are made of CQUAD4 elements from the NASTRAN solver (3 nodes for the small edges and 7 for the bigger ones). The model has a total of 5346 Degree Of Freedom .

The cage is not modelled in this study (considered rigid); therefore, neither the added mass nor the added inertia is considered.

Eventually, in this case, in addition to the thermoelastic dissipation, the bulk and surface losses will be considered.

### 5.1.1 Plates descriptions

In this case there are two types of panels : The bigger ones called 'huge plates' and the smaller ones called 'small plates'. Their dimensions are illustrated in Figure 5.3.

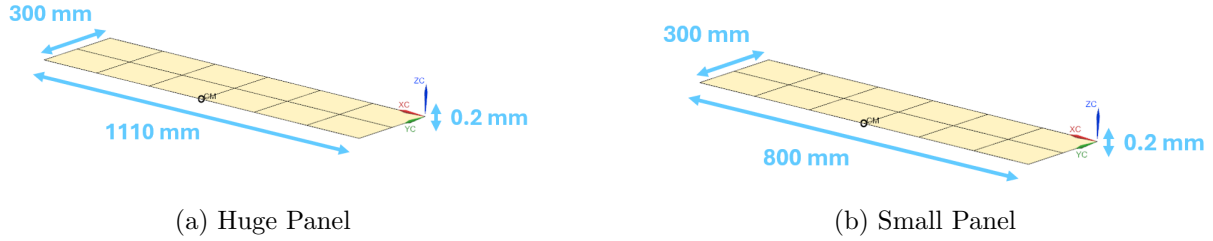


Figure 5.3: Panels of the E-TEST model (dimensions) [9]

The plates are reinforced (see Figure 5.4) and not simply 2D elements. In order to get a model that describes it, a fin model which fits with experimental data will be build.

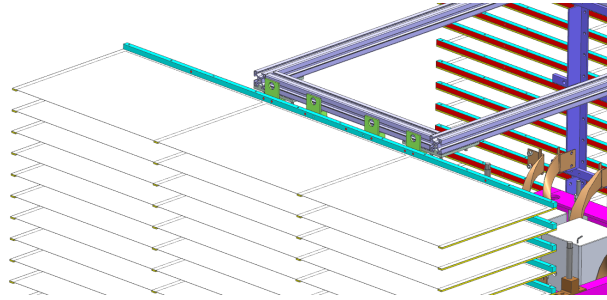


Figure 5.4: Reinforced plates of E-TEST [9]

Eventually, the plates are considered to be made of Al1050 with the following masses :

- Huge plate : 0.721 kg
- Small plate : 0.543 kg

#### 5.1.1.1 Experimental Data

In [24], an experimental analysis of the plates was performed. This analysis determined the resonance frequencies of a huge plate that can be seen in Table 5.3.

| Mode | Res. Freq. [Hz] |
|------|-----------------|
| 1    | 10.8            |
| 2    | 12.8            |
| 3    | 16.6            |
| 2    | 18.8            |

Table 5.3: Measured resonance frequencies of a huge plate for the E-TEST model [24]

#### 5.1.1.2 Adapted FEM model

The resonances for the huge plates can be matched. The reinforcements are assumed to be fully rigid (see Figure 5.6). This is a quite strong assumption but still acceptable to catch the first resonances. Moreover, a small mass is added at the centre of the panel to take into account the non-uniform distance between the reinforcements in the experimental plates (see Figure 5.5).

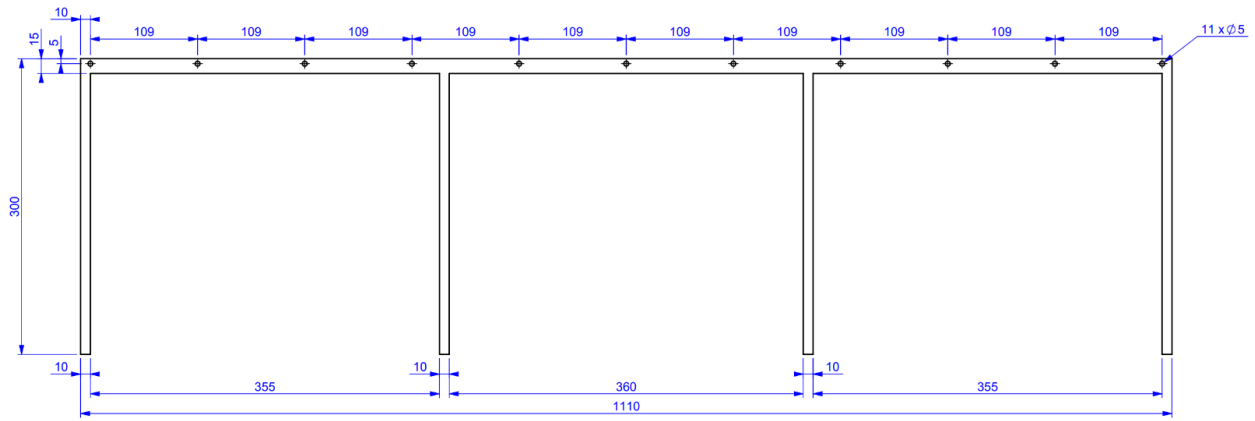


Figure 5.5: Huge Plates Dimensions with the reinforcements (Technical drawing) [9]

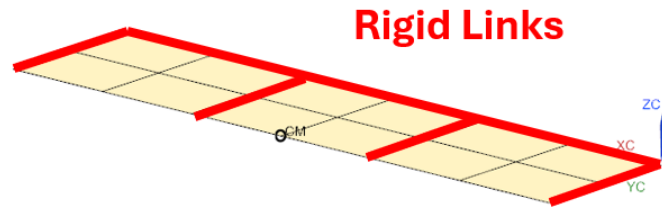


Figure 5.6: Rigid links applied to the panels

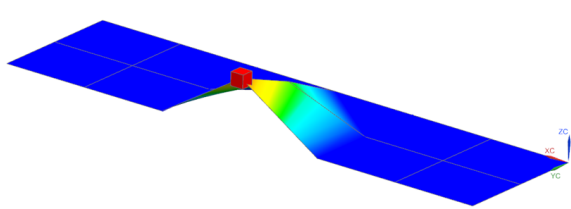
It should be noted that the goal of this adapted FEM model is to reach the resonances (in the same fixed configuration) of the panels with the less elements in the mesh. Indeed, since the computation has to be performed at each frequency, the computation can become important for models with a lot of DoFs. That is why the plates are represented with few elements.

Using the following properties for the plate :

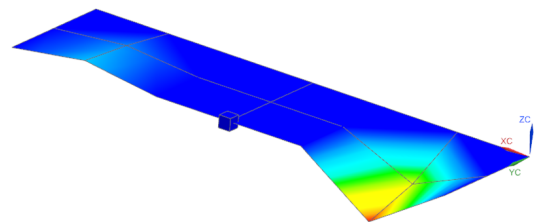
- Effective thickness : 0.29 [mm] ( $> 2$  mm, since there is a painting on the experimental panel)
- Young Modulus : 70.5 [GPa]
- Poisson Ratio : 0.33 [-]
- Density : 7258.98 [ $kg/m^3$ ]
- Mass of the point mass : 0.02 [kg]

The first two resonances can be caught (see Figure 5.7)

Note that the comparison with [24] is made for the fixed rear edge.



(a) Mode 1 : 10.81 [Hz]



(b) Mode 2 : 12.79 [Hz]

Figure 5.7: Two first modes of the adapted FEM model of the Huge plate

This corresponds to the FEM model developed in [24].

The purpose is just to have a rough value of the resonances of the plates to see their impact on the thermal noise. To perform a more precise analysis, the modes should be measured experimentally and a MAC matrix should be performed to validate the exactness of the numerical model.

This model is not precise at all and this is not the objective. The goal is to show that the thermal noise determination method can be applied to any complex system.

Eventually, as no experimental data exist for the small plates, the exact same properties were chosen except for the density, which is equal to  $5415.76 \text{ [kg/m}^3\text{]}$  to follow the defined mass.

### 5.1.2 Material Properties

The material properties used in the model can be defined. For the Ti6Al4V and the CuCrZr, they are the ones used for the mechanical model and for the dissipation computation. For the aluminium, the mechanical properties of the plates were presented in the previous section. For the dissipation, the properties of the Al6061 were chosen due to a lack of data. For the TE dissipation of the plates, a thickness of 0.2 mm was chosen.

Properties can be found in Table 5.4.

| Materials | $\rho \text{ [kg/m}^3\text{]}$ | $E \text{ [GPa]}$ | $\nu \text{ [-]}$ | $\lambda \text{ [W/m/K]}$ | $c_p \text{ [J/kg/K]}$ | $\alpha \text{ [K}^{-1}\text{]}$ | $\beta \text{ [K}^{-1}\text{]}$ | $\phi_{bulk} \text{ [-]}$ | $\alpha_{surf} \text{ [m]}$ |
|-----------|--------------------------------|-------------------|-------------------|---------------------------|------------------------|----------------------------------|---------------------------------|---------------------------|-----------------------------|
| Ti6Al4V   | 4540 [22]                      | 122.62 [8]        | 0.3               | 0.843 [28]                | 9.68 [28]              | 6.2e-6 [37]                      | -5.22e-06 [8]                   | 1e-4 [3]                  | 0 [22]                      |
| Al6061    | 2700                           | 77.8 [30]         | 0.3               | 28.43 [28]                | 8.85 [28]              | 3.5e-7 [31]                      | -4.43e-06 [30]                  | 9.1e-07 [20]              | 0 [22]                      |
| CuCrZr    | 8900                           | 130               | 0.3               | 300                       | 2                      | 3e-6                             | 0                               | 1e-4                      | 0                           |

Table 5.4: Material properties of the E-TEST model (values taken at 20 K)

The distribution over the temperature of the properties can be found in Appendix 2.5.2.1 (Figures 8.12, 8.13, 8.14, 8.15, 8.16 & 8.17).

Note that :

- the *poisson* ratio  $\nu$  is considered to be 0.3 [-] for all materials following the suggestion of [11].
- $\alpha_{surf}$  of the Al6061 is considered to be the same as to one of the Al5056 referenced in [22].
- the density of the Al6061 is a common value that was chosen.
- the properties of the CuCrZr were totally arbitrary chosen due to a lack of data. They are still representative of the material with the right order of magnitude. Again, this is a limitation but is not that important since the goal is not to be fully precise but to show that from such a complex system, the thermal noise response of the mirror can be computed.

## 5.2 Results

The top wire is considered fixed at the top extremity and the whole system is assumed to be at 20 K. This is realistic following the E-TEST measurements (see Figure 1.6).

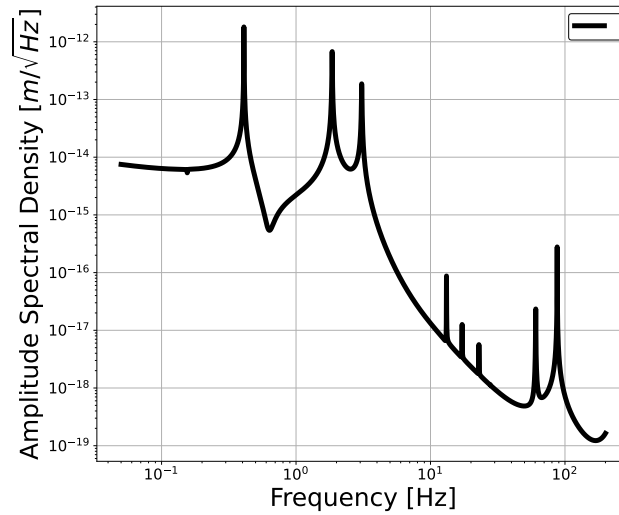


Figure 5.8: Thermal noise Response ASD - E-TEST model (in the x axis)

Figure 5.8 exhibit the same shape as the First Model. The two pendulum modes can be seen at 0.4 and 1.8 Hz, respectively, and are followed by the rocking mode of the cold platform. The torsion modes of the huge plates create a motion in the x direction (by a rotation around y). However, there are also the small fins. Due to their location, they will also introduce resonances in the x axis through their out-of-phase flexion modes. The first torsion and out-of-phase flexion modes of the plates are represented in Figure 5.9.

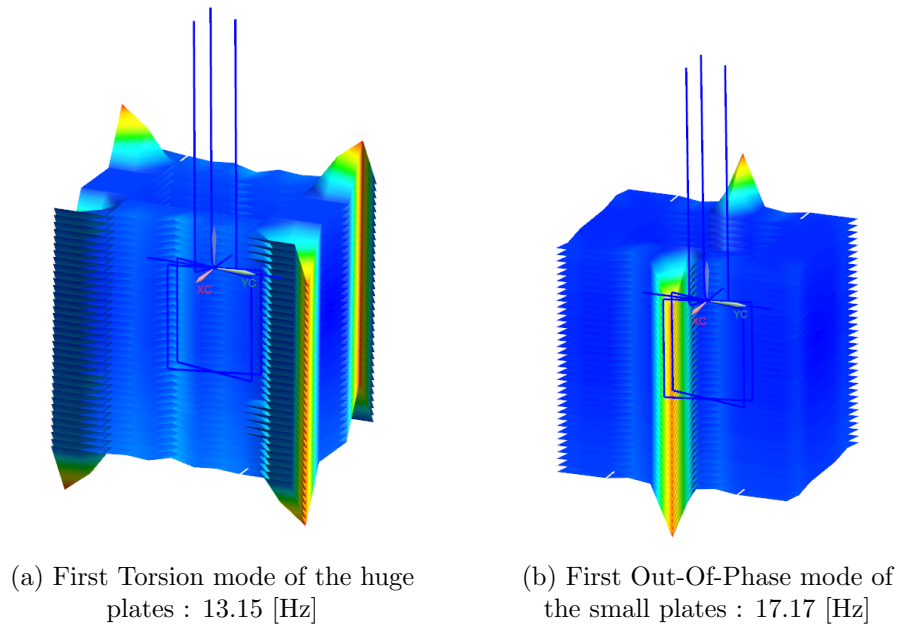


Figure 5.9: First Torsion and Out-Of-Phase flexion modes of resp. the huge and small panels for the E-TEST model

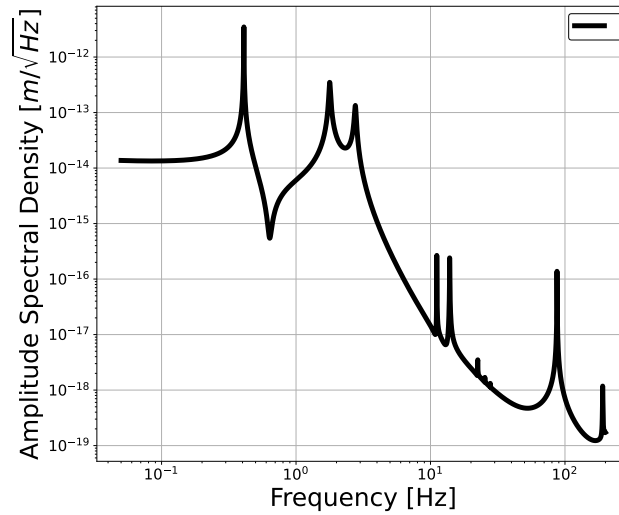


Figure 5.10: Thermal noise Response ASD - E-TEST model (in the y axis)

For the y axis, from Figure 5.10, the same conclusions can be drawn. There is additional resonance from the torsion modes of the small fins (rotation around the x axis).

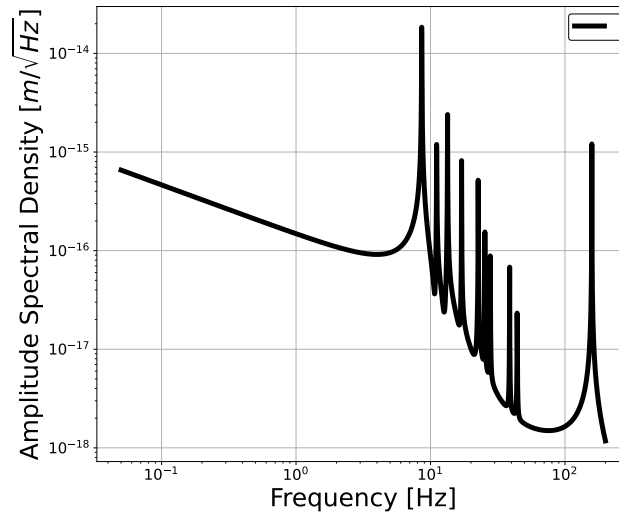


Figure 5.11: Thermal noise Response ASD - E-TEST model (in the z axis)

In the z-axis, Figure 5.11 shows that the in-phase flexion modes of the plates appear. In this case, the first in-phase flexion mode of the huge plate is no more 10.80 Hz, but 11.11 Hz since the edge is no longer fixed. Moreover, the modes of the fins are near the axial ones (at 8.6 Hz and 158.4 Hz) meaning that the axial modes (between the Cold Platform and the Mirror in the z axis) are coupled with the fin modes. The coupling can be observed in Figure 5.12.

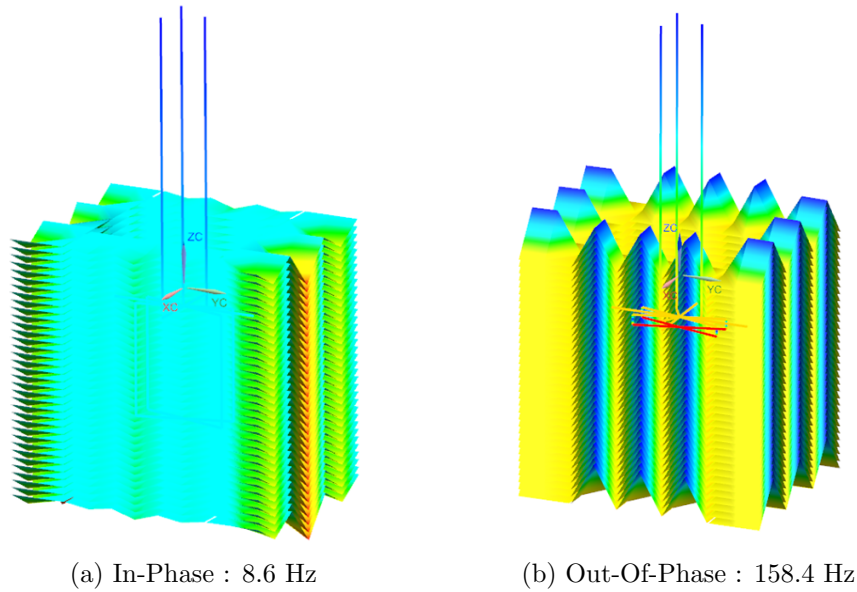


Figure 5.12: Axial modes of the E-TEST model

Eventually, at low frequency, the TN response in Figure 5.11 seems to increase. This effect comes from the constant high loss angles (bulk and surface dissipation). Indeed, as discussed in Section 2.5.2, considering a loss angle which is constant leads to an infinite TN PSD at low frequency. The curves computed are probably not exact, since the material properties of the CuCrZr are assumed, but prove that such a system can be studied by the FEM.

It should be noted that the cage to fix the plates is not represented. Again, this is a simplified model and it is expected that it will not completely change the results. Indeed, if the cage is considered to be a rigid body, only the mass and inertia of the Cold Platform will change. This modification will mainly affect the rocking modes (that can be coupled with the modes of the plates).

### 5.3 Conclusion on the E-TEST model

The E-TEST model shows that the FEM method can be applied on complex systems. The definition of the model was based on actual measurements and can be used to compute the TN noise response of the mirror in such a context. Moreover, the model enabled to highlight the behaviour of this huge RHE, compared to the simple model presented in the previous section. Eventually, the study shows the importance of experimental analysis correlation, since the modes of the fins increase locally the thermal noise. The modes need to be well known as they might have a significant TN impact in the frequency range of sensibility.

## Chapter 6

### ET-LF model

Now that complex systems can be constructed, the ET-LF payload can be studied. The baseline design is represented in Figure 6.1.

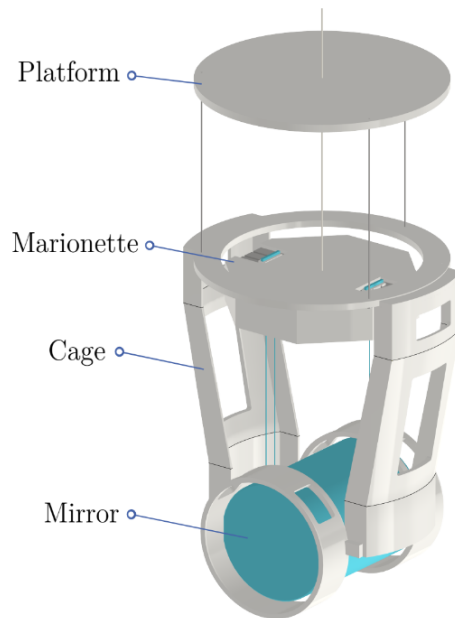


Figure 6.1: Baseline design of the ET-LF payload based on AdVirgo's double pendulum design [11]

The model consists of a triple pendulum system with, in the order :

- The Platform (PF)
- The Marionette (MA)
- The Mirror (MI)

Where their properties can be found in Table 6.1.

|                        | Mirror | Marionette | Platform |
|------------------------|--------|------------|----------|
| m [kg]                 | 211    | 211 + 200  | 300      |
| $I_{xx}$ [ $kg\ m^2$ ] | 5.3    | 6.57       | 18       |
| $I_{yy}$ [ $kg\ m^2$ ] | 4.15   | 6.57       | 18       |
| $I_{zz}$ [ $kg\ m^2$ ] | 4.15   | 12.9       | 34       |

Table 6.1: Masses properties of the ET-LF model [11]

The added 200 kg of the Marionette comes from the Cage that is not modelled in this manuscript. In this study, only the mass is added but not the inertia of the Cage. These masses are suspended by wires of different shapes and materials that can be obtained in Table 6.2. The wires disposition is the same as defined in [11].

| Wires              | $L$ [m] | $d$ [mm] | $n$ [-] | Materials |
|--------------------|---------|----------|---------|-----------|
| F <sup>1</sup> -PF | 2.5     | 3.9      | 1       | Ti6Al4V   |
| PF-MA              | 1.1     | 8.1      | 1       | Silicon   |
| MA-MI              | 1.2     | 3        | 4       | Silicon   |

Table 6.2: Wires properties of the ET-LF model [11]

In this study, only the case of wires made of Silicon will be studied, but it should be noted that a configuration with Sapphire wires exists.

The properties of the materials can be found in Table 6.3.

| Materials | $\rho$ [kg/m <sup>3</sup> ] | $E$ [GPa]  | $\nu$ [-] | $\lambda$ [W/m/K] | $c_p$ [J/kg/K] | $\alpha$ [K <sup>-1</sup> ] | $\beta$ [K <sup>-1</sup> ] | $\phi_{bulk}$ [-] | $\alpha_{surf}$ [m] |
|-----------|-----------------------------|------------|-----------|-------------------|----------------|-----------------------------|----------------------------|-------------------|---------------------|
| Ti6Al4V   | 4540 [22]                   | 122.62 [8] | 0.3       | 0.843 [28]        | 9.68 [28]      | 6.2e-6 [37]                 | -5.22e-06 [8]              | 1e-4 [3]          | 0 [22]              |
| Al6061    | 2700                        | 77.8 [30]  | 0.3       | 28.43 [28]        | 8.85 [28]      | 3.5e-7 [31]                 | -4.43e-06 [30]             | 9.1e-07 [20]      | 0 [22]              |
| Silicon   | 2330 [22]                   | 130 [22]   | 0.3       | 4940 [22]         | 3.4 [22]       | -2.97e-9 [22]               | -7.9e-6 [22]               | 1e-9 [22]         | 5e-13 [22]          |

Table 6.3: Material properties of the ET-LF model (values taken at 20 K)

Note that :

- the *poisson* ratio  $\nu$  is considered to be 0.3 [-] for all materials following the suggestion of [11].
- $\alpha_{surf}$  of the Al6061 is considered to be the same as to one of the Al5056 referenced in [22].
- the density of the Al6061 is a common value that was chosen.

The Al6061 will be used later in the manuscript.

## 6.1 Basic model

Considering first the basic model with only the triple pendulum. The model realised is represented in Figure 6.2, where the axes used are also defined.

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<sup>1</sup>The 'F' means Fixed.

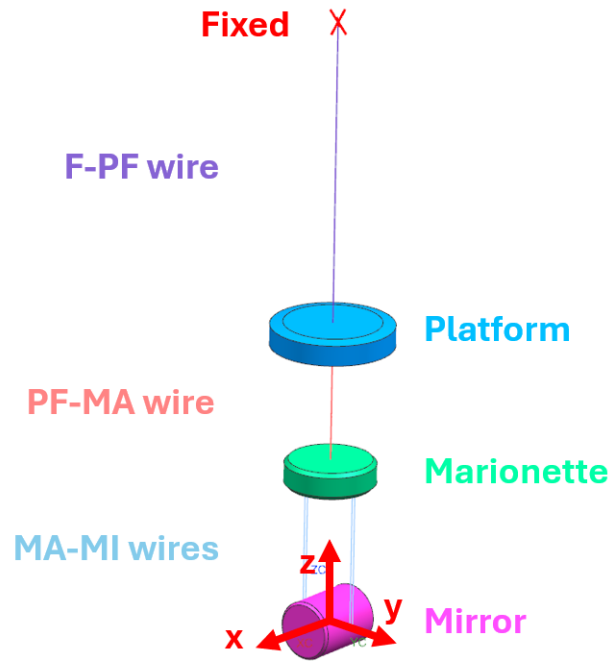


Figure 6.2: ET-LF Basic model scheme

The wires are still beam elements and the number of elements per wire are :

- F-PF : 20 beam elements
- PF-MA : 10 beam elements
- MA-MI : 10 beam elements

The model has a total of 402 Degree Of Freedom .

This model will be used, as the name suggests, as a basis for comparing the next models.

## 6.2 ET-LF with Heat Link

One way to cool down the mirror is to rely on conductive cooling using Heat Links, as explained in Section 1.3.

In this study, a model with 1 Heat Link connected to the platform is considered. The other extremity of the wire is considered to be fixed. The FE model developed is represented in Figure 6.3.

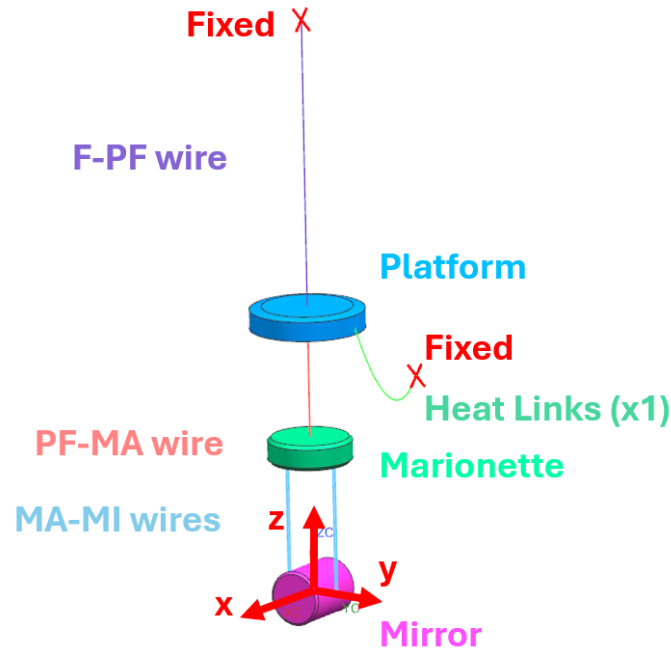


Figure 6.3: ET-LF Heat Links model scheme

The Heat link is made up of 20 beam elements and is considered to have the following properties [27] :

- $E = 69$  [GPa]
- $\nu = 0.3$  [-] (*Poisson* coefficient)
- $\rho = 2700$  [kg/m]
- $d = 2$  [mm]
- $L = 1$  [m]
- $\phi = 0.5$  [-] (Total loss) [22]

The loss angle is chosen to be high in a same way than in [22]. This choice comes from an estimation of the strong dissipation that should be induced by the friction in the HL, considering it as twisted wires.

The model has a total of 516 Degree Of Freedom .

### 6.3 ET-LF with Radiative Heat Exchanger (RHE)

Another way to cool down the mirror is to rely on radiative cooling. This method has been proved to reach the desired temperature in the case of the E-TEST prototype (see Figure 1.6). The interest of using such a cooling strategy is to avoid vibrations induction from the cooling system, to the mirror. The RHE is vertical and fixed with RBEs to the platform. This concept comes from the study [24], which developed a way to adapt the working RHE of E-TEST to the ET-LF cryogenic payload. The FE model developed is represented in Figure 6.4.

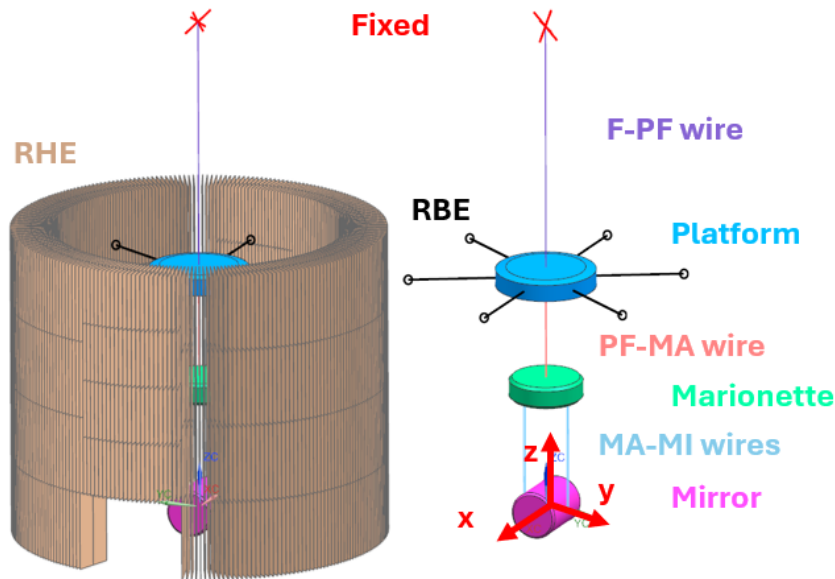


Figure 6.4: ET-LF RHE model scheme

The concept used in this manuscript, follows the idea of [24] with adaptations. Similarly to E-TEST, there are two types of fins :

- The huge fins with a length of 3 m
- The small fins with a length of 2.25 m

Both types are made of Al6061 and have a width of 0.4 m and a thickness of 1 mm.

The RHE is made up of 208 huge panels and 46 small panels. It leads to a total effective area of  $582 \text{ m}^2$ , which should be enough from a thermal point of view [24]. The total mass of the RHE is 785.7 kg.

It has to be remembered that this is a concept that does not consider any masses for the system that will attach the RHE to the Platform, that is why a thickness of 1 mm has been considered. In future work on the fins, they might not be flat but composed of a structural surface. For many reasons such as to increase the stiffness (or reduce the mass) but also to increase the emissivity of the plates [18]. In this study, the RHE mass was expected to not be higher than the initial total mass of the system. Eventually, the plates are placed at a distance of 1.4 m from the centre of the Platform.

Considering the FEM mesh, the panels are still 2D elements (CQUAD4) with 3 nodes on their smaller edges and 5 nodes on their bigger ones. They are considered to be reinforced, still with rigid elements. The rigid edges are the two small ones and the internal big edge. This assumption has an impact on the resonances of the panels and might be modified in future work.

The model has a total of 9546 Degree Of Freedom .

Note that considering RBEs connections between the PF and the RHE might be too optimistic. This effect will be studied in Section 6.4.5.

## 6.4 Results

### 6.4.1 Comparison of the cooling strategies

First, the TN response of the mirror for the three cases, defined in the previous sections, will be compared. The basic model will be used as a reference. In this comparison, all systems will be considered at 20 K.

The TN response of the mirror in the x axis (axis of the laser beam) will be compared to the ET-D-LF sensitivity curve [16] (see Figure 1.10 and Section 1.6.5).

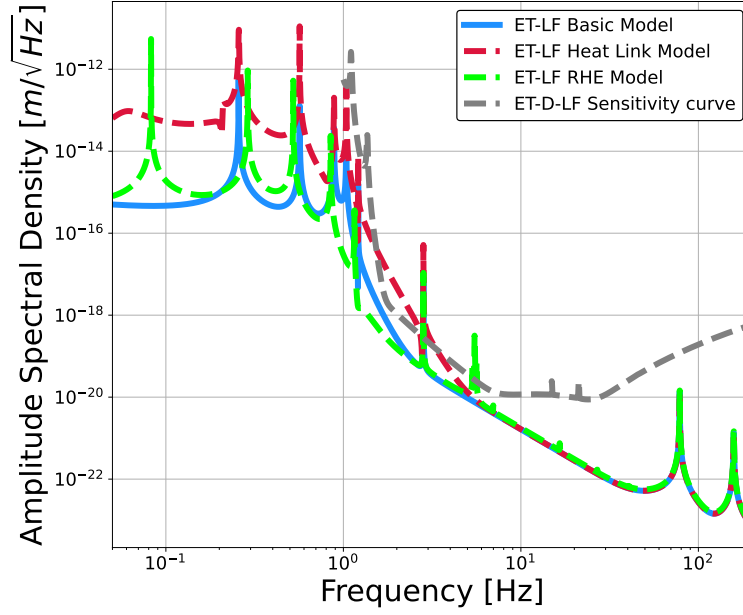


Figure 6.5: Thermal noise Response ASD - Comparison of the cooling strategies for the ET-LF concept (in the x axis)

Figure 6.5 displays that the heat link model follows the resonances of the basic model. This was expected since the stiffness and mass added by the Heat Link are negligible. However, at low frequency, some small resonances appear at 0.06 Hz and 0.21 Hz. These are the torsion modes around the z axis. Previously, they appeared only for the rotational DoF around z. However, the HL is considered to be fixed at its extremity. Then, the torsion motion is transmitted to the Heat Link, which converts it to a small motion in the x direction (as the HL is in the OXZ plane).

The main impact of the Heat Link model is to highly increase the TN response of the mirror. This is due to :

1. a high loss angle : Indeed, having a loss angle of 0.5 [-] is extremely high and leads to a lot of dissipation.
2. the fixed boundary condition : Having a high loss angle is not a problem if the heat link is not deforming at all. Considering it fixed at its extremity makes it deform every time the PF is moving. This is why the increase in TN is higher at low frequency. At high frequency, it is mainly the violin modes of the suspensions that generate resonances.

The behaviour is similar to the one observed in [22].

The basic model is a triple pendulum, so the three pendulum ( $\approx 0.25, 0.56$  &  $0.88$  Hz, respectively for the basic model) and rocking modes ( $\approx 1.04, 1.21$  &  $2.82$  Hz, respectively for the basic model) are appearing in addition to the violin modes of the wires. The RHE model has a new resonance appearing at  $0.08$  Hz. This mode is the rocking mode (around y-axis) of the platform (PF), which is at  $1.04$  Hz for the two other models (see Figure 6.6).

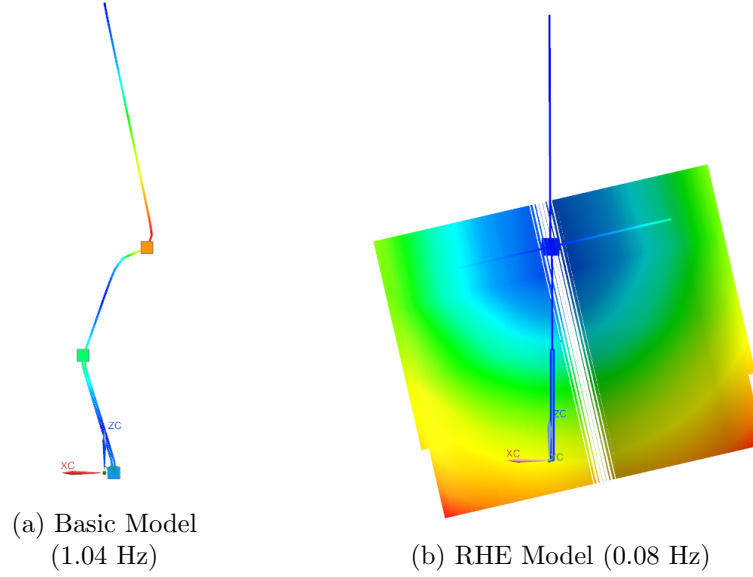


Figure 6.6: First Rocking modes of the PF around the y axis for the ET-LF models

This is due to the important addition of inertia brought by the RHE. Otherwise, the rocking modes of the mirror and the marionette are still at nearly the same frequency (for the three cases). The two first pendulum modes of the RHE, respectively, increase and decrease in frequency, which is expected with the increase in mass. The higher is the mass, the lower should be the resonance for a classical stiffness ( $\omega_0 \sim k/m$ ). However, it should be remembered that for such suspended wires, the higher is the mass, the higher is the stiffness due to the tension. For an infinite mass, the upper wire is infinitely stiff, then only two pendulum modes should result. For the RHE model, new resonances, due to the modes of the fins, appear. The first resonances, can be seen at  $\approx 5$  Hz. Moreover, the corresponding modes are represented in Figure 6.7.

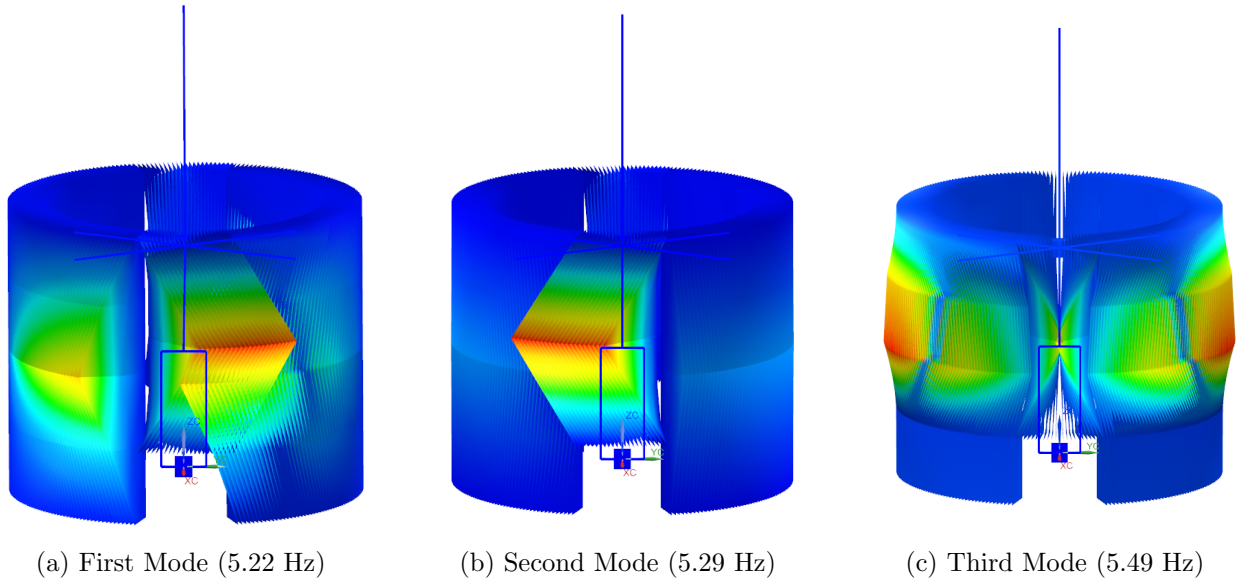


Figure 6.7: First modes of the RHE for the ET-LF model

These modes correspond to the coupling of the first flexion modes of the two types of fins. Other modes appear but are not that important in amplitude. The RHE is connected far from the mirror, reducing the transmitted motion. Moreover, the amplitude is reduced by the configuration in circle. Indeed, in the First Model or in the E-TEST model, the plates at each side were moving in phase in the same direction. In this case, the plates are not all moving in a specific translation axis but around an axis.

The configuration is then quite good for reducing the amplitude from the plate modes.

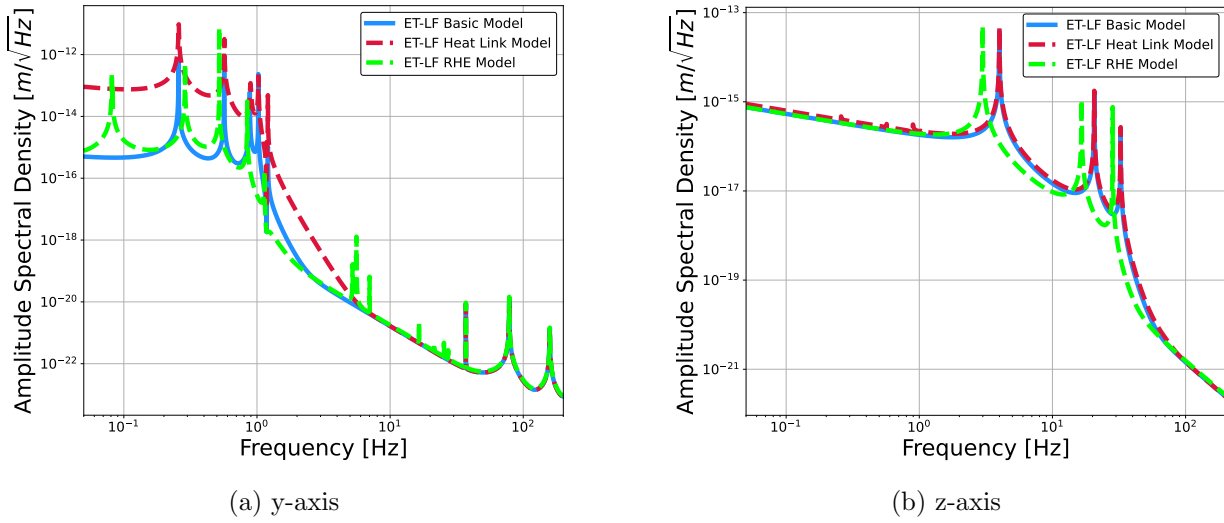


Figure 6.8: Thermal noise Response ASD - Comparison of the cooling strategies for the ET-LF concept (y and z axes)

In the y axis, Figure 6.8a shows that it is quite similar to the x axis. For the z axis, since the RHE is vertical, Figure 6.8b shows that no plate modes appear in this frequency range. Moreover, the axial modes appear at lower frequency as a result of the increase in mass (for the RHE model).

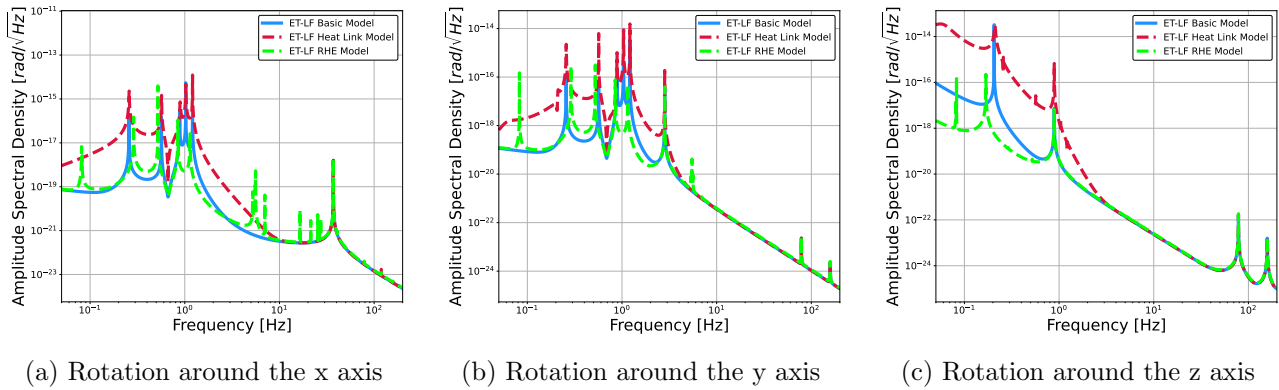


Figure 6.9: Thermal noise Response ASD - Comparison of the cooling strategies for the ET-LF concept (Rotational axis)

Figure 6.9 displays that the expected modes appear and the same conclusions can be made as before. However, it can be added that some violin modes appear for the rotation around the z axis. These violin modes are due to the configuration of the bottom wires which create a torsion motion of the mirror for the first out-of-phase violin modes (see Figure 6.10).

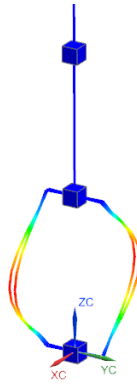


Figure 6.10: Torsion mode (around z) due to the first out-of-phase violin mode of the bottom wires for the ET-LF model

The in-phase violin modes are present in the x and y axes (see Figures 6.5, & 6.8a).

### 6.4.2 Temperature of the top wire

An interesting effect to study is the temperature in the wire. Indeed, considering the wire at 20 K is a huge approximation. Considering the upper extremity of the wire that should be at 300 K <sup>2</sup>.

The two following extreme cases will be computed :

- The first one is to consider that the whole system is at 20 K. This is the best case.
- The second one is to consider that the top wire is fully at 300 K and the rest at 20 K. This is the worst case.

In future work, it would be interesting, to determine the actual temperature distribution in the wire through a thermal analysis.

Then, as the method allows it, evaluate the real impact of having a temperature distribution in the top wire.

Before starting, the dissipation of the top wire should be evaluated at 300 K.

#### 6.4.2.1 Temperature dependence of the dissipation of the Ti6Al4V wire

Changing the temperature has a direct impact not only on the TN but also on the dissipation of the wire through material properties. Indeed, there are huge changes especially for the specific heat and thermal conductivity.

The temperature variation curves of the material properties can be found in Appendix 8.6. The bulk losses also vary with temperature, and the curves can also be found in Appendix 8.6 .

Following this, the global loss angle (considering bulk and TE dissipation) can be computed for different temperatures.

For this study, the parameters of the basic ET-LF model are used.

<sup>2</sup>300 K is an order of magnitude, not a specification

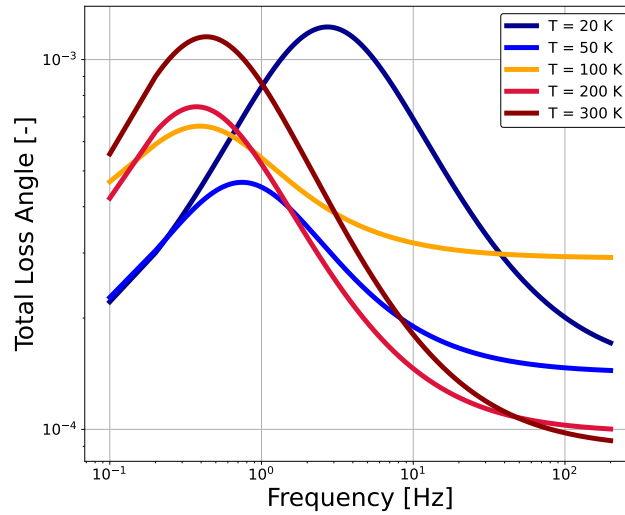


Figure 6.11: Total loss angle of the Ti6Al4V wire for different temperature

Figure 6.11 highlights that, contrary to logical thinking, at low temperature, dissipation is not necessarily lower. Starting at 300 K, when the temperature is reduced, the dissipation is decreasing. However, there is a turning point at a certain temperature that makes the dissipation increase again. This is mainly due to an increase in the Young Modulus, but also an important decrease in the specific heat (see Figure 8.13). Indeed, at high temperatures, the slope of the specific heat is weak compared to the one at low temperatures. This effect might depend on the material, on the diameter of the wire and on its tension.

The TN PSD is also directly linked to the temperature. So, having a lower dissipation does not necessarily mean that the TN response will be lower. It still follows the logical thinking of : the lower is the temperature, the lower is the TN.

Eventually, at 300 K,  $\beta$  is no longer negligible since the slope of the Young Modulus - Temperature curve is sharper (see Figure 8.14). Indeed, at 300 K, the order of magnitude of  $\beta$  is now  $-1e-4$ . Following the computation made in Section 2.5.2.1, for a diameter of  $1e-3$  m and considering Ti6Al4V, a suspended mass of 1e2 kg is enough to make the nonlinear contribution no longer negligible. For the ET-LF mass of 922 kg, the nonlinear contribution of the TE dissipation plays an important role at higher temperatures (see Figure 6.12).

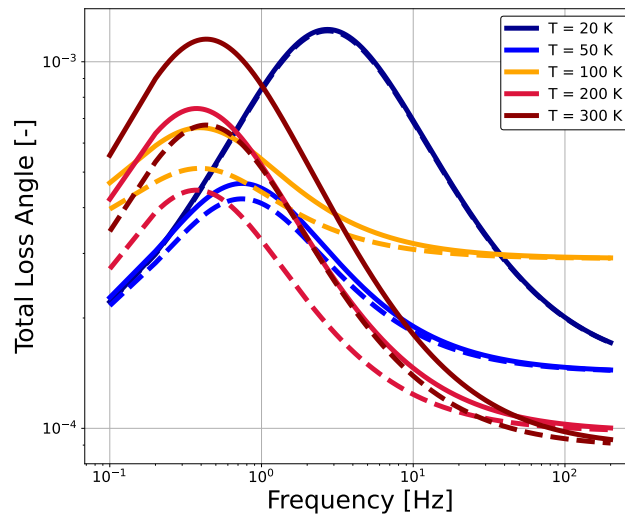


Figure 6.12: Total loss angle of the Ti6Al4V wire for different temperature - Comparison between the linear (dashed line) and nonlinear (full line) thermoelastic dissipation

Figure 6.12 shows that, at low temperature, the nonlinear contribution is negligible. However, not considering it at higher temperature leads to a large underestimation of the dissipation. This effect depends on the wire diameter, material, and tension.

#### 6.4.2.2 TN computation

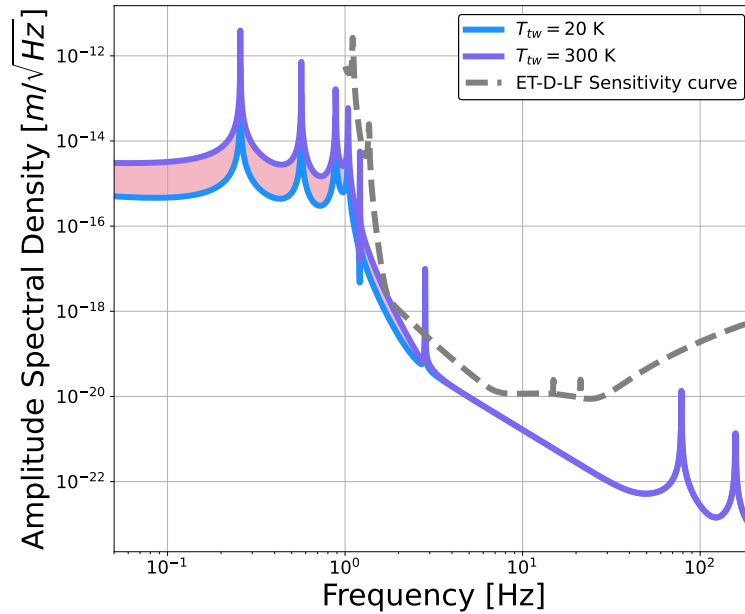


Figure 6.13: Thermal noise Response ASD - Effect of the temperature in the top wire of the ET-LF basic model (in the x axis)

Figure 6.13 depicts that having a higher temperature leads to a higher TN response of the mirror at low frequency. At higher frequency, the top wire is isolated from the mirror. These results were expected, but show that the increase is not that catastrophic. Indeed, this is probably due to the isolation between the top wire and the mirror. The curve for a temperature distribution in the top wire should be in between these two curves (pink area in Figure 6.13). Such a curve is still interesting to compute in future work.

#### 6.4.3 Diameter of the top wire

Another interesting computation is to increase the top wire diameter for the RHE ET-LF model. Indeed, as an important mass is added to the system, the tension in the top wire will increase and the wire might not support it without braking. Then considering the addition of an RHE, the impact might also be to change the top wire.

Considering that the mass is nearly doubled by adding the RHE concept, it is assumed that doubling the diameter would be enough to support all the mass.

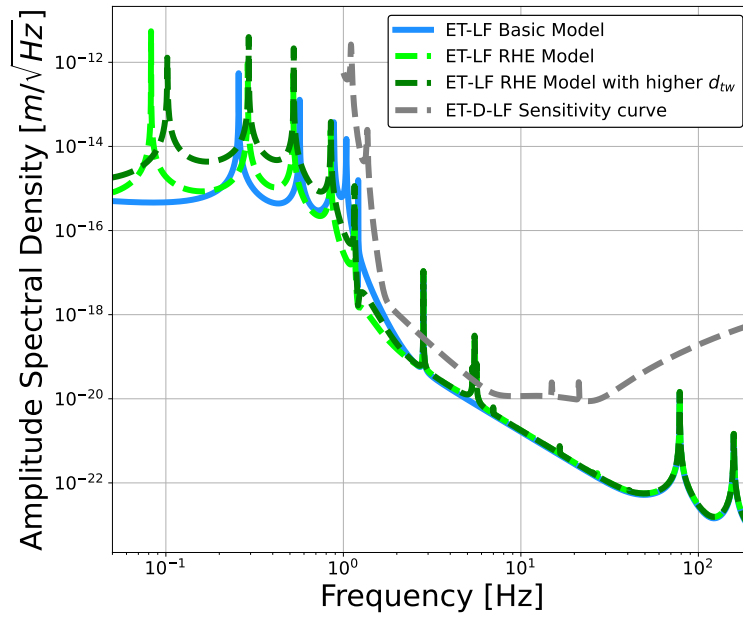


Figure 6.14: Thermal noise Response ASD - Effect of the diameter of the top wire of the ET-LF RHE model (in the x axis)

Figure 6.14 highlights that double the diameter mainly increases the TN response of the mirror at low frequency, which was expected. Indeed, a larger diameter means a higher relaxation time and then a dissipation peak that appears sooner and is smaller (in width). Furthermore, the top wire deforms mainly at low frequencies.

The increase in diameter also slightly modifies the rocking mode of the PF.

Despite all of this, double the diameter reasonably increases the TN response. This means that a larger diameter for the top wire might be envisaged if the added mass of the RHE is too important.

#### 6.4.4 Reinforcements of the RHE

Considering the panels as they were done is perhaps too optimistic. Indeed, for simplicity, the RHE might simply be suspended without any reinforcements for the fins.

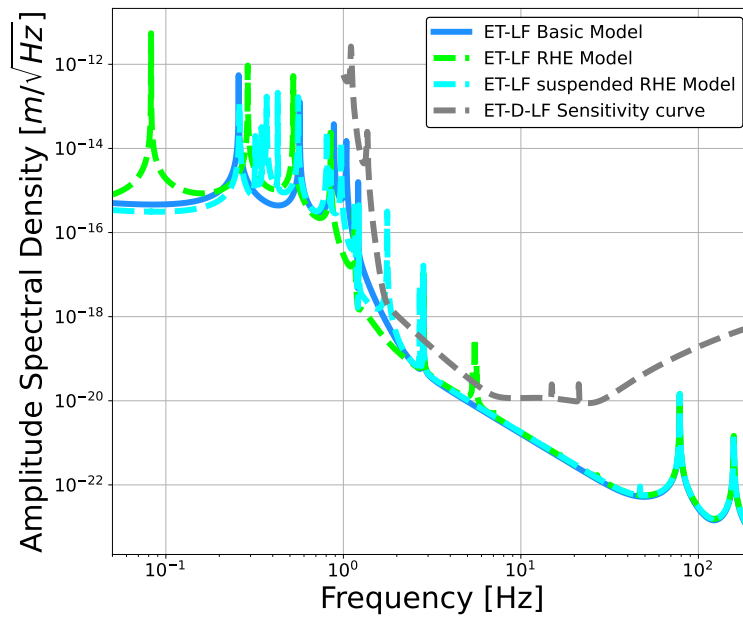


Figure 6.15: Thermal noise Response ASD - Comparison of two attachment configuration for the ET-LF RHE model (in the x axis)

Figure 6.15 shows that considering simply suspended plates, without reinforcements, makes them less stiffer. Then, in Figure 6.15, more modes from the panels appear. The first flexion mode of the suspended case is different and appears sooner (0.32 Hz). It is different since it has a higher effective modal mass, resulting in a higher amplitude. At such a frequency, the mode is coupled with the PF rocking mode. This is represented in Figure 6.16.

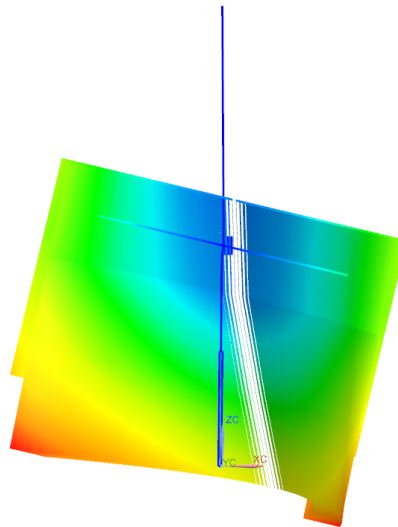


Figure 6.16: First flexion mode of the RHE for the suspended RHE ET-LF model

Generally, it can be concluded that having plates with the highest stiffness possible is the best case to minimize TN response. This conclusion follows well the ones drawn in Section 4.

#### 6.4.5 PF-RHE Connections

The assumption of having the RHE rigidly linked to the PF might be too optimistic. Knowing that the PF should have a diameter of  $\approx 952.2$  mm, it means that the 6 links between the PF and the fins should have a length of 0.9239 m. Considering these as RBEs is perhaps a strong assumption. These links can be replaced by beam elements. As a first idea, since no design exist, the beam can

have a rectangular cross-section of 160 x 80 mm. The beam are considered to be made of Al6061 (see properties in Table 6.3).

The added mass of such connections is equal to 191.58 kg.

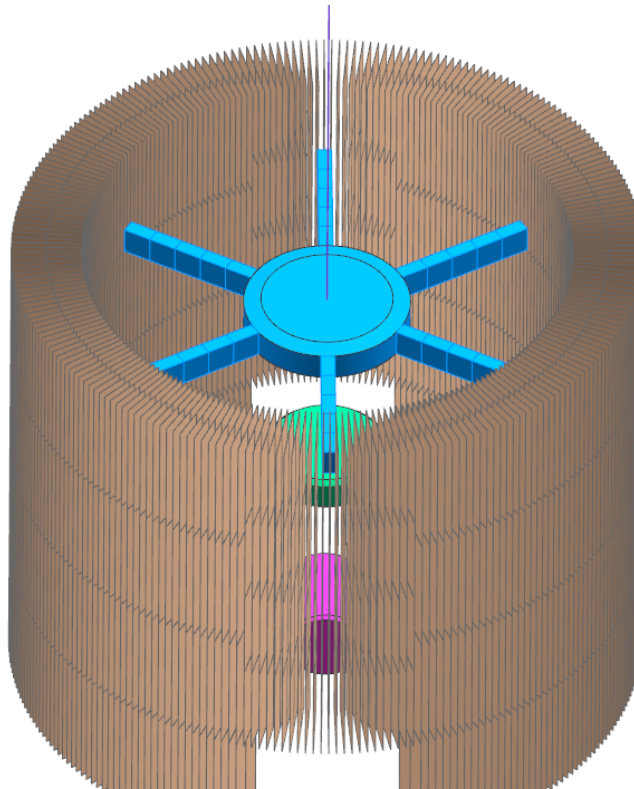


Figure 6.17: FE mesh for the ET-LF RHE model, considering beam connection between the PF and the RHE

Figure 6.17 represents the adapted FE mesh. The 6 links are composed of 5 beam elements each. The model has a total of 9726 DoFs. Since post-processing the matrices becomes complicated, the assumption of having the same dissipation for the beam links and the fins is made. Thus, the maximum value between both will be chosen for all their DoFs. In this case, the dissipation is computed with the bulk of the Al6061 and the TE damping from the beams (considering the highest thickness of 160 mm).

This is the worst case, as the fins dissipate the same amount of energy as they were 160 mm thick. The TN response is expected to be highly increased. However, the purpose of this study is to identify the effects, in terms of resonances. For the amplitudes, it is already known that the values are overestimated.

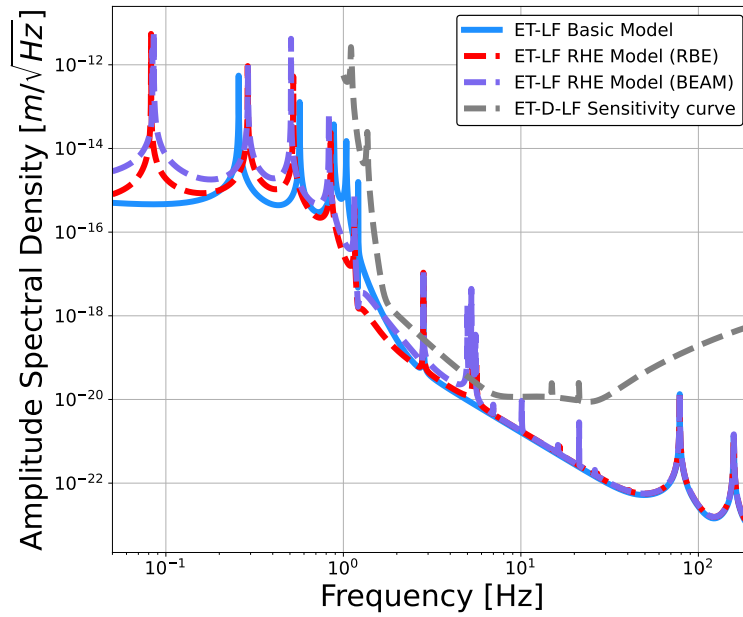


Figure 6.18: Thermal noise Response ASD - Comparison between RBEs and beam elements that connect the PF and the RHE for the ET-LF RHE model (in the x axis)

Figure 6.18 highlights the effects of considering beam links rather than RBEs. First, as expected, the rocking and pendulum modes are slightly changed when considering beam since masses and inertia are added. Next, the fin modes are also modified, as the connections between the RHE and the PF are no longer rigid. Due to the decrease in stiffness, the resonances appear at lower frequencies. However, the amplitudes of the response for these modes are increased. This might be explained by the overestimation of the dissipation. The violin modes are unchanged.

Figure 6.18 also shows that new resonances appear when having beam links. They can be seen notably at  $\approx 10$  Hz and 21.3 Hz (see Figure 6.19), for example. These resonances come from the non rigid links.

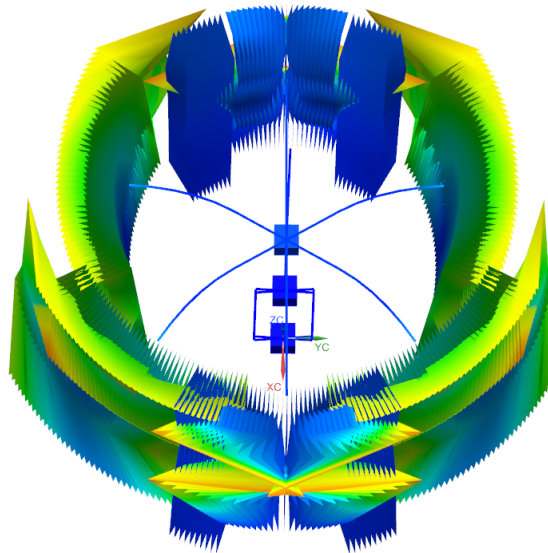


Figure 6.19: Bending mode of the beam links at 21.3 Hz

These modes corresponds to flexion modes of the beams coupled with fin modes. They were not appearing in the previous model since the link were fully rigid.

Figure 6.18 depicts that, even with the worst dissipation, the global increase in the TN response

is not that important. The results give only an insight and need to be improved in terms of dissipation. They also highlight the slight changes in resonances that appear when considering more realistic connections. This shows that the previous assumption was already a good estimate in terms of resonances. For the amplitudes, they should be verified with the dissipation.

Changes in resonances are expected to be reduced with stiffer connections. In this case, simple full filled beams were used. The dimensions were limited to not induce a high mass increase. For future work, stiffer beams, with the same mass, might be investigated. This can be done through new shapes, reinforcements, or new materials.

#### 6.4.6 Comparison with *Koroveshi et al.*

In the paper of *Koroveshi et al.* [22], they developed an analytical model of the ET-LF concept. The model is a double pendulum that has nearly the same properties as in Tables 6.1 & 6.2. To analytically describe the wires, they considered the approximated solution of the stiffness for the top wire (since the tension should be high). For the 4 bottom wires, they solved the *Euler-Bernoulli* Equation (3.15) with the following BC :

1.  $X(0, \omega) = 0$  (wire clamped at the top)
2.  $\left. \frac{\partial X(y, \omega)}{\partial y} \right|_{y=0} = 0$  (wire clamped at the top)
3.  $X(L, \omega) = X_1(\omega)$  (wire connected to the centre of mass (CoM) of the suspended mass, the DoF of this mass being  $X_1(\omega)$ )
4.  $\left. \frac{\partial X(y, \omega)}{\partial y} \right|_{y=L} = 0$

These are nearly the same as in Section 3.2, only the last BC differs. The BC determined in Section 3.2 was determined following the paper of *Gonzalez & Saulson* for a 1 DoF pendulum [12]. *Koroveshi et al.* consider that the wires bend only at the attachment point, such as :  $\left. \frac{\partial X(y, \omega)}{\partial y} \right|_{y=L} = 0$ . This assumption has a huge impact on the EoM of the suspended mass since, looking at Equation (3.16), the contribution of the tension in the wire is equal to zero. Then, considering such a BC is the same as considering no tension from the wire in the EoM of the suspended mass.

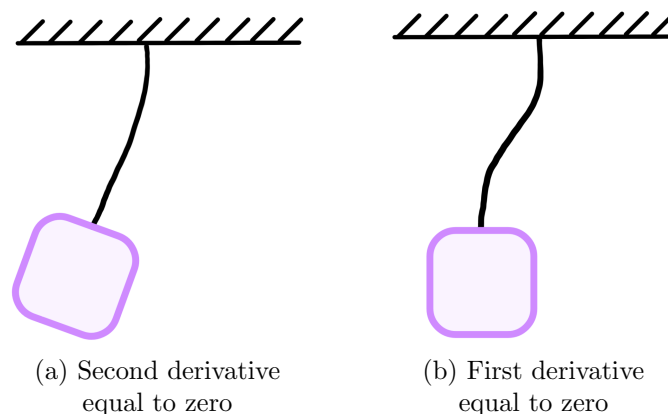


Figure 6.20: Comparison of the lower extremity BC of a wire for the pendulum mode of a 1 DoF system

Figure 6.20 shows the difference between the two BC in the case of a pendulum mode of a 1 DoF system.

This Section aims to show the impact of this boundary condition, in addition to the approximated formula used for the top wire.

In this context, three cases will be compared :

1. The model defined by *Koroveshi et al.*
2. An approach of the model from *Koroveshi et al.* realised with the FEM. This model aims to reach nearly the same results as in [22].
3. The ET-LF basic model.

The model approached will never meet the same curve as in [22], as the exact stiffness of the top wire is computed with FEM. In the software, the rotational DoFs of the last node of the wire were blocked to meet the BC of no slope (see Figure 6.20b). For this model, the exact same properties of [22], for the wires and the masses, were used. Similarly to [22], the temperatures of the upper and lower wires are equal to 15 K and 20 K, respectively.

The ET-LF basic model aims to show the differences between these simplified models and a more advanced one. In this model, a triple pendulum is considered with masses that have inertia. Moreover, the four bottom wires are not connected to the CoM of the mirror.

Note that for this model, the masses and inertia are slightly different but should not modify the order of magnitude of the TN response. For the double pendulum of the bottom, the exact same length, diameter, and materials are used. The temperature is considered to be equal to 20 K for the entire system. Last but not least, the model does not use the BC of [22] at the end of the wire. The comparison of the assumption will be made with this model.

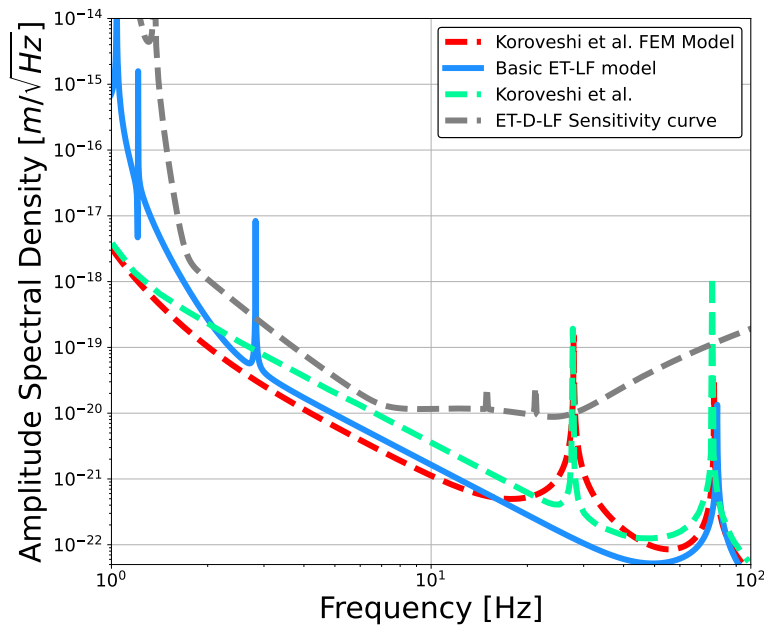


Figure 6.21: Thermal noise Response ASD - Comparison with *Koroveshi et al.* [22] of the ET-LF model

Figure 6.21 shows that the FEM model has the same resonances as the *Koroveshi et al.* curve. These correspond to the two first violin modes of the lower wires. The amplitudes of the curves are not the same. This effect was already seen in Figure 3.16, where the estimate TN response of the intermediate model was higher in amplitude. This comes from the approximated solution, since in Figure 3.16, the same BCs were used.

In the basic model, as expected, there are more resonances at low frequencies. These are the rocking modes of the masses. The most important is that the first violin mode of the lower wires

completely differs from the two other models. The mode passes from  $\approx 28$  Hz to  $\approx 78$  Hz. This can be explained by Equation (3.16), since when the slope of the lower extremity of the wire is equal to zero, it has the same effect as having no tension. However, tension is essential, as it increases the stiffness of the wires. Then, when the slope is non-zero, the tension plays a role in the EoM such that the wire is stiffer. This leads to a resonance that appears at a higher frequency.

Figure 6.21 highlights not only the impact of using an approximated solution but also the importance of the boundary conditions.

## 6.5 Conclusion on the ET-LF model

This Section enabled to apply the FE method to an actual case. A first estimate of the TN was found for many cases, notably considering a radiative cooling strategy. It has been seen that, similar to previous Sections, the RHE mainly impacts the TN response through the added mass and inertia and through fin modes. Section shows that using the radiative cooling strategy might still successfully satisfy the requirements of ET [16]. This was expected since the RHE does not globally increase the TN response.

A comparison was made between the two cooling strategies, highlighting the important increases in TN for conductive cooling (same as in [22]).

In addition, many other studies have been done.

The temperature in the top wire might have a significant impact on the TN at low frequencies (still satisfying ET requirements [16]). Even if the worst case has been considered for this study, the temperature distribution in the wire should still be important. Although this distribution was not implemented, the method should allow it. The section also identified the importance of the temperature dependence of the dissipation through the material properties. Moreover, the need to consider the nonlinear contribution of the thermoelastic dissipation has been highlighted.

Next, a study was performed to see the impact of having a higher diameter for the top wire. Indeed, for the added mass of the RHE, the top diameter might not have been design to support this amount of mass. The study shows that the TN is increased but not highly and still satisfies ET requirements [16].

Next, the need to have high stiffness for the fins and the PF-RHE connections has been highlighted to minimize the TN response.

Eventually, the model of *Koroveshi et al.* was compared with a model based on FE method. This last study mainly shows the huge importance of the boundary conditions and approximations imposed. These can modify both the amplitude and the resonances of the TN response of the mirror.

## Chapter 7

# Conclusion and Future work

### 7.1 Conclusion

The study performed in this manuscript was focused, in a first time, on the theoretical description of the Thermal Noise based on the Fluctuation and Dissipation Theorem. This step was essential to highlight the main assumptions of this theory. This Chapter also described the phenomenon of the internal dissipation and the theory behind. All dissipation contributions were developed based on actual knowledge, since this field of study is still in development.

Next, this theory was applied to basic pendulum systems. This study, mainly analytical, was aimed at highlighting the influence of all the parameters. A discussion was held about the approximate solution for the stiffness of the suspended wires. The importance of the violin modes was then introduced. But the main purpose was to introduce the new FE description of the TN based on the state space matrices of a system. This method was first compared with the analytical solutions to verify the exactness of the results. These verifications were performed for the following cases :

- 1 DoF pendulum
- 1 DoF pendulum with inertia
- 2 DoF pendulum

The last case was also used to compare the model with the one developed in [22]. It highlighted the importance of the assumptions made in the paper of *Korovesi et al.*

In addition, a mesh convergence study was performed for the basic 1 DoF pendulum

Once that basic pendulum systems were studied, the purpose was to investigate the addition of fins on a double pendulum. This was realised in the context of the radiative cooling strategy for the mirror. This study shows that the introduction of fins mainly impacts the TN response through the addition of mass and inertia. These mainly change the pendulum and rocking modes of the system. Moreover, new resonances from the fins appeared in the TN response. It was found that they depend mainly on the stiffness and dissipation of the panels. The TN response of the mirror should be 'minimized' for high stiffness of the plates, since no resonances will appear in the frequency range of interest. Otherwise, the RHE does not increase the global amplitude of the TN, but increases it locally. This conclusion was drawn on the basis of a sensitivity analysis performed. Finally, a section introduced the AFDT that enables the description of non-uniform temperature in the system, which was the main limiting assumption of the FDT.

The Chapter shows that the FE method can be applied for more complex systems. In the same purpose, the E-TEST model was studied. The objective was to show that the method can be applied to this model (that was the first to introduce the concept of radiative cooling). The method was successful in estimating the TN response of the mirror.

Eventually, a study of the ET-LF model was performed. This model being a concept of the next generation of gravitational wave detector, it constitutes the best context of application for the FEM TN estimation. Through this analysis, the comparison of the two cooling strategies was performed. The results show the interest of using a radiative, rather than conductive, cooling strategy. Moreover, additional studies were performed to highlight the potential sources of discussion for the next TN studies of the model. Eventually, a Section was dedicated to the comparison with the model of *Koroveshi et al.* [22] that shows the huge impacts of the approximations and assumptions that can be considered in TN studies.

The FE method developed in this manuscript is based on knowledge of the state space matrices of a system. A way of applying the dissipation to these matrices was defined, highlighting some advantages of the method. Since the structure is discretised, the dissipation can also be discretised. This shows that the description of temperature gradients, local dissipation, non-uniform geometry, etc, can be envisaged.

In general, the advantages of the method are :

- ✓ Easy to implement.
- ✓ Easily adaptable to any complex model, which is favorable for TN design.
- ✓ Can compute the TN for non-uniform temperature in the system. It should also be able to take into account temperature gradients.
- ✓ Can consider non-uniform cross sections for the wires or the fins.
- ✓ Might include local dissipations due to joining or local defects.
- ✓ Can compute the TN in each DoFs.
- ✓ Can be implemented in FE software that would make the estimate of the TN easier.
- ✓ Was verified to be exact through the comparison with analytical results.
- ✓ From the FEA, the mode shapes can be determined. The model might then be correlated with experimental modal measurements.

The actual disadvantages are :

- ✗ High computation time since the computations must be done at each frequency. This drawback might be mitigated using other computation techniques or by using parallel computation.
- ✗ High computation cost since the complete inverse of the transfer matrix should be known for the AFDT. The cost increases with the number of DoF considered.
- ✗ The method is not yet applicable to complex systems, since the matrices are post-processed. In future work, this drawback should be mitigated by the implementation on FE solvers.
- ✗ The method is still not adapted to 3D meshes.

The FE method developed in this thesis highlighted many important advantages that will be more than necessary in the TN design of future gravitational wave detectors. Some drawbacks still exist, but they might be easily mitigated in future work.

## 7.2 Future work

Future work should mainly focus on the development of the FE method directly on solvers rather than post-processing the matrices. This should allow to fully benefit from the advantages of the method developed in this manuscript.

Next, since computations must be done at each frequency, using parallel computing should be an

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New approach of Thermal Noise FE modelling  
for the ET cryogenic payload

Future work

option to minimize computation time.

Once done, more complex systems should be easier to study.

Introducing temperature gradients might be quite interesting (See Section 6.4.2). On the basis of FE advantages, non-uniform wires/fins might be envisaged. Local dissipations might also be considered. Future works should also focus on the adaptation of the FE method to 3D meshes. Such adaptation might be more than interesting in future estimates of the TN for complex systems. The hard work lies in relating the stiffness matrix of 3D meshes to dissipation theory.

The method developed in this manuscript might also be adapted for TN estimates of other systems. Indeed, this noise is not only a problem for gravitational wave detectors. Thermal noise has to be mitigated in the context of quantum computers, for example. A better understanding and estimation of this noise might reduce the cooling cost of such complex systems.

Globally, it can be concluded that the development of the method should be more than necessary for future characterisations of the Thermal Noise.

## Chapter 8

# Appendices

### 8.1 Use of Artificial Intelligence

Artificial Intelligence was used to improve Figures and illustrations of this master's thesis.

### 8.2 Useful Relations

1.  $\frac{e^{ixt}-1}{ix} = e^{i\frac{x}{2}t} \left( e^{i\frac{x}{2}t} - e^{-i\frac{x}{2}t} \right) \frac{1}{ix} = e^{i\frac{x}{2}t} \frac{2i}{ix} \sin(x/2 t) = e^{i\frac{x}{2}t} \frac{2 \sin(x/2 t)}{x}$
2. *Heisenberg Representation* :  $\frac{d\hat{A}}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{A}] + \frac{\partial \hat{A}}{\partial t}$   
This is equivalent to the *Schrödinger Representation* Definition [2.1.1](#)
3.  $\frac{1+e^{-x}}{1-e^{-x}} = 2 \frac{e^{-x}}{1-e^{-x}} + \frac{1-e^{-x}}{1-e^{-x}} = 1 + \frac{2}{e^x-1}$

### 8.3 Limits Study - Thermal Noise PSDs of a basic 1-DoF system with a constant loss angle

Considering that the loss angle is a constant value such as :  $\phi(\omega) \approx \phi$ . The PSD of the excitation at low frequency becomes :

$$\lim_{\omega \rightarrow 0} S_f(\omega, T) = \lim_{\omega \rightarrow 0} \frac{4k_B T k \phi}{\omega} = \infty \quad (8.1)$$

And, as expected, the PSD response at low frequency is :

$$\lim_{\omega \rightarrow 0} S_x(\omega, T) = \lim_{\omega \rightarrow 0} \frac{4k_B T \phi k}{\omega \underbrace{\left[ (k - \omega^2 m)^2 + \phi^2 k^2 \right]}_{\rightarrow (1+\phi^2)k^2}} = \infty \quad (8.2)$$

### 8.4 Representations

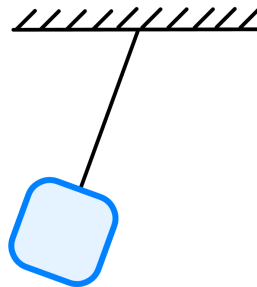


Figure 8.1: 1-degree-of-freedom Pendulum Mode Representation

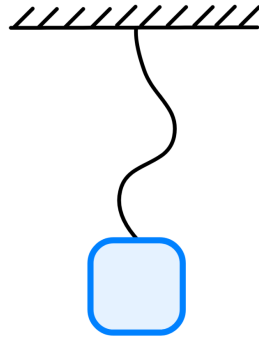


Figure 8.2: Violin Mode Representation

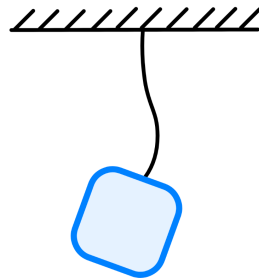


Figure 8.3: Rocking Mode Representation

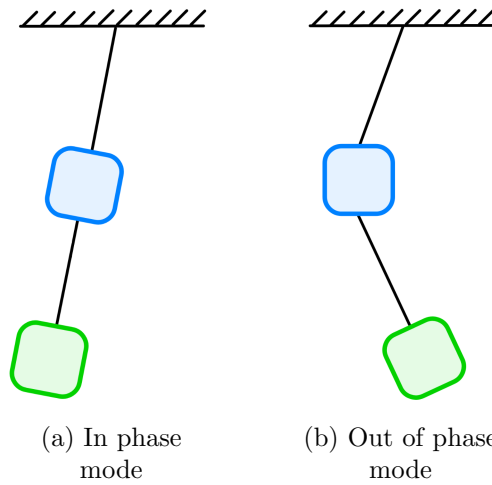


Figure 8.4: 2-Degree Of Freedom Pendulum Modes Representation

## 8.5 Additional Figures

### 8.5.1 1-degree-of-freedom pendulum sensitivity analysis - diameter variation

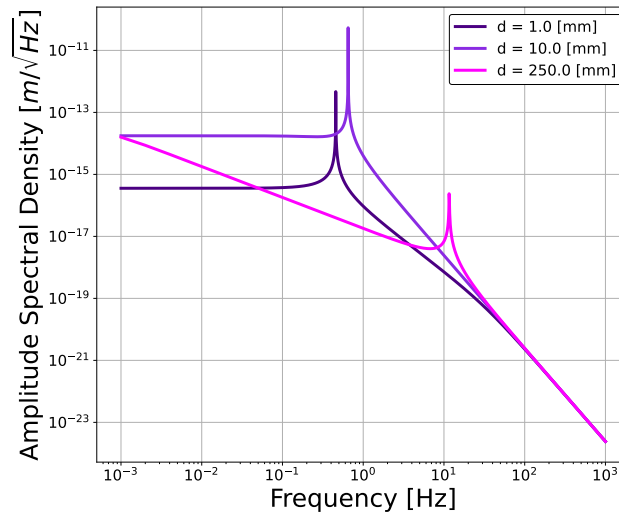


Figure 8.5: Thermal noise for Diameter of wires variation comparison for the 1-Degree Of Freedom pendulum system - second comparison

### 8.5.2 First Model - Sensitivity Analysis - appendices Figures

#### 8.5.2.1 RHE Configuration investigation - Rotational Degree Of Freedom

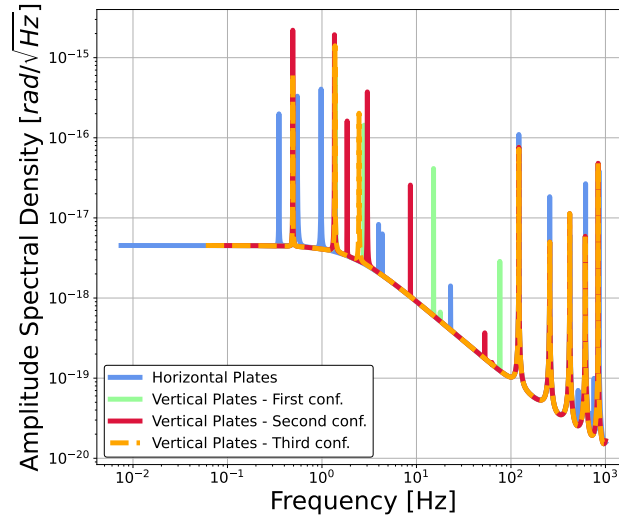


Figure 8.6: Thermal noise Response ASD - comparison for different plate configurations for the First Model system (Rotation around the x axis)

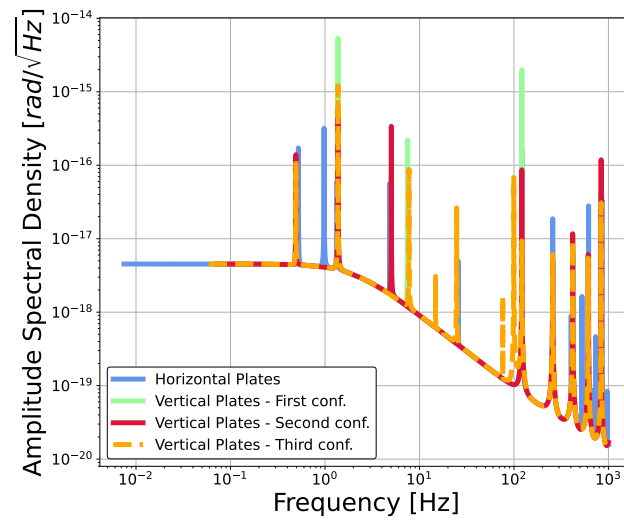


Figure 8.7: Thermal noise Response ASD - comparison for different plate configurations for the First Model system (Rotation around the y axis)

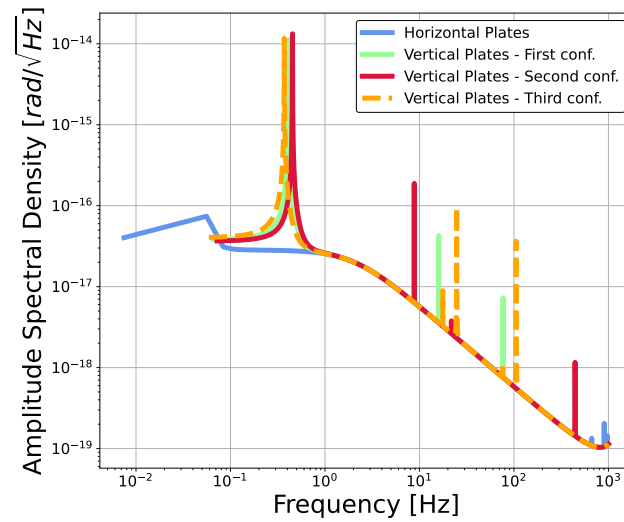


Figure 8.8: Thermal noise Response ASD - comparison for different plate configurations for the First Model system (Rotation around the z axis)

### 8.5.3 FDT adapted to Temperature Distribution - Verification

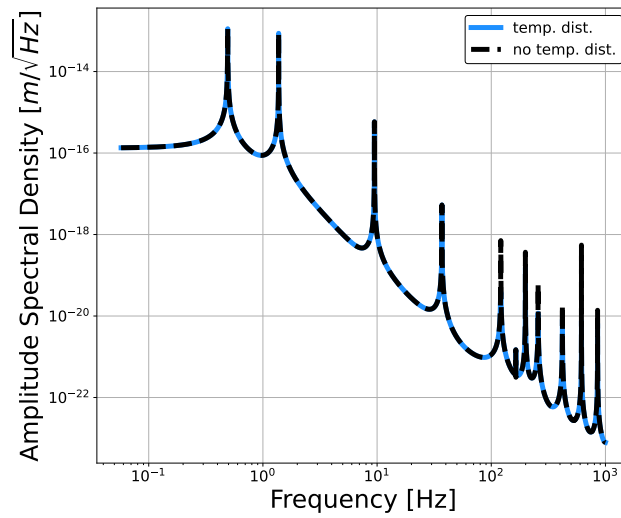


Figure 8.9: Thermal noise Response ASD - comparison between the FDT (every element at 20 K) and ADFT (wires at 20 K and panels at 20 K) for the First Model system (in the x axis)

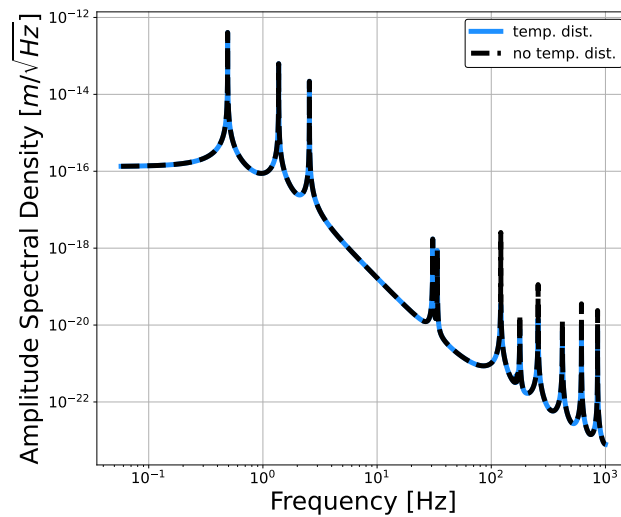


Figure 8.10: Thermal noise Response ASD - comparison between the FDT (every element at 20 K) and ADFT (wires at 20 K and panels at 20 K) for the First Model system (in the y axis)

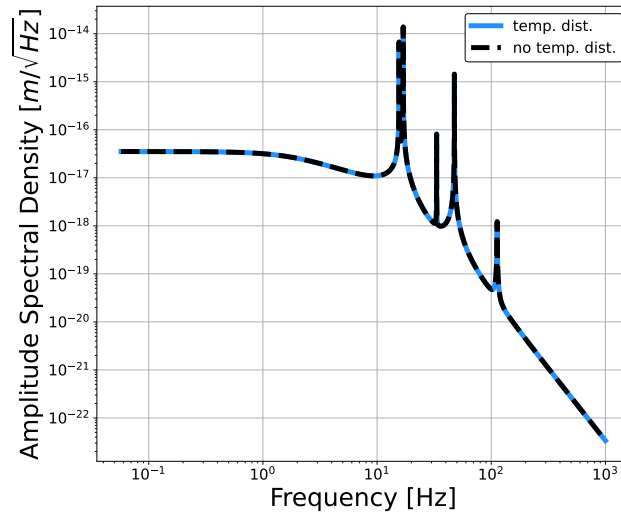


Figure 8.11: Thermal noise Response ASD - comparison between the FDT (every element at 20 K) and ADFT (wires at 20 K and panels at 20 K) for the First Model system (in the z axis)

From these figures the following maximum relative difference ( $\varepsilon$ ) between the two curves can be computed for each axis :

- x - axis :  $\varepsilon = 3.33 \cdot 10^{-10}$  [-]
- y - axis :  $\varepsilon = 1.03 \cdot 10^{-9}$  [-]
- z - axis :  $\varepsilon = 4.99 \cdot 10^{-12}$  [-]

## 8.6 Material Properties temperature distribution

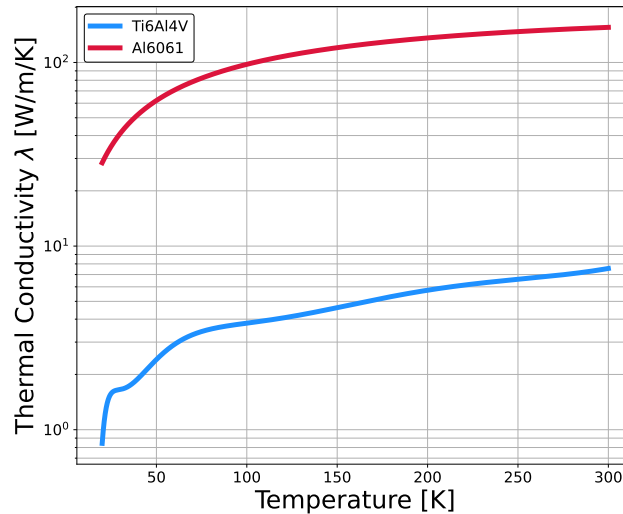


Figure 8.12: Thermal Conductivity distribution over the temperature for the Ti6Al4V and Al6061 from [28]

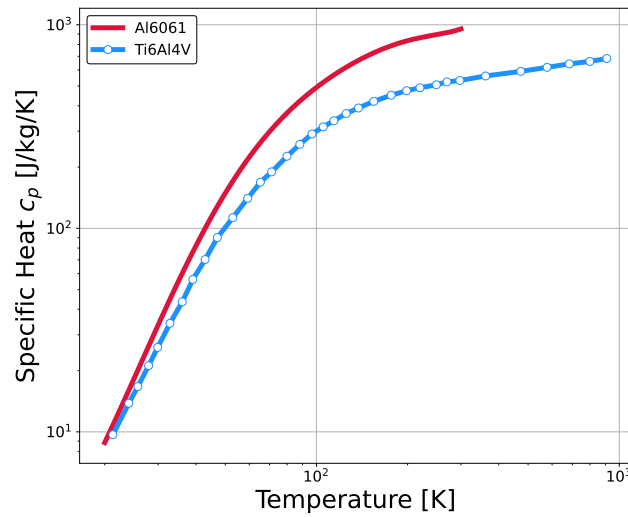


Figure 8.13: Specific Heat distribution over the temperature for the Ti6Al4V (from [38]) and Al6061 (from [28])

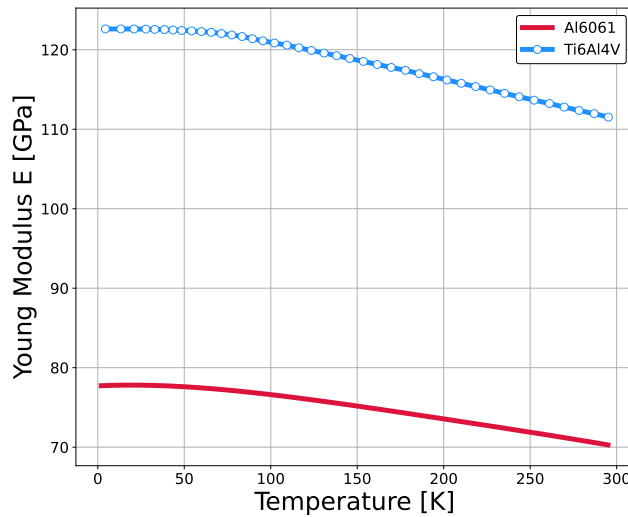


Figure 8.14: Young Modulus distribution over the temperature for the Ti6Al4V (from [8]) and Al6061 (from [30])

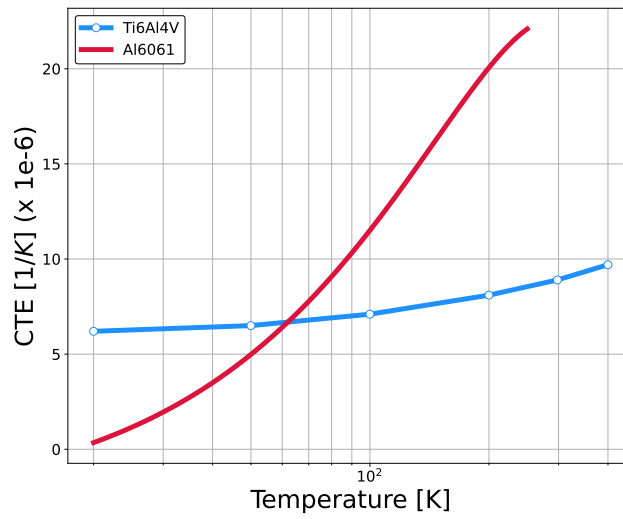


Figure 8.15: Coefficient of Thermal Expansion distribution over the temperature for the Ti6al4V (from [37]) and for the Al6061 (from [31])

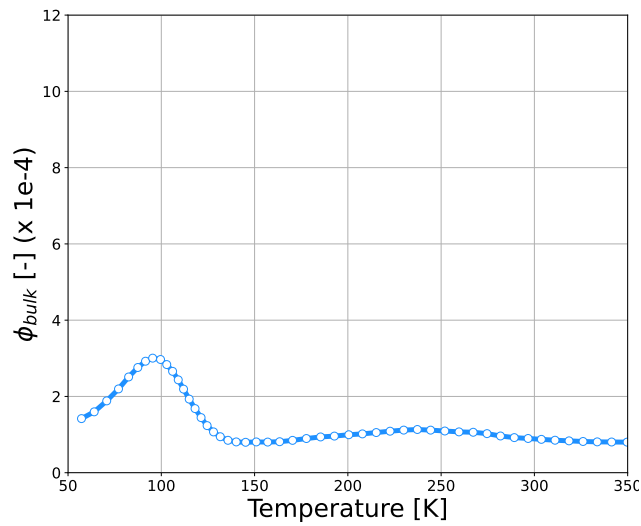


Figure 8.16: Bulk loss distribution over the temperature for the Ti6al4V (from [3])

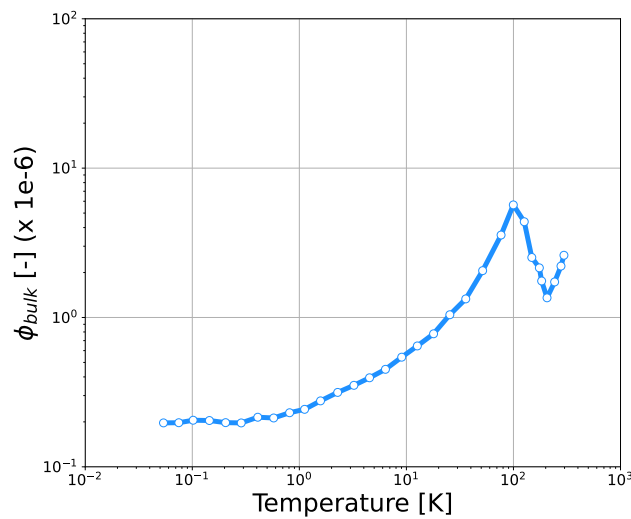


Figure 8.17: Bulk loss distribution over the temperature for the Al6061 (from [20])

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