Quantum Entanglement: a Study of Recent Separability Criteria

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Introduction

Quantum entanglement is a feature of quantum mechanics that has risen numerous philosophical, physical and mathematical questions since the early days of the quantum theory. It can be seen as the most non-classical feature of quantum mechanics and has absolutely no classical equivalent. Consequently, debates took place when it has been described in 1935 by Einstein, Podolsky and Rosen in Ref. [1] and by Schrödinger in Ref. [2] for the first time. In the latter paper, Schrödinger wrote the now famous citation 'I would not call [entanglement] *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought.' Indeed, Schrödinger acknowledged the existence of global states of bipartite systems (systems made of two subsystems) that cannot be factorized, i.e. that cannot be written as a tensor product of states of the subsystems. Therefore, for entangled states, only a common description of the subsystems exists.

Einstein, Podolsky and Rosen proposed in Ref. [1] a thought experiment known today as the *EPR paradox*, with the aim of proving the incompleteness of quantum mechanics. The thought experiment goes as follows: consider two particles prepared in an entangled state. The particles are then spatially separated and one performs a measurement on one of them, say the first particle. Given the laws of quantum mechanics, the state of the other particle collapses in a state dictated by both the outcome of the measurement on the first particle and the initial state of the composite system. Then, it appears that the correlations between both particles are stronger then the correlations one could classically expect. So, EPR argued that the measurement outcomes of both particles were determined at the creation of the pair and that quantum mechanics missed some (local) hidden variable that should make the theory a causal and local one. This was in contradiction with Bohr and Heisenberg's interpretation of quantum mechanics, known as Copenhagen interpretation.

This paradox induced many debates within the scientific community but even *outside* of it. Indeed, a daily newspaper also reported the EPR paper [3]. The EPR paradox was resolved in 1964 by Bell, by showing that a local hidden variable theory is incompatible with the statistical predictions of quantum mechanics [4]. In order to do this, he proved an inequality (known as *Bell's inequality*) that all local theories have to verify. However, he also showed that the predictions of quantum mechanics violate this inequality. It meant that the quantumly correlations between entangled states are impossible to obtain within a classical theory. Then, in 1982, Aspect, Grangier and Roger carried out the first experimental violation of this inequality [5]. This experiment, called *Aspect's experiment*, confirmed the predictions of quantum mechanics, and thus confirmed its incompatibility with local theories.

Aspect's experiment helped the transition of quantum entanglement from purely theoretical considerations and philosophical debates to practical experiments, and it began to be considered as a powerful resource that enables tasks not permitted by classical resources. For instance, one can mention quantum cryptography [6], quantum teleportation [7] and quantum computing [8]. Additionally to the fundamental reason, all these practical perspectives were a reason why a strong entanglement theory was needed.

However, determining whether a given state is entangled or not is still an open problem today, both from the theoretical and experimental points of view. This is known as the *separability problem* (states with no entanglement are called separable and form a convex subset of all the quantum states). Theoretically, entanglement is defined by a mathematical property of quantum states that are described by density operators acting on Hilbert spaces. Although a general solution is still lacking, the separability problem has been solved for pure states [9], and for 2×2 and 2×3 systems [10]. We also note that from a philosophical point of view, characterising the set of separable states could answer the question whether the world is more quantum or more classical, i.e. does the set of states contain more quantum correlated (i.e. entangled) or classically correlated states?

The aim of this work is to give a selective but up to date review of the separability problem. Indeed, the two most-cited reviews on this topic were published in 2009 [11, 12]. Keeping this goal in mind, we present and analyse several separability criteria that appeared relevant to us. We focus on the theoretical perspective of the problem. The manuscript is structured as follows.

In the first chapter, we present some basic notions of quantum mechanics used throughout the Chapters 2 and 3. We first introduce quantum states and make the distinction between pure states (described by state vectors) and mixed states (described by density operators). Then, we present several concepts used to treat bipartite systems, namely partial trace, Bloch representation and Schmidt decomposition of quantum states. In the same section, we get to the heart of the matter by giving the mathematical definitions of entanglement for bipartite systems. Finally, we generalize these definitions to multipartite systems (systems composed of two or more subsystems).

The second chapter is dedicated to the first separability criteria that have historically been presented in literature. We begin with the celebrated positive partial transpose criterion, then introduce criteria based on entanglement witnesses and entanglement measures. The latter are used to detect entanglement, but also to quantify it. The next section of this chapter is devoted to concurrences, which, as we will see, solve the separability problem for multipartite pure states. We close the second chapter by introducing another celebrated criterion, namely the computable cross-norm or realignment criterion. The positive partial transpose criterion and the computable cross-norm or realignment criterion are the most well-known separability criteria.

Finally, Chapter 3 is devoted to more recent separability criteria, developed from 2007 to the present year. We begin this last chapter with the correlation matrix criterion and with the covariance matrix criterion, which are both strong separability criteria that make use of the Bloch representation of quantum states. Then comes an enhanced version of the computable cross-norm or realignment criterion, followed by a family of separability criteria. The next section of this chapter is dedicated to a necessary and sufficient criterion, but to our knowledge non-computable, developed in 2018. Then we present a criterion based on positive operator-valued measures. Ultimately, the last section is again dedicated to a family of criteria, which appears to unify several of the above mentioned criteria. We conclude our work by comparing and contrasting the criteria exposed in both Chapters 2 and 3.

Notations

Throughout this manuscript, we encounter the following sets build on *finite* dimensional Hilbert spaces $\mathcal{H}$:

- $\mathcal{L}(\mathcal{H})$: the complex vector space of all linear operators acting on $\mathcal{H}$[1];
- $\mathcal{HS}(\mathcal{H})$: (called the *Hilbert-Schmidt* space) the real vector space of all Hermitian operators acting on $\mathcal{H}$;
- $\mathcal{S}(\mathcal{H})$: the set[2] of all positive operators acting on $\mathcal{H}$ with unit trace, i.e. the set of all density operators acting on $\mathcal{H}$.

Note that $\mathcal{S}(\mathcal{H}) \subset \mathcal{HS}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$. For an operator $\hat{A} \in \mathcal{L}(\mathcal{H})$, we denote its complex conjugate by $\hat{A}^*$, its transpose by $\hat{A}^T$ and its Hermitian conjugate (or adjoint) by $\hat{A}^\dagger$. Let $|\Psi\rangle \in \mathcal{H}$, $\hat{A}, \hat{B} \in \mathcal{L}(\mathcal{H})$ and $\{\lambda_i\}$, $\{\sigma_i\}$ be the sets of the eigen- and singular values of $\hat{A}$ respectively (see Appendix A.1). On $\mathcal{L}(\mathcal{H})$, we define the standard *Hilbert-Schmidt* (or *Frobenius*) *inner product* $\langle \hat{A}|\hat{B}\rangle = \mathrm{Tr}(\hat{A}^\dagger \hat{B})$. This inner product is well defined since we are working on finite dimensional spaces and therefore the trace is well defined. We also define the following norms:

- *Euclidian norm*: $||\Psi|| \equiv \sqrt{\langle\Psi|\Psi\rangle} = \sqrt{\Psi^\dagger\Psi}$;
- *Trace* (or *Ky Fan*) *norm*: $||\hat{A}||_{\mathrm{Tr}} \equiv \mathrm{Tr}\left(\sqrt{\hat{A}^\dagger \hat{A}}\right) = \sum_i \sigma_i$;
- *Hilbert-Schmidt* (or *Frobenius*) *norm*: $||\hat{A}||_{\mathrm{HS}} \equiv \sqrt{\mathrm{Tr}(\hat{A}^\dagger \hat{A})} = \sqrt{\sum_i \sigma_i^2}$;

Since for a Hermitian operator $\hat{A}$, its singular values are the absolute values of its eigenvalues, one has $||\hat{A}||_{\mathrm{Tr}} = \sum_i |\lambda_i|$ and $||\hat{A}||_{\mathrm{HS}} = \sqrt{\sum_i \lambda_i^2}$. We denote with $\mathcal{M}_n(\mathbb{C})$ ($\mathcal{M}_n(\mathbb{R})$) and with $\mathcal{M}_{n,m}(\mathbb{C})$ ($\mathcal{M}_{n,m}(\mathbb{R})$) the set of $n \times n$ and $n \times m$ complex (real) matrices respectively.

[1] $\mathcal{L}(\mathcal{H})$ is equal to the complex vector space of all bounded operators acting on $\mathcal{H}$ since all linear operators defined on finite-dimensional Hilbert spaces are bounded.

[2] Note that $\mathcal{S}(\mathcal{H})$ is not a vector space.

Chapter 1

Basic notions of quantum mechanics

In this chapter, basic notions of quantum mechanics are introduced, in order to enable the full understanding of the separability criteria exposed in Chapters 2 and 3. We also introduce notations, that hold throughout all chapters. In the first section, we introduce density operators and their properties. We also introduce the notions of pure and mixed states, and present a special representation of quantum states, namely the Bloch representation. The second section is devoted to bipartite systems and we define in it the partial trace, expand the Bloch representation to bipartite states, present the Schmidt decomposition and then define the notion of quantum entanglement that sets the ground for Chapters 2 and 3. Finally, in the last section we expand the definitions of quantum entanglement to multipartite systems.

1.1 Quantum states and density operators

1.1.1 Pure states

In classical mechanics, a system is completely characterised by its position and momentum at any time. Given these two quantities at a certain time t_0, one can determine the position and momentum of the system at any other time by solving the classical equations of motion as given by Newton's Second Law,

$$\frac{d\mathbf{p}}{dt} = \sum_i \mathbf{F}_i, \tag{1.1}$$

with $\mathbf{p}$ being the momentum of the system and $\{\mathbf{F}_i\}$ being the set of all external forces applied to the system. In quantum mechanics, this is no longer true. The concepts of position and momentum lose their classical meanings and the system is now characterised by a normalised *state vector* belonging to a Hilbert space. The evolution of the state vector is given by the Schrödinger Equation,

$$i\hbar\frac{d}{dt}\left|\Psi\right\rangle = \hat{H}\left|\Psi\right\rangle, \tag{1.2}$$

with $\left|\Psi\right\rangle$ being the state vector of the system and $\hat{H}$ being the Hamiltonian of the system. If the state of the system is truly known, the system is said to be in a *pure state*. However, pure states are not the most general states one could think about. A quantum system can also be in a *classical* probabilistic mixture of pure states. The system is then said to be in a *mixed state* [13].

Let $|\Psi\rangle$ be the state vector (also called the *state* $|\Psi\rangle$) of a system belonging to a Hilbert space $\mathcal{H}$ of dimension d. As we have seen before, state vectors are normalised. Thus we only consider vectors with unit norm in this work. We can define the *density operator* $\hat{\rho}$, an operator acting on $\mathcal{H}$ associated to the state vector $|\Psi\rangle$, as the projection operator

$$\hat{\rho} \equiv |\Psi\rangle \langle\Psi| . \tag{1.3}$$

Its action on state vectors is

$$\hat{\rho} |\phi\rangle = |\Psi\rangle \langle\Psi|\phi\rangle = \langle\Psi|\phi\rangle |\Psi\rangle , \quad \forall |\phi\rangle \in \mathcal{H}. \tag{1.4}$$

The density operator, also often called *density matrix* or even *state of the system*, has a unit trace and is positive semi-definite. Indeed, if $\{|i\rangle : i = 1, ..., d\}$ is an orthonormal basis of $\mathcal{H}$, we have

$$\mathrm{Tr}(\hat{\rho}) = \sum_{i=1}^{d} \langle i|\hat{\rho}|i\rangle = \sum_{i=1}^{d} \langle i|\Psi\rangle \langle\Psi|i\rangle = \sum_{i=1}^{d} \langle\Psi|i\rangle \langle i|\Psi\rangle = \langle\Psi|\mathbb{1}|\Psi\rangle = 1, \tag{1.5}$$

$$\langle\phi|\hat{\rho}|\phi\rangle = \langle\phi|\Psi\rangle \langle\Psi|\phi\rangle = \langle\phi|\Psi\rangle \langle\phi|\Psi\rangle^* = |\langle\phi|\Psi\rangle|^2 \geq 0, \quad \forall\phi \in \mathcal{H} \tag{1.6}$$

and

$$\begin{aligned} \langle\phi|\hat{\rho}|\chi\rangle^* &= (\langle\phi|\Psi\rangle \langle\Psi|\chi\rangle)^* = \langle\Psi|\phi\rangle \langle\chi|\Psi\rangle \\ &= \langle\chi|\Psi\rangle \langle\Psi|\phi\rangle = \langle\chi|\hat{\rho}|\phi\rangle \quad \forall |\phi\rangle , |\chi\rangle \in \mathcal{H} \Leftrightarrow \hat{\rho} = \hat{\rho}^\dagger \end{aligned} \tag{1.7}$$

where in Eq. (1.5) we used the completeness relation of basis in vector states, $\sum_i |i\rangle \langle i| = \mathbb{1}$. Since the density operator is a projector, one has $\mathrm{Tr}(\hat{\rho}^2) = 1$. In the next section, we see that this is no longer true for mixed states. The expectation value of an observable represented by a Hermitian operator $\hat{A}$ acting on $\mathcal{H}$ is given by $\langle\hat{A}\rangle_\Psi = \langle\hat{A}\rangle_\rho = \mathrm{Tr}(\hat{A}\hat{\rho})$ and if the state $|\Psi\rangle$ evolves accordingly to Eq. (1.2), then the evolution of $\hat{\rho}$ is given by

$$i\hbar\frac{d}{dt}\hat{\rho} = [\hat{H}, \hat{\rho}]. \tag{1.8}$$

1.1.2 Mixed states

Now, let us suppose that the state of a system is not perfectly known and that the system is in a classical probabilistic mixture of pure states. This means that the system is in some state $|\psi_1\rangle$ with a probability p_1, in some state $|\psi_2\rangle$ with a probability p_2, and so on, with all probabilities summing to one and all states belonging to the same Hilbert space $\mathcal{H}$. This is what is called a mixed state. The state can then no longer be described by a single state vector since it is characterised by a set of states vectors, together with their respective probabilities. A convenient object to represent these states is the mixed state density operator, an operator acting on the Hilbert space $\mathcal{H}$ defined as

$$\hat{\rho} \equiv \sum_{i=1}^{L} p_i |\psi_i\rangle \langle\psi_i| = \sum_{i=1}^{L} p_i\hat{\rho}_i \tag{1.9}$$

with $L > 1$, $p_i > 0$, $\sum_{i=1}^{L} p_i = 1$ (the p_is are called convex weights) and where $\hat{\rho}_i$ is the density operators of the pure state $|\psi_i\rangle \in \mathcal{H}$, $i = 1, \ldots, L$. The density operator of a mixed state has also a unit trace and is also positive semi-definite. Indeed,

$$\mathrm{Tr}(\hat{\rho}) = \sum_{i=1}^{L} p_i \mathrm{Tr}(\hat{\rho}_i) = \sum_{i=1}^{L} p_i = 1, \tag{1.10}$$

$$\langle \phi|\hat{\rho}|\phi\rangle = \sum_{i=1}^{L} p_i \langle \phi|\hat{\rho}_i|\phi\rangle \geq 0, \quad \forall\, |\phi\rangle \in \mathcal{H} \tag{1.11}$$

and $\hat{\rho}$ is Hermitian since it is a real linear combination of Hermitian operators. As mentioned before, density operators of mixed states are not projectors and one has, in general, $\mathrm{Tr}(\hat{\rho}^2) \leq 1$ with $\mathrm{Tr}(\hat{\rho}^2) = 1$ for pure states and $\mathrm{Tr}(\hat{\rho}^2) < 1$ for mixed states.

Proof. Let $\hat{\rho}$ be a state as in Eq. (1.9). One has

$$\begin{aligned}
\mathrm{Tr}(\hat{\rho}^2) &= \sum_{i=1}^{d} \langle i|\hat{\rho}^2|i\rangle = \sum_{i=1}^{d} \sum_{j,k=1}^{L} p_j p_k \langle i|\psi_j\rangle \langle \psi_j|\psi_k\rangle \langle \psi_k|i\rangle \\
&= \sum_{i=1}^{d} \sum_{j,k=1}^{L} p_j p_k \langle \psi_j|\psi_k\rangle \langle \psi_k|i\rangle \langle i|\psi_j\rangle \\
&= \sum_{j,k=1}^{L} p_j p_k |\langle \psi_j|\psi_k\rangle|^2 \leq \sum_{j,k=1}^{L} p_j p_k = 1,
\end{aligned} \tag{1.12}$$

where the equality holds if and only if $L = 1$, i.e. for pure states. □

It is proven in Section 1.1.3 that $\mathrm{Tr}(\hat{\rho}^2) \geq 1/d$, where the equality holds only for states proportional to the identity, i.e. for $\hat{\rho} = \mathbb{1}/d$. So, one can write

$$\frac{1}{d} \leq \mathrm{Tr}(\hat{\rho}^2) \leq 1. \tag{1.13}$$

Quantum states can be either pure or mixed, and in order to quantify *how much a given state is mixed*, the notion of *purity* has been defined.

Definition 1 (Purity [14])**.** The (standard) purity Π of a state $\hat{\rho}$ acting on a Hilbert space $\mathcal{H}$ of dimension d is

$$\Pi(\hat{\rho}) \equiv \frac{d\mathrm{Tr}(\hat{\rho}^2) - 1}{d - 1}.$$

Eq. (1.13) yields

$$0 \leq \Pi(\hat{\rho}) \leq 1 \tag{1.14}$$

with

$$\Pi(\hat{\rho}) = 0 \Leftrightarrow \hat{\rho} \text{ is pure} \tag{1.15}$$

and

$$\Pi(\hat{\rho}) = 1 \Leftrightarrow \hat{\rho} = \frac{1}{d} \sum_i |i\rangle \langle i| = \frac{1}{d}\mathbb{1}. \tag{1.16}$$

The latter state is called *maximally mixed state*.

For mixed states, the expectation value of an observable $\hat{A}$ is also given by $\mathrm{Tr}(\hat{A}\hat{\rho})$ and the evolution of $\hat{\rho}$ is given by Eq. (1.8) as well. So far, we wrote that a density operator is defined as in Eq. (1.9) and has, as properties, a unit trace and semi-definite positivity. Conversely, if an arbitrary operator is positive semi-definite and has a unit trace, it may represent a system in a probabilistic mixture of pure states.

Proof. Let $\hat{\rho}$ be a positive semi-definite operator with unit trace. Using the spectral decomposition of operators, one can write

$$\hat{\rho} = \sum_{i=1}^{d} \lambda_i \left|\phi_i\right\rangle \left\langle\phi_i\right|, \tag{1.17}$$

where λ_i $(i = 1, \ldots, d)$ and $\left|\phi_i\right\rangle$ $(i = 1, \ldots, d)$ are respectively the eigenvalues and eigenvectors of $\hat{\rho}$. The right hand side of Eq. (1.17) is indeed a density operator since

$$\langle\chi|\hat{\rho}|\chi\rangle \geq 0 \quad \forall \left|\chi\right\rangle \in \mathcal{H} \Rightarrow \langle\phi_i|\hat{\rho}|\phi_i\rangle = \lambda_i \langle\phi_i|\phi_i\rangle = \lambda_i \geq 0, \quad \forall i \tag{1.18}$$

and

$$\mathrm{Tr}(\hat{\rho}) = 1 \Leftrightarrow \sum_i \lambda_i \mathrm{Tr}(\left|\phi_i\right\rangle \left\langle\phi_i\right|) = 1 \Leftrightarrow \sum_i \lambda_i \langle\phi_i|\phi_i\rangle = 1 \Leftrightarrow \sum_i \lambda_i = 1. \tag{1.19}$$

Therefore, $\hat{\rho}$ represents a system in a probabilistic mixture of pure states with a probability λ_i to be in the state $\left|\phi_i\right\rangle$, $\forall i$. It should be noted that some eigenvalues may be zero. The states represented by eigenvectors associated to zero eigenvalues are obviously not a part of the probabilistic mixture. □

1.1.3 Bloch representation of quantum states

The Bloch (sphere) representation of quantum states was first introduced in the context of quantum information theory, for two-dimensional systems. It turned out to be a very useful tool in this context and has thus been expanded to higher dimensional systems [15]. Respecting that, in this section we first introduce this representation for two-dimensional systems (*qubits*), then generalise the concept to arbitrary d-dimensional systems ($d \geq 2$, *qudits*).

Single qubit

In quantum information, the basic object is the *quantum bit*, shortened *qubit*. A qubit is a two-level (two-dimensional) system described by a state vector belonging to a Hilbert space $\mathcal{H}$ of dimension 2. First, let us focus on pure states. The state of a qubit can be written as

$$\left|\Psi\right\rangle = \alpha \left|0\right\rangle + \beta \left|1\right\rangle \tag{1.20}$$

with α, β being complex numbers that verify $|\alpha|^2 + |\beta|^2 = 1$ (called *normalisation condition*, since state vectors are normalised) and where $\{\left|0\right\rangle, \left|1\right\rangle\}$ is an orthonormal basis

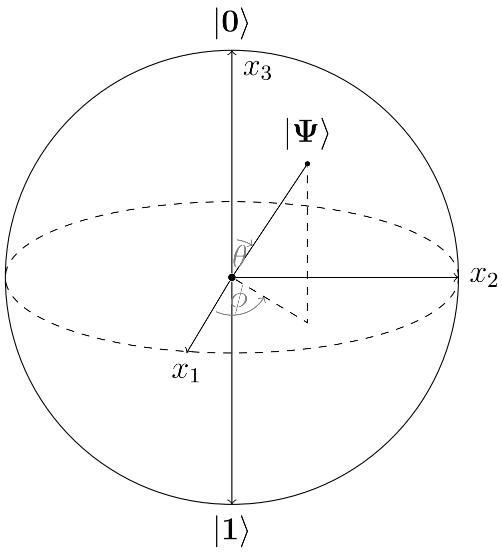

Figure 1.1: Bloch sphere

of the Hilbert space called *computational basis*[1]. From the normalisation condition, one can choose $\alpha = e^{i\phi_1}\cos\left(\frac{\theta}{2}\right)$ and $\beta = e^{i\phi_2}\sin\left(\frac{\theta}{2}\right)$ (with $\phi_1, \phi_2 \in [0, 2\pi[$ and $\theta \in [0, \pi]$ being real parameters) and therefore rewrite Eq. (1.20) as

$$\begin{aligned} |\Psi\rangle &= e^{i\phi_1}\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi_2}\sin\left(\frac{\theta}{2}\right)|1\rangle \\ &= e^{i\phi_1}\left[\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right] \end{aligned} \tag{1.21}$$

with $\phi = \phi_2 - \phi_1$, and since two quantum states are indistinguishable from one another if they only differ from a global phase [13], one can write

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle . \tag{1.22}$$

The real parameters θ and ϕ can be interpreted as angles and can therefore be used to represent the state $|\Psi\rangle$ on a 3D-sphere of radius 1. This sphere is called *Bloch sphere* and is represented in Figure 1.1. The vector $(1; \theta; \phi)$ is called *Bloch vector* (or *coherence vector*) of the state $|\Psi\rangle$.

As we have seen in Section 1.1.2, some quantum systems cannot be described by state vectors and hence all qubits states cannot be written as in Eq. (1.20). Let us consider a qubit in a mixed state described by a density operator $\hat{\rho}$. This density operator can be written as (see Ref. [16] or see single qudit case hereafter)

$$\hat{\rho} = \frac{1}{2}\mathbb{1} + \frac{1}{2}\mathbf{r}\cdot\hat{\sigma}, \tag{1.23}$$

[1] In a two-dimensional Hilbert space $\mathcal{H}$, the computational basis states are $|0\rangle$ and $|1\rangle$. They form an orthogonal basis of $\mathcal{H}$, $\{|0\rangle, |1\rangle\}$. This basis generalises to d-dimensional Hilbert spaces as $\{|i\rangle : i = 0, \ldots, d-1\}$, where all basis states are orthogonal. Then, for N-partite systems with Hilbert space $\mathcal{H}_{tot} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and with d_n, dimension of $\mathcal{H}_n$ ($n = 1, \ldots, N$), the computational basis reads $\{|i_1, \ldots, i_N\rangle \equiv |\mathbf{i}\rangle : i_n = 0, \ldots, d_n - 1 : n = 1, \ldots, N\}$, again with an orthonormality relation between all basis elements [16].

where $\hat{\sigma} = (\hat{\sigma}_1; \hat{\sigma}_2; \hat{\sigma}_3)$ is a vector of operators containing the three Pauli operators (see Appendix A.2.1) and where $\mathbf{r} \in \mathbb{R}^3$ is a real vector called the Bloch vector of the state $\hat{\rho}$. We note that together with the identity operator, the Pauli operators from a basis of $\mathcal{HS}(\mathcal{H})$ and therefore any hermitian operator of $\mathcal{HS}(\mathcal{H})$ can be written as a linear combination of these four operators. The decomposition of Eq. (1.23) is called Bloch representation of the state $\hat{\rho}$. The Pauli operators are traceless, which assures the unit trace of $\hat{\rho}$ in Eq. (1.23). However, in order to assure the positivity of $\hat{\rho}$ in Eq. (1.23), one needs to introduce the following condition on the Bloch vector $\mathbf{r}$:

$$|\mathbf{r}| \leq 1, \tag{1.24}$$

where the equality holds if and only if the state represented by the Bloch vector is pure.

Proof. Let $\hat{\rho}$ be the state of a qubit system and consider its Bloch representation as in Eq. (1.23). Since $\mathrm{Tr}(\hat{\rho}^2) \leq 1$, one has

$$\begin{aligned} \mathrm{Tr}(\hat{\rho}^2) =& \frac{1}{4}\mathrm{Tr}(\mathbb{1}) + \frac{1}{2}\sum_{i=1}^{3} r_i \mathrm{Tr}(\hat{\lambda}_i) + \frac{1}{4}\sum_{i,j=1}^{3} r_i r_j \underbrace{\mathrm{Tr}(\hat{\lambda}_i\hat{\lambda}_j)}_{2\delta_{ij}} \\ =& \frac{1}{2} + \frac{1}{2}\underbrace{\sum_{i=1}^{3} r_i^2}_{|\mathbf{r}|^2} \leq 1 \\ \Leftrightarrow |\mathbf{r}| \leq& 1, \end{aligned} \tag{1.25}$$

using $\hat{\sigma_i}^2 = \mathbb{1}$, $\forall i$, and the trace properties of the Pauli matrices, that is (see Appendix A.2.1)

$$\mathrm{Tr}(\hat{\sigma}_i) = 0, \quad \forall i \tag{1.26}$$

$$\mathrm{Tr}(\hat{\sigma}_i\hat{\sigma}_j) = 2\delta_{ij} \quad \forall i, j. \tag{1.27}$$

We note that for two-dimensional systems, the positivity condition is equivalent to $\mathrm{Tr}(\hat{\rho}^2) \leq 1$ [17]. For higher dimensional systems, this is no longer true. □

This means that pure states are on the surface of the Bloch sphere of Figure 1.1, whereas mixed states are inside the sphere. We call *Bloch-vector space* the set of all vectors with $|\mathbf{r}| \leq 1$, i.e. all vectors inside the ball delimited by the Bloch sphere.

Single qudit $(d \geq 2)$

Now, let us expand the notion of Bloch representation to qudits. First, we notice that the set of traceless Hermitian $SU(d)$ generators (see Appendix A.2)

$$\{\hat{\lambda}_i : i = 1, \ldots, d^2 - 1\} \tag{1.28}$$

is a set of $d^2 - 1$ linearly independent Hermitian operators, which verify the following orthogonality relation:

$$\mathrm{Tr}(\hat{\lambda}_i\hat{\lambda}_j) = 2\delta_{ij}. \tag{1.29}$$

Together with the identity operator $\mathbb{1}$, they form a basis of the real Hilbert space $\mathcal{HS}(\mathcal{H})$. Indeed, the set

$$\mathcal{B} = \{\mathbb{1}, \hat{\lambda}_i : i = 1, \ldots, d^2 - 1\} \tag{1.30}$$

is made of d^2 linearly independent Hermitian operators. Since the generators are traceless, they all are orthogonal to the identity operator,

$$\mathrm{Tr}(\mathbb{1}\hat{\lambda}_i) = 0 \quad \forall i = 1, \ldots, d^2 - 1. \tag{1.31}$$

Within this basis $\mathcal{B}$, every density operator $\hat{\rho}$ acting on $\mathcal{H}$ of dimension d can be decomposed as[2]

$$\hat{\rho} = \frac{1}{d}\mathbb{1} + \frac{1}{2}\sum_{i=1}^{d^2-1} r_i \hat{\lambda}_i, \tag{1.32}$$

where the real coefficients r_i are equal to $\mathrm{Tr}(\hat{\lambda}_i \hat{\rho})$. Indeed,

$$\mathrm{Tr}(\hat{\lambda}_i \hat{\rho}) = \frac{1}{d}\mathrm{Tr}(\hat{\lambda}_i) + \frac{1}{2}\sum_j r_j \mathrm{Tr}(\hat{\lambda}_i \hat{\lambda}_j) = r_i. \tag{1.33}$$

We notice that for any operator $\hat{A} \in \mathcal{HS}(\mathcal{H})$ developed in the basis $\mathcal{B}$, the coefficient in front of the identity operator is always equal to the trace of $\hat{A}$ divided by the dimension of $\mathcal{H}$ (here d).

The decomposition of Eq. (1.32) is the Bloch representation of the state $\hat{\rho}$ and the vector $\mathbf{r} = (r_1, ..., r_{d^2-1})$ is the Bloch vector of the state $\hat{\rho}$. Bloch vectors are real vectors, since their components are nothing but the expectation values of the generators of $SU(d)$, which are Hermitian operators. Note that the Bloch vector $\mathbf{r}$ completely characterises the state $\hat{\rho}$. There is a one-to-one correspondence between $\mathbf{r}$ and $\hat{\rho}$. When $d = 2$, one naturally finds Eq. (1.23).

From Eq. (1.32) it is easy to show that $\mathrm{Tr}(\hat{\rho}^2) \geq 1/d$. Indeed, one has

$$\begin{aligned} \mathrm{Tr}(\hat{\rho}^2) &= \frac{1}{d^2}\mathrm{Tr}(\mathbb{1}) + \frac{1}{d}\sum_i r_i \mathrm{Tr}(\hat{\lambda}_i) + \frac{1}{4}\sum_{i,j} r_i r_j \mathrm{Tr}(\hat{\lambda}_i \hat{\lambda}_j) \\ &= \frac{1}{d} + \frac{1}{2}|\mathbf{r}|^2 \geq \frac{1}{d}. \end{aligned} \tag{1.34}$$

This also shows that $\mathrm{Tr}(\hat{\rho}^2) = 1/d \Leftrightarrow \hat{\rho} = \mathbb{1}/d$.

Positivity condition

As stated above, every density operator of a qudit system can be written in the form of Eq. (1.32). However, the converse is not necessarily true, i.e. every operator of the form of Eq. (1.32) does not automatically represent a physical system. This derives from the conditions that operators have to verify in order to be density operators. The Hermiticity and unit trace conditions are immediately satisfied by Eq. (1.32), but the positivity condition adds restrictions to the Bloch vectors. For two-dimensional systems,

[2] We recall that $\hat{\rho} \in \mathcal{HS}(\mathcal{H})$.

the positivity condition is equivalent to $\mathrm{Tr}(\hat{\rho}^2) \leq 1$ [17], which has been proven to be equivalent to

$$|\mathbf{r}| \leq 1, \tag{1.35}$$

where the equality holds if and only if $\hat{\rho}$ is a pure state. This leads to the Bloch sphere (of radius 1), which contains all physical states of dimension 2 and thus is called Bloch-vector space. For d-dimensional systems, the condition expressed by Eq. (1.35) generalises as

$$|\mathbf{r}| \leq \sqrt{\frac{2(d-1)}{d}}. \tag{1.36}$$

Proof.

$$\begin{aligned}
\mathrm{Tr}(\hat{\rho}^2) =& \frac{1}{d^2}\mathrm{Tr}(\mathbb{1}) + \frac{1}{d}\sum_i r_i \mathrm{Tr}(\hat{\lambda}_i) + \frac{1}{4}\sum_{ij} r_i r_j \underbrace{\mathrm{Tr}(\hat{\lambda}_i\hat{\lambda}_j)}_{2\delta_{ij}} \\
=& \frac{1}{d} + \frac{1}{2}\underbrace{\sum_i r_i^2}_{|\mathbf{r}|^2} \leq 1 \\
\Leftrightarrow |\mathbf{r}| \leq& \sqrt{\frac{2(d-1)}{d}}
\end{aligned} \tag{1.37}$$

□

However, for $d > 2$, the condition $\mathrm{Tr}(\hat{\rho}^2) \leq 1$ is no longer equivalent to the positivity of $\hat{\rho}$ and therefore Eq. (1.36) is only a necessary condition on Bloch vectors. This means that the Bloch-vector space for d-dimensional systems ($d > 2$) is a subset of a ball of radius $\sqrt{2(d-1)/d}$, in which all vectors give rise to positive and traceless operators, i.e. density operators. The Bloch-vector space is given by the following theorem:

Theorem 1 (Bloch-vector space theorem [17])**.** *Let $\mathbf{r} = (r_1, \dots, r_{d^2-1})$ be the Bloch vector of a state $\hat{\rho} \in \mathcal{S}(\mathcal{H})$ and let $\{\hat{\lambda}_i : i = 1, \dots, d^2-1\}$ be the set of the $SU(d)$ generators. Let $a_i(\mathbf{r})$ be the coefficients of the characteristic polynomial*

$$\det(x\mathbb{1} - \hat{\rho}) = \det\left(\left(x - \frac{1}{d}\right)\mathbb{1} - \frac{1}{2}\sum_i r_i\hat{\lambda}_i\right).$$

The Bloch-vector space for a d-dimensional system is

$$B(\mathbb{R}^{d^2-1}) = \{\mathbf{r} \in \mathbb{R}^{d^2-1} | a_i(\mathbf{r}) \geq 0, \forall i = 1, \dots, d\}.$$

In order to represent a physical system, a vector $\mathbf{r}$ associated to an operator $\hat{\rho}$ through Eq. (1.32) has to be in $B(\mathbb{R}^{d^2-1})$, i.e. verify the d conditions $a_i(\mathbf{r}) \geq 0$. These conditions have been explicitly written down in Ref. [17] and read

$$a_1(\mathbf{r}) = 1 \geq 0, \tag{1.38}$$

$$a_2(\mathbf{r}) = \frac{1}{2!}\Big(1 - \mathrm{Tr}(\hat{\rho}^2)\Big) \geq 0, \tag{1.39}$$

$$a_3(\mathbf{r}) = \frac{1}{3!}\Big(1 - 3\mathrm{Tr}(\hat{\rho}^2) + 2\mathrm{Tr}(\hat{\rho}^3)\Big) \geq 0, \tag{1.40}$$

$$a_4(\mathbf{r}) = \frac{1}{4!}\Big(1 - 6\mathrm{Tr}(\hat{\rho}^2) + 8\mathrm{Tr}(\hat{\rho}^3) + 3\big(\mathrm{Tr}(\hat{\rho}^2)\big)^2 - 6\mathrm{Tr}(\hat{\rho}^4)\Big) \geq 0, \tag{1.41}$$

$$a_5(\mathbf{r}) = \dots \tag{1.42}$$

Note that the first inequality is trivially satisfied, which leaves us with $d-1$ inequalities and that second inequality is nothing else than $\text{Tr}(\hat{\rho}^2) \leq 1$ which is equivalent to Eq. (1.36). This set of inequalities can also be expressed using the Bloch vector $\mathbf{r}$ and the constants g_{ijk} of $SU(d)$ group given in Eq. (A.6) (again, see Appendix A.2). One gets

$$a_1(\mathbf{r}) = 1 \geq 0, \tag{1.43}$$

$$a_2(\mathbf{r}) = \frac{1}{2!}\left(\frac{d-1}{d} - \frac{1}{2}|\mathbf{r}|^2\right) \geq 0, \tag{1.44}$$

$$a_3(\mathbf{r}) = \frac{1}{3!}\left(\frac{(d-1)(d-2)}{d^2} - \frac{3(d-2)}{2d}|\mathbf{r}|^2 + \frac{1}{2}\sum_{ijk} g_{ijk}\hat{\lambda}_i\hat{\lambda}_j\hat{\lambda}_k\right) \geq 0, \tag{1.45}$$

$$\begin{aligned} a_4(\mathbf{r}) = \frac{1}{4!}\Bigg(&\frac{(d-1)(d-2)(d-3)}{d^3} - \frac{3(d-2)(d-3)}{d^3}|\mathbf{r}|^2 + \frac{3(d-2)}{4d}|\mathbf{r}|^4 \\ &+ \frac{2(d-2)}{d}\sum_{ijk} g_{ijk}\hat{\lambda}_i\hat{\lambda}_j\hat{\lambda}_k - \frac{3}{4}\sum_{ijklm} g_{ijk}g_{klm}\hat{\lambda}_i\hat{\lambda}_j\hat{\lambda}_l\hat{\lambda}_m\Bigg) \geq 0, \end{aligned} \tag{1.46}$$

$$a_5(\mathbf{r}) = \ldots \tag{1.47}$$

Note that a_i has meaning only for $i \leq d$. For $d = 2$, we obtain the norm condition on $\mathbf{r}$ leading to the Bloch sphere. For d-dimensional systems with $d > 2$, the structures generated by this set of inequalities are not symmetric [17]. It means that unlike the two-dimensional scenario where the Bloch-vector space is a ball, higher dimensional Bloch-vector spaces are non-trivial structures. We can conclude this section by saying that any operator of the form of Eq. (1.32) with $\mathbf{r} = (r_1, \ldots, r_{d-1})$ satisfying Eqs. (1.43) to (1.47) is a density operator, i.e. that it represents a physical system.

1.2 Bipartite systems

In this section, we define notions linked to *bipartite systems*, i.e. systems that are composed of two subsystems. In Section 1.2.4, we define bipartite entanglement, which is at the heart of this work.

Consider a bipartite system AB composed of two subsystems, A and B. The Hilbert space $\mathcal{H}_{AB}$ associated to the global system is given by the tensor product of the Hilbert spaces associated to the subsystems i.e.

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B, \tag{1.48}$$

where $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Hilbert spaces associated to the subsystems A and B, respectively. The dimension d_{AB} of $\mathcal{H}_{AB}$ is given by the product of the dimensions d_A of $\mathcal{H}_A$ and d_B of $\mathcal{H}_A$, $d_{AB} = d_A d_B$. These notations hold throughout the whole manuscript.

1.2.1 Partial trace

Let $\hat{\rho}$ be the state of the global system AB and let

$$\{|a_i\rangle : i = 1, \dots, d_A\}, \quad \{|b_k\rangle : k = 1, \dots, d_B\} \tag{1.49}$$

be orthonormal basis of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Hence the set of product states[3] $\{|a_i, b_k\rangle : i = 1, \dots, d_A; k = 1, \dots, d_B\}$ is a basis of $\mathcal{H}_{AB}$ [13]. The state $\hat{\rho}$ can be expanded as

$$\hat{\rho} = \sum_{i,j=1}^{d_A} \sum_{k,l=1}^{d_B} c_{ijkl} \, |a_i, b_k\rangle \langle a_j, b_l| \, , \tag{1.50}$$

where c_{ijkl} are complex coefficients. We define the *partial trace operators* Tr_B and Tr_A by their actions on the density operator,

$$\mathrm{Tr}_B(\hat{\rho}) \equiv \hat{\rho}^{(A)}, \quad \langle a_i|\hat{\rho}^{(A)}|a_j\rangle = \sum_{k=1}^{d_B} \langle a_i, b_k|\hat{\rho}|a_j, b_k\rangle \quad \forall i, j = 1, \dots, d_A \tag{1.51}$$

and

$$\mathrm{Tr}_A(\hat{\rho}) \equiv \hat{\rho}^{(B)}, \quad \langle b_k|\hat{\rho}^{(B)}|b_l\rangle = \sum_{i=1}^{d_A} \langle a_i, b_k|\hat{\rho}|a_i, b_l\rangle \quad \forall k, l = 1, \dots, d_B. \tag{1.52}$$

So, from Eq. (1.50), we get

$$\hat{\rho}^{(A)} = \sum_m \sum_{ijkl} c_{ijkl} \langle b_m|a_i, b_l\rangle \langle a_j, b_k|b_m\rangle = \sum_{ijm} c_{ijmm} \, |a_i\rangle \langle a_j| \, , \tag{1.53}$$

and analogously for $\hat{\rho}^{(B)}$. The operators $\hat{\rho}^{(A)}$ and $\hat{\rho}^{(B)}$ are positive and of trace 1, so they potentially represent states in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

Proof. The operators $\hat{\rho}^{(A)}$ and $\hat{\rho}^{(B)}$ are indeed positive, since

$$\begin{aligned}
&\langle \chi|\hat{\rho}|\chi\rangle \geq 0 \quad \forall \, |\chi\rangle \in \mathcal{H}_{AB} \Rightarrow \sum_m \langle \phi, b_m|\hat{\rho}|\phi, b_m\rangle \geq 0 \quad \forall \, |\phi\rangle \in \mathcal{H}_A \\
\Leftrightarrow &\sum_m \sum_{ijkl} c_{ijkl} \langle \phi, b_m|a_i, b_k\rangle \langle a_j, b_l|\phi, b_m\rangle \geq 0 \quad \forall \, |\phi\rangle \in \mathcal{H}_A \\
\Leftrightarrow &\sum_m \sum_{ijkl} c_{ijkl} \langle \phi|a_i\rangle \langle b_m|b_k\rangle \langle a_j|\phi\rangle \langle b_l|b_m\rangle \geq 0 \quad \forall \, |\phi\rangle \in \mathcal{H}_A \\
\Leftrightarrow &\sum_{ijm} c_{ijmm} \langle \phi|a_i\rangle \langle a_j|\phi\rangle \geq 0 \quad \forall \, |\phi\rangle \in \mathcal{H}_A \\
\Leftrightarrow &\langle \phi|\hat{\rho}^{(A)}|\phi\rangle \quad \forall \, |\phi\rangle \in \mathcal{H}_A
\end{aligned} \tag{1.54}$$

and analogously for $\hat{\rho}^{(B)}$. They are of trace 1,

$$\mathrm{Tr}(\hat{\rho}^{(A)}) = \sum_i \langle a_i|\hat{\rho}^{(A)}|a_i\rangle = \sum_i \sum_k \langle a_i, b_k|\hat{\rho}|a_i, b_k\rangle = \mathrm{Tr}(\hat{\rho}) = 1 \tag{1.55}$$

and analogously for $\hat{\rho}^{(B)}$. □

[3] A product state $|\psi\rangle$ is a state in $\mathcal{H}_{AB}$ that can be written as $|\psi\rangle = |\phi\rangle \otimes |\chi\rangle \equiv |\phi \otimes \chi\rangle = |\phi, \chi\rangle$, with $|\phi\rangle \in \mathcal{H}_A$ and $|\chi\rangle \in \mathcal{H}_B$.

The operators $\hat{\rho}^{(A)}$ and $\hat{\rho}^{(B)}$ are called the *reduced states* of the state $\hat{\rho}$. The point in these operators is that, for operators of the form $\hat{A} \otimes \mathbb{1}$ ($\mathbb{1} \otimes \hat{B}$), the subsystem A (B) can be seen as if it were in the state $\hat{\rho}^{(A)}$ ($\hat{\rho}^{(B)}$).

Proof. Let $\hat{\rho}$ be a bipartite density operator and let $\hat{\rho}^{(A)}$ be its partial trace on the second subsystem.

$$\begin{aligned} \mathrm{Tr}((\hat{A} \otimes \mathbb{1})\hat{\rho}) &= \sum_{ik} \langle a_i, b_k|(\hat{A} \otimes \mathbb{1})\hat{\rho}|a_i, b_k\rangle \\ &= \sum_{ik}\sum_{jl} \langle a_i, b_k|\hat{A} \otimes \mathbb{1}|a_j, b_l\rangle \langle a_j, b_l|\hat{\rho}|a_i, b_k\rangle \\ &= \sum_{ik}\sum_{jl} \langle a_i|\hat{A}|a_j\rangle \langle b_k|b_l\rangle \langle a_j, b_l|\hat{\rho}|a_i, b_k\rangle \\ &= \sum_{ij} \langle a_i|\hat{A}|a_j\rangle \sum_{k} \langle a_j, b_k|\hat{\rho}|a_i, b_k\rangle \\ &= \sum_{ij} \langle a_i|\hat{A}|a_j\rangle \langle a_j|\hat{\rho}^{(A)}|a_i\rangle \\ &= \sum_{i} \langle a_i|\hat{A}\hat{\rho}^{(A)}|a_i\rangle \\ &= \mathrm{Tr}(\hat{A}\hat{\rho}^{(A)}). \end{aligned} \tag{1.56}$$

This proof goes analogously for $\hat{\rho}^{(B)}$. □

We note that in general, $\hat{\rho} \neq \hat{\rho}^{(A)} \otimes \hat{\rho}^{(B)}$.

1.2.2 Bipartite Bloch representation

The Bloch representation introduced in Section 1.1.3 can be expanded to bipartite systems, and is widely used in separability criteria. Let us consider a density operator $\hat{\rho}$ associated to a $d_A \times d_B$ bipartite system AB. This operator acts on $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Let

$$\mathcal{B}_A = \left\{\mathbb{1}, \hat{\lambda}_i : i = 1, \ldots, d_A^2 - 1\right\} \quad \text{and} \quad \mathcal{B}_B = \left\{\mathbb{1}, \hat{\sigma}_j : j = 1, \ldots, d_B^2 - 1\right\} \tag{1.57}$$

be orthogonal basis of the real Hilbert spaces $\mathcal{HS}(\mathcal{H}_A)$ and $\mathcal{HS}(\mathcal{H}_B)$ respectively, with $\{\hat{\lambda}_i : i = 1, \ldots, d_A^2 - 1\}$, generators of $SU(d_A)$ and $\{\hat{\sigma}_j : j = 1, \ldots, d_B^2 - 1\}$, generators of $SU(d_B)$ as in Section 1.1.3 (see Appendix A.2). Then

$$\mathcal{B}_{AB} = \left\{\mathbb{1} \otimes \mathbb{1}, \hat{\lambda}_i \otimes \mathbb{1}, \mathbb{1} \otimes \hat{\sigma}_j, \hat{\lambda}_i \otimes \hat{\sigma}_j : i = 1, \ldots, d_A^2 - 1; j = 1, \ldots, d_B^2 - 1\right\} \tag{1.58}$$

is a basis of $\mathcal{HS}(\mathcal{H}_{AB})$[4]. One can develop $\hat{\rho}$ on $\mathcal{B}_{AB}$ as the following real linear combination

$$\hat{\rho} = \frac{1}{d_A d_B}\mathbb{1} \otimes \mathbb{1} + \frac{1}{2d_B}\sum_{i=1}^{d_A^2-1} r_i \hat{\lambda}_i \otimes \mathbb{1} + \frac{1}{2d_A}\sum_{j=1}^{d_B^2-1} s_j \mathbb{1} \otimes \hat{\sigma}_j + \frac{1}{4}\sum_{i=1}^{d_A^2-1}\sum_{j=1}^{d_B^2-1} \mathcal{T}_{ij}\hat{\lambda}_i \otimes \hat{\sigma}_j, \tag{1.59}$$

[4]We note that $\mathcal{HS}(\mathcal{H}_{AB}) = \mathcal{HS}(\mathcal{H}_A) \otimes \mathcal{HS}(\mathcal{H}_B)$. Indeed, both $\mathcal{HS}(\mathcal{H}_{AB})$ and $\mathcal{HS}(\mathcal{H}_A) \otimes \mathcal{HS}(\mathcal{H}_B)$ are of dimension $(d_A d_B)^2$ and one can easily show that $\mathcal{HS}(\mathcal{H}_A) \otimes \mathcal{HS}(\mathcal{H}_B) \subseteq \mathcal{HS}(\mathcal{H}_{AB})$.

where $r_i = \mathrm{Tr}\Big((\hat{\lambda}_i \otimes \mathbb{1})\hat{\rho}\Big)$, $s_j = \mathrm{Tr}\Big((\mathbb{1} \otimes \hat{\sigma}_j)\hat{\rho}\Big)$ and $\mathcal{T}_{ij} = \mathrm{Tr}\Big(\hat{\lambda}_i \otimes \hat{\sigma}_j)\hat{\rho}\Big)$. Indeed,

$$\begin{aligned}\mathrm{Tr}((\hat{\lambda}_i \otimes \mathbb{1})\hat{\rho}) =& \frac{1}{d_A d_B}\mathrm{Tr}(\hat{\lambda}_i \otimes \mathbb{1}) + \frac{1}{2d_B}\sum_{j=1}^{d_A^2-1} r_j \mathrm{Tr}(\hat{\lambda}_j\hat{\lambda}_i \otimes \mathbb{1}) \\ &+ \frac{1}{2d_A}\sum_{j=1}^{d_B^2-1} s_j \mathrm{Tr}(\hat{\lambda}_i \otimes \hat{\sigma}_j) + \frac{1}{4}\sum_{j=1}^{d_A^2-1}\sum_{k=1}^{d_B^2-1} \mathcal{T}_{jk}\mathrm{Tr}(\hat{\lambda}_j\hat{\lambda}_i \otimes \hat{\sigma}_k) \\ =& \frac{1}{2d_B}\sum_{j=1}^{d_A^2-1} r_j \mathrm{Tr}(\hat{\lambda}_j\hat{\lambda}_i)\mathrm{Tr}(\mathbb{1}) = r_i, \qquad \forall i = 1, \ldots, d_A^2 - 1,\end{aligned} \tag{1.60}$$

with analogous proofs for s_j and $\mathcal{T}_{ij}$ ($i = 1, \ldots, d_A^2 - 1; j = 1, \ldots, d_B^2 - 1$). Eq. (1.59) is the bipartite Bloch representation of the bipartite state $\hat{\rho}$. In general, the N-partite Bloch representations of N-partite systems[5] are easy to find, although tedious to write down for $N > 2$. The vectors $\mathbf{r} = (r_1, \ldots, r_{d_A^2-1})$ and $\mathbf{s} = (s_1, \ldots, s_{d_B^2-1})$ are the Bloch vectors of the reduced states of $\hat{\rho}$.

Proof. Keeping in mind that the generators of $SU(d)$ groups are traceless, one gets

$$\hat{\rho}^{(A)} = \mathrm{Tr}_B(\hat{\rho}) = \frac{1}{d_A}\mathbb{1} + \frac{1}{2}\sum_{i=1}^{d_A^2-1} r_i \hat{\lambda}_i \tag{1.61}$$

and

$$\hat{\rho}^{(B)} = \mathrm{Tr}_A(\hat{\rho}) = \frac{1}{d_B}\mathbb{1} + \frac{1}{2}\sum_{j=1}^{d_B^2-1} s_j \hat{\sigma}_j, \tag{1.62}$$

which is nothing but the Bloch representation of states of $\mathcal{S}(\mathcal{H}_A)$ and $\mathcal{S}(\mathcal{H}_B)$ respectively (see Eq. (1.32)). □

The matrix $\mathcal{T}$ of matrix elements $\mathcal{T}_{ij}$ accounts for the possible correlations between the subsystems and is therefore called the *correlation matrix.* Note that if $\mathcal{T} = 0$ then the state is separable, but the converse is not true.

Normal form

As we will see in Chapters 2 and 3, Bloch representations of density operators are widely used in separability criteria. However, Eq. (1.59) is somewhat heavy and can be lightened. Let $\hat{\rho}$ be a density operator associated to a $d_A \times d_B$ bipartite system AB in the form of Eq. (1.59). It has been shown in Ref. [18] that every full local rank density operator can be transformed into a density operator with maximally mixed subsystems[6], i.e., for

[5] Systems composed of N subsystems.

[6] All bipartite mixed states with non-full local ranks are either reducible to states with full local rank or entangled [19], so we will not investigate non-full local ranks density operators, for which the normal form does not always exist. Moreover, for non-full local ranks density operators, normal forms can be reached arbitrarily closely.

bipartite states,

$$\hat{\rho} \longrightarrow \hat{\tilde{\rho}} = \frac{1}{d_A d_B}\mathbb{1} + \frac{1}{4}\sum_{i=1}^{d_A^2-1}\sum_{j=1}^{d_B^2-1} \tilde{\mathcal{T}}_{ij}(\hat{\lambda}_i \otimes \hat{\sigma}_j). \tag{1.63}$$

This transformation is achieved through stochastic local operations assisted by classical communication, SLOCC[7] (also called local filtering operations). This is called the *filter normal form* (or simply *normal form*) of the state $\hat{\rho}$. The normal form is unique (up to local unitary transformations) and always non-zero for full-rank states [18]. In practice, one gets to the normal form using invertible operators $\hat{F}_A$ and $\hat{F}_B$ with determinants equal to one[8] and applying them to $\hat{\rho}$ accordingly to

$$\hat{\tilde{\rho}} = \frac{(\hat{F}_A \otimes \hat{F}_B)\hat{\rho}(\hat{F}_A \otimes \hat{F}_B)^\dagger}{\mathrm{Tr}\left((\hat{F}_A \otimes \hat{F}_B)\hat{\rho}(\hat{F}_A \otimes \hat{F}_B)^\dagger\right)}. \tag{1.64}$$

The transformation preserves the entanglement properties of the state $\hat{\rho}$ and from a separable decomposition of $\hat{\tilde{\rho}}$, it is easy to obtain a separable decomposition of $\hat{\rho}$.

Proof. First, we notice that Eq. (1.64) can be rewritten as

$$\hat{\rho} = \frac{(\hat{F}_A \otimes \hat{F}_B)^{-1}\hat{\tilde{\rho}}\left((\hat{F}_A \otimes \hat{F}_B)^\dagger\right)^{-1}}{\mathrm{Tr}\left((\hat{F}_A \otimes \hat{F}_B)^{-1}\hat{\tilde{\rho}}\left((\hat{F}_A \otimes \hat{F}_B)^\dagger\right)^{-1}\right)}. \tag{1.65}$$

We recall that if $\hat{\tilde{\rho}}$ is separable, it can be written as in Definition 5

$$\hat{\tilde{\rho}} = \sum_i \tilde{p}_i \hat{\tilde{\rho}}_i^{(A)} \otimes \hat{\tilde{\rho}}_i^{(B)} \tag{1.66}$$

with $\hat{\tilde{\rho}}_i^{(A)}$ and $\hat{\tilde{\rho}}_i^{(B)}$ being pure states. So, if $\hat{\tilde{\rho}}$ is separable, then

$$\begin{aligned}
\hat{\rho} &= \frac{\sum_i \tilde{p}_i \left(\hat{F}_A^{-1}\hat{\tilde{\rho}}_i^{(A)}(\tilde{F}_A^\dagger)^{-1}\right) \otimes \left(\hat{F}_B^{-1}\hat{\tilde{\rho}}_i^{(B)}(\tilde{F}_B^\dagger)^{-1}\right)}{\sum_j \tilde{p}_j \mathrm{Tr}\left(\hat{F}_A\hat{\tilde{\rho}}_j^{(A)}(\hat{F}_A^\dagger)^{-1}\right)\mathrm{Tr}\left(\hat{F}_B\hat{\tilde{\rho}}_j^{(B)}(\hat{F}_B^\dagger)^{-1}\right)} \\
&= \sum_i \frac{\tilde{p}_i \mathrm{Tr}(A_i)\mathrm{Tr}(B_i)}{\sum_j \tilde{p}_j \mathrm{Tr}(A_j)\mathrm{Tr}(B_j)} \hat{\rho}_i^A \otimes \hat{\rho}_i^B \\
&= \sum_i p_i \hat{\rho}_i^A \otimes \hat{\rho}_i^B
\end{aligned} \tag{1.67}$$

where we defined

$$\hat{A}_i \equiv \hat{F}_A^{-1}\hat{\tilde{\rho}}_i^{(A)}(\tilde{F}_A^\dagger)^{-1}, \quad \hat{B}_i \equiv \hat{F}_B^{-1}\hat{\tilde{\rho}}_i^{(B)}(\tilde{F}_B^\dagger)^{-1}, \tag{1.68}$$

[7] *Local operations* are operations of the form $\hat{A}\otimes\hat{B}$, i.e. an operation $\hat{A}$ on the first subsystem and an operation $\hat{B}$ on the second subsystem, that can be applied independently. The term *stochastic* means that these operations give unpredictable outcomes. Then, *assisted by classical communication* means that the results of the measures of $\hat{A}$ and $\hat{B}$ can be communicated through classical communication. When the outcomes are predictable, these operations are simply called local operations assisted by classical communication, LOCC.

[8] There exist algorithms to obtain to these operators [18].

$$\hat{\rho}_i^A \equiv \frac{\hat{F}_A^{-1}\hat{\tilde{\rho}}_i^{(A)}(\tilde{F}_A^\dagger)^{-1}}{\mathrm{Tr}(\hat{A}_i)}, \quad \hat{\rho}_i^B \equiv \frac{\hat{F}_B^{-1}\hat{\tilde{\rho}}_i^{(B)}(\tilde{F}_B^\dagger)^{-1}}{\mathrm{Tr}(B_i)}, \tag{1.69}$$

$$p_i \equiv \frac{\tilde{p}_i \mathrm{Tr}(A_i)\mathrm{Tr}(B_i)}{\sum_j \tilde{p}_j \mathrm{Tr}(A_j)\mathrm{Tr}(B_j)}. \tag{1.70}$$

The decomposition of Eq. (1.67) is indeed the decomposition of a density operator since

$$\sum_i p_i = \sum_i \frac{\tilde{p}_i \mathrm{Tr}(A_i)\mathrm{Tr}(B_i)}{\sum_j \tilde{p}_j \mathrm{Tr}(A_j)\mathrm{Tr}(B_j)} = 1, \tag{1.71}$$

$$p_i \geq 0 \quad \forall i = 1, ..., L. \tag{1.72}$$

and the sets $\{\hat{\rho}_i^{(A)}\}$ and $\{\hat{\rho}_i^{(B)}\}$ are sets of unit trace and positive operators. □

Normal forms are often used to enhance separability criteria, hence are of a great use.

1.2.3 Schmidt decomposition

Let's consider a bipartite state vector $|\Psi\rangle \in \mathcal{H}_{AB}$ and without loss of any generality let us assume $d_A \leq d_B$. The state can be decomposed in an arbitrary basis $\{|a_i, b_k\rangle\}$ of $\mathcal{H}_{AB}$ as

$$|\Psi\rangle = \sum_{i=1}^{d_A}\sum_{k=1}^{d_B} c_{ik} |a_i, b_k\rangle, \tag{1.73}$$

where $c_{ik} = \langle a_i, b_k|\Psi\rangle$ are complex coefficients. For any state $|\Psi\rangle$, there exist two orthonormal basis $\{|u_i\rangle : i = 1, \ldots, d_A\}$ and $\{|v_k\rangle : k = 1, \ldots, d_B\}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively such that

$$|\Psi\rangle = \sum_{i=1}^{d_A} \sigma_i |u_i\rangle \otimes |v_i\rangle = \sum_{\substack{i=1:\\ \sigma_i \neq 0}}^{r} \sigma_i |u_i\rangle \otimes |v_i\rangle, \quad \sigma_i = \langle u_i, v_i|\Psi\rangle, \tag{1.74}$$

where the coefficients $\sigma_i \geq 0$ $(i = 1, \ldots, d_A)$ satisfy $\sum_i \sigma_i^2 = 1$ and are the singular values of the matrix C, with matrix elements c_{ij} of Eq. (1.73) [16]. These coefficients σ_i are called *Schmidt coefficients* of $|\Psi\rangle$. The number r of non-zero Schmidt coefficients is called *Schmidt rank* of $|\Psi\rangle$ and Eq. (1.74) is called *Schmidt decomposition* of $|\Psi\rangle$. The Schmidt rank $r \leq d_A$ gives an idea of the amount of entanglement of a system [16]. Indeed, pure product states are the only pure states with Schmidt rank of 1 and thus the Schmidt rank of a bipartite pure state can be used to determine whether the state is separable or entangled. We say that a pure state is *maximally entangled* if $r = d_A$ and $\sigma_i = 1/\sqrt{d_A}$ for all $i = 1, \ldots, d_A$. For bipartite mixed states, the notion of Schmidt rank generalises as the *Schmidt number*.

Definition 2 (Schmidt number [20])**.** The Schmidt number of a bipartite density operator $\hat{\rho}$ is the number k

$$k \equiv \min_{\{p_i, |\psi_i\rangle\}} \left\{ \max_i \left\{ r\big(\,|\psi_i\rangle\,\big) \right\} \right\}$$

where the minimum is taken over all decompositions $\sum_i p_i |\psi_i\rangle\langle\psi_i|$ of $\hat{\rho}$.

It follows from Definition 2 that for a pure state, its Schmidt rank and Schmidt number are equal. One can also write the Schmidt decomposition for density operators. For a density operator $\hat{\rho}$ acting on $\mathcal{H}_{AB}$ with dimension $d_A \times d_B$ (again assuming $d_A \leq d_B$), there exist two orthonormal basis $\{\hat{G}_i^{(A)} : i = 1, \ldots, d_A^2\}$ and $\{\hat{G}_k^{(B)} : k = 1, \ldots, d_B^2\}$ of $\mathcal{SH}(\mathcal{H}_A)$ and $\mathcal{SH}(\mathcal{H}_B)$ respectively such that [11]

$$\hat{\rho} = \sum_{i=1}^{(d_A)^2} \sigma_i \hat{G}_i^{(A)} \otimes \hat{G}_i^{(B)}, \quad \sigma_i = \langle \hat{G}_i^{(A)} \otimes \hat{G}_i^{(B)} \rangle_\rho \,, \tag{1.75}$$

where $\sigma_i \geq 0$ are the Schmidt coefficients of $\hat{\rho}$. Elements of $\{\hat{G}_i^{(A)}\}$ and $\{\hat{G}_k^{(B)}\}$ verify the following orthonormality relation:

$$\mathrm{Tr}\left(\hat{G}_i^{(A)} \hat{G}_j^{(A)}\right) = \mathrm{Tr}\left(\hat{G}_i^{(B)} \hat{G}_j^{(B)}\right) = \delta_{ij}. \tag{1.76}$$

Remark 1. The state $\hat{\rho}$ may be written as

$$\hat{\rho} = \sum_{i=1}^{d_A^2} \sum_{k=1}^{d_B^2} \mathcal{C}_{ik} \hat{A}_i \otimes \hat{B}_k, \tag{1.77}$$

where $\{\hat{A}_i \otimes \hat{B}_k : i = 1, \ldots, d_A^2; k = 1, \ldots, d_B^2\}$ is an orthonormal basis of $\mathcal{HS}(\mathcal{H}_{AB})$. The σ_is are the singular values of the matrix $\mathcal{C}$, with entries $\mathcal{C}_{ij}$ [20].

We note that the Schmidt decomposition of any state is easily computable. Indeed, it is done by applying a singular value decomposition on the matrix C for pure states, and on the matrix $\mathcal{C}$ for mixed states [16, 11].

1.2.4 Bipartite entanglement

In this subsection we define quantum entanglement for bipartite systems. We first focus on pure states, then expand the definitions to mixed states.

Pure states

As stated previously, a bipartite system can be described by a state vector $|\Psi\rangle$ belonging to the Hilbert space $\mathcal{H}_{AB}$, which can be written as in Eq. (1.73), i.e.

$$|\Psi\rangle = \sum_{i,j=1}^{d_A, d_B} c_{ij} \left| a_i, b_j \right\rangle, \tag{1.78}$$

where c_{ij} are complex coefficients, $\{|a_i\rangle : i = 1, ..., d_A\}$ is a basis of $\mathcal{H}_A$ and $\{|b_i\rangle : i = 1, ..., d_B\}$ is a basis of $\mathcal{H}_B$. The definition of entanglement of bipartite pure states is the following:

Definition 3 (Separable pure state)**.** A bipartite pure state $|\Psi\rangle \in \mathcal{H}_{AB}$ is called *separable* (or *product state*) if there exist two states $|\phi\rangle \in \mathcal{H}_A$ and $|\chi\rangle \in \mathcal{H}_B$ such that $|\Psi\rangle = |\phi\rangle \otimes |\chi\rangle$. Otherwise, the state is called *entangled.*

This definition is purely mathematical. From a physical point of view, if a bipartite state $|\Psi\rangle$ is separable, it means that its two subsystems are physically independent. A separable state $|\Psi\rangle$ can be produced locally using classical communication, i.e. produced by creating two states $|\psi\rangle$ and $|\chi\rangle$ independently. The global state is then $|\Psi\rangle = |\phi\rangle \otimes |\chi\rangle$. The two subsystems may only be classically correlated. So, if one makes a measure $\hat{A}$ the first subsystem and a measure $\hat{B}$ the second subsystem, i.e. a measure $\hat{A} \otimes \hat{B}$ on $|\Psi\rangle$, one gets

$$\langle \hat{A} \otimes \hat{B} \rangle_\Psi = \langle \Psi | \hat{A} \otimes \hat{B} | \Psi \rangle = \langle \phi, \chi | \hat{A} \otimes \hat{B} | \phi, \chi \rangle = \langle \phi | \hat{A} | \phi \rangle \langle \chi | \hat{B} | \chi \rangle = \langle \hat{A} \rangle_\phi \langle \hat{B} \rangle_\chi \,, \tag{1.79}$$

i.e. the measures on the two subsystems can only be classically correlated. On the other hand, if the bipartite state $|\Psi\rangle$ is entangled, the measurement outcomes are quantumly correlated. Since entanglement is a purely quantum feature, one says that the measurement outcomes are *quantumly* correlated (as opposed to classically correlated). The two subsystems are no longer physically independent and an interaction between them is needed in order to create the global state $|\Psi\rangle$.

Mixed states

A bipartite system in a mixed state is described by a density operator $\hat{\rho}$ acting on the Hilbert space $\mathcal{H}_{AB}$,

$$\hat{\rho} = \sum_{i=1}^{L} p_i \, |\psi_i\rangle \, \langle \psi_i| \,, \tag{1.80}$$

where $L > 0$ and where p_is are convex weights.

Definition 4 (Product state)**.** A bipartite mixed state represented by a density operator $\hat{\rho}$ acting on $\mathcal{H}_{AB}$ is called a *product state* if there exist two density operators $\hat{\rho}_A$ and $\hat{\rho}_B$ acting on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively such that $\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B$.

Definition 5 (Separable mixed states)**.** A bipartite mixed state represented by a density operator $\hat{\rho}$ acting on $\mathcal{H}_{AB}$ is called *separable* if there exist two sets of L pure density operators $\{\hat{\rho}_i^{(A)}\}$ and $\{\hat{\rho}_i^{(B)}\}$ and a set of convex weights $\{p_i > 0 : \sum_{i=1}^{L} p_i = 1\}$ such that $\hat{\rho} = \sum_{i=1}^{L} p_i \, \hat{\rho}_i^{(A)} \otimes \hat{\rho}_i^{(B)}$, i.e. if $\hat{\rho}$ can be written as a convex combination[9] of separable pure states. Otherwise, the state is called *entangled*.

From these definitions, we see that a product state is always a separable state.

1.3 Multipartite systems

In this section, we expand the definitions of quantum entanglement of bipartite systems to *multipartite systems*. A multipartite system is a system composed of N subsystems (called a *N-partite system*). The Hilbert space $\mathcal{H}_{tot}$ associated to the global system is the tensor product of the Hilbert spaces associated to all the parts of the multipartite system i.e.

$$\mathcal{H}_{tot} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N, \tag{1.81}$$

[9] A convex combination is a linear combination where all coefficients are non-negative and sum to one.

where $\mathcal{H}_n$ is the Hilbert space associated to the subsystem n ($n = 1, \ldots, N$). The dimension d_{tot} of $\mathcal{H}_{tot}$ is given by the product of the dimensions d_n of all $\mathcal{H}_n$, $d_{tot} = \prod_{n=1}^{N} d_n$.

1.3.1 Multipartite entanglement

Again, we first define entanglement for pure states and then expand the definitions to mixed states.

Pure states

A pure N-partite system can be described by a state vector belonging to the Hilbert space $\mathcal{H}_{tot}$

$$|\Psi\rangle = \sum_{i_1,\ldots,i_N=1}^{d_1,\ldots,d_N} c_{i_1,\ldots,i_N} \, |a_{i_1}^{(1)}\rangle \otimes \ldots \otimes |a_{i_N}^{(N)}\rangle \,, \tag{1.82}$$

where $c_{i_1,\ldots,i_N}$ are complex coefficients and where $\{|a_{i_n}^{(n)}\rangle \,; i = 1, \ldots, d_n\}$ is a basis of $\mathcal{H}_n$ ($n = 1, \ldots, N$). This state can be either *fully separable*, *m-separable* or entangled.

Definition 6 (Fully separable pure state)**.** A N-partite pure state $|\Psi\rangle \in \mathcal{H}_{tot}$ is called *fully separable* (or *product state*) if there exist N states $|\phi_n\rangle \in \mathcal{H}_n$ ($n = 1, \ldots, N$) such that $|\Psi\rangle = |\phi_1\rangle \otimes \ldots \otimes |\phi_N\rangle$.

If a state is not fully separable, it contains some entanglement, i.e. some (or all) of the subsystems are entangled.

Definition 7 (m-separable pure state)**.** A N-partite pure state $|\Psi\rangle \in \mathcal{H}_{tot}$ is called m-*separable* with respect to a given partition $\{I_1, \ldots, I_m\}$ ($1 < m < N$) where I_i are disjoint subsets of $I = \{1, \ldots, N\}$, $\bigcup_{j=1}^{k} I_j = I$ if there exists m states $|\phi_{I_i}\rangle \in \mathcal{H}_{I_i}$ such that $|\Psi\rangle = |\phi_{I_1}\rangle \otimes \ldots \otimes |\phi_{I_m}\rangle$.

Definition 8 (Entangled pure state)**.** A N-partite pure state is called *entangled* if it is neither fully-separable nor m-separable ($1 < m < N$).

Mixed states

In order to obtain the definitions of entanglement of a N-partite mixed state $\hat{\rho}$ acting on $\mathcal{H}_{tot}$, we generalise the definitions of entanglement for pure states using convex combinations.

Definition 9 (Fully separable mixed state)**.** A N-partite mixed state $\hat{\rho}$ acting on $\mathcal{H}_{tot}$ is called *fully separable* if $\hat{\rho}$ can be written as a convex combination of fully separable pure states.

Definition 10 (m-separable mixed state)**.** A N-partite mixed state $\hat{\rho}$ acting on $\mathcal{H}_{tot}$ is called m-*separable* with respect to a given partition if it can be written as a convex combination of m-separable (with respect to the same partition) pure states.

Finally, we have the definition of entangled mixed states.

Definition 11 (Entangled mixed states)**.** A N-partite mixed state is called *entangled* if it is neither fully-separable nor m-separable ($1 < m < N$).

Chapter 2

First separability criteria

In this chapter, we expose several relevant separability criteria that have historically been the first to appear in literature. We focus on the well-known separability criteria that are essential to the topic of quantum entanglement. So far, since the separability problem is still an open one, separability criteria are either necessary *and* sufficient, but not practical (i.e. redefinitions of entanglement) or are (easily) computable, but only necessary *or* sufficient. For each criterion, we explicitly state to which category it belongs. All criteria exposed here first focus on bipartite entanglement, and some are later generalised to multipartite entanglement. In this chapter, we try to present the separability criteria in chronological order, which makes it easier to highlight the links between them. In order to compare the different criteria, we define the following relative strengths: two separability criteria C_1 and C_2 can be

- equivalent: C_1 and C_2 detect exactly the same states;
- complementary: C_1 can detect states not detected by C_2 and vice versa;
- C_1 is stronger than C_2: C_1 can detect all states detected by C_2 and at least one more;
- C_1 is weaker than C_2: all states detected by C_1 are detected by C_2 and C_2 can detect at least one more.

Throughout this chapter, we mainly consider bipartite systems. Unless otherwise stated, the states representing the systems belong to a Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ of dimension $d_{AB} = d_A d_B$, with $\mathcal{H}_A$ ($\mathcal{H}_B$) of dimension d_A (d_B) being the Hilbert space associated to the first (the second) subsystem. We will also denote by $\{|a_i\rangle : i = 1, \ldots, d_A\}$ ($\{|b_k\rangle : k = 1, \ldots, d_B\}$) an arbitrary basis of $\mathcal{H}_A$ ($\mathcal{H}_B$). In the case of multipartite systems, systems are composed of N subsystems, each one associated to a d_n-dimensional Hilbert space $\mathcal{H}_n$ ($n = 1, \ldots, N$). The Hilbert space associated to the global multipartite system is $\mathcal{H}_{tot} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, of dimension $d_{tot} = d_1 \ldots d_N$.

2.1 Positive partial transpose criterion

The most remarkable entanglement criterion is probably the *positive partial transpose* (PPT) criterion (also called *Peres-Horodecki* criterion), due to its simplicity and efficiency. In order to write the criterion, we first need to introduce the operation of *partial* *transposition*[1]. Consider a bipartite system of two subsystems, namely A and B. Let us recall that any bipartite density operator $\hat{\rho}$ acting on $\mathcal{H}_{AB}$ can be written in the form of Eq. (1.50), that is to say

$$\hat{\rho} = \sum_{i,j} \sum_{k,l} c_{ijkl} \left| a_i, b_k \right\rangle \left\langle a_j, b_l \right| . \tag{2.1}$$

The partial transposition of $\hat{\rho}$ is an operation on the subsystem A or on the subsystem B. We denote the partial transposition on A (on B) by $\hat{\rho}^{T_A}$ ($\hat{\rho}^{T_B}$). These operations are defined by their actions on $\hat{\rho}$ as

$$\begin{aligned} \hat{\rho}^{T_A} &\equiv \sum_{i,j} \sum_{k,l} c_{jikl} \left| a_i, b_k \right\rangle \left\langle a_j, b_l \right| , \\ \hat{\rho}^{T_B} &\equiv \sum_{i,j} \sum_{k,l} c_{ijlk} \left| a_i, b_k \right\rangle \left\langle a_j, b_l \right| . \end{aligned} \tag{2.2}$$

We note that partial transpositions are related to the usual transposition operation through

$$\left(\hat{\rho}^{T_A}\right)^{T_B} = \hat{\rho}^{T}, \tag{2.3}$$

and that $\left(\hat{\rho}^{T_A}\right)^{T_A} = \hat{\rho} = \left(\hat{\rho}^{T_B}\right)^{T_B}$. Now, a state $\hat{\rho}$ is said to be PPT (or that it has a positive partial transpose) when its partial transposition is positive semi-definite, i.e.

$$\hat{\rho}^{T_A} \geq 0 \tag{2.4}$$

which is equivalent to $\hat{\rho}^{T_B} \geq 0$. The PPT criterion reads

Theorem 2 (PPT criterion [10])**.** *If a state $\hat{\rho}$ is separable, then it is PPT.*

In other words, if a state $\hat{\rho}$ is not PPT (called NPT, for *negative partial transpose*), then it is entangled. This simple criterion was first introduced by Peres in 1996 [21] as a necessary condition for the separability of a quantum state. In the same paper, he conjectured that it was also a sufficient condition. The same year, Horodecki *et al.* proved that it is indeed a necessary and sufficient condition for the separability of 2×2 and 2×3 systems, but not for higher dimensional systems (the condition is then only necessary, as in Theorem 2 [10]), making the PPT criterion an easy computable necessary criterion for separability. This means there exist PPT entangled states[2]. Thus, we may consider that the separability problem is reduced to checking whether a PPT state is entangled or separable. We note that in general, the knowledge of the whole density operator is needed in order to apply the PPT criterion.

[1]Not to be mistaken with the *partial trace*, introduced in Section 1.2.1.

[2]PPT entangled states are sometimes abusively referred to as *bound entangled* states. Bound entangled states are undistillable entangled states, i.e. states that are entangled but for which there exists no LOCC protocol that allows us to extract a maximally entangled state (see Ref.[15]) from them, or their copies [15]. All PPT entangled states are indeed bound entangled but the converse is not true, there exists NPT bound entangled states [15].

2.2 Entanglement witnesses

One way to characterise entanglement is to call on an *entanglement witness* [20, 11]. This term, first used by Terhal in 2000 [20], refers to operators that can detect entangled states. The separability criterion based on entanglement witnesses belongs to the first category of criteria, i.e. necessary and sufficient criteria but not practical. An important property of this criterion is that it does not require the knowledge of the whole density operator, but only the expectation value of one operator, the entanglement witness. We fist discuss these operators for bipartite entanglement, then analyse their relations with positive maps and finally briefly discuss multipartite entanglement.

2.2.1 Definitions and properties

In order to accurately define entanglement witnesses, we first need to introduce the concept of *block-positive operators*

Definition 12 (Ref. [20]). A Hermitian operator $\hat{W}$ acting on $\mathcal{H}_{AB}$ is called *block-positive* if

$$\langle \phi \otimes \chi | \hat{W} | \phi \otimes \chi \rangle \geq 0 \quad \forall \, |\phi\rangle \otimes |\chi\rangle \in \mathcal{H}_{AB}.$$

From this definition, it is clear that if a Hermitian operator is positive, it is also block-positive. We can now properly define entanglement witnesses

Definition 13 (Ref. [20]). A Hermitian operator $\hat{W}$ acting on $\mathcal{H}_{AB}$ is called an *entanglement witness* if it is block-positive but not positive, i.e.

$$\begin{aligned} &\langle \phi \otimes \chi | \hat{W} | \phi \otimes \chi \rangle \geq 0 \quad \forall \, |\phi\rangle \otimes |\chi\rangle \in \mathcal{H}_{AB} \\ \exists \, |\psi\rangle \in \mathcal{H}_{AB} : \quad &\langle \psi | \hat{W} | \psi \rangle < 0. \end{aligned}$$

Definition 13 of entanglement witnesses is equivalent to the following definition:

Definition 14 (Ref. [11]). A Hermitian operator $\hat{W}$ acting on $\mathcal{H}_{AB}$ is called an *entanglement witness* if

$$\begin{aligned} \mathrm{Tr}(\hat{W}\hat{\rho}_s) \geq 0 \quad & \text{for all separable states } \hat{\rho}_s \in \mathcal{S}(\mathcal{H}_{AB}), \\ \mathrm{Tr}(\hat{W}\hat{\rho}_e) < 0 \quad & \text{for at least one entangled state } \hat{\rho}_e \in \mathcal{S}(\mathcal{H}_{AB}). \end{aligned}$$

Proof. Let $\{|jk\rangle\}$ be a basis of $\mathcal{H}_{AB}$, let $\hat{W}$ be an entanglement witness as in Definition 13 and let $\hat{\rho}_s = \sum_i p_i \, |\psi_i^{(A)}, \psi_i^{(B)}\rangle \langle \psi_i^{(A)}, \psi_i^{(B)}|$ be an arbitrary separable state. One has

$$\begin{aligned} \mathrm{Tr}(\hat{W}\hat{\rho}_s) &= \mathrm{Tr}\left(\hat{W} \sum_i p_i \, |\psi_i^{(A)}, \psi_i^{(B)}\rangle \langle \psi_i^{(A)}, \psi_i^{(B)}| \right) \\ &= \sum_i p_i \mathrm{Tr}\left(\hat{W} \, |\psi_i^{(A)}, \psi_i^{(B)}\rangle \langle \psi_i^{(A)}, \psi_i^{(B)}| \right) \\ &= \sum_i p_i \sum_{jk} \langle jk | \hat{W} | \psi_i^{(A)}, \psi_i^{(B)}\rangle \langle \psi_i^{(A)}, \psi_i^{(B)} | jk \rangle \\ &= \sum_i p_i \, \langle \psi_i^{(A)}, \psi_i^{(B)} | \hat{W} | \psi_i^{(A)}, \psi_i^{(B)}\rangle \\ &\geq 0 \quad \text{for all separable } \hat{\rho}_s \end{aligned} \tag{2.5}$$

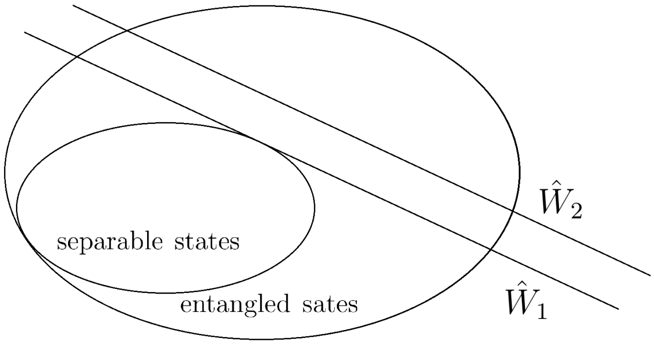

Figure 2.1: Set of all bipartite states $\mathcal{S}(\mathcal{H}_{AB})$ and 2 witnesses, $\hat{W}_1$ and $\hat{W}_2$.

and

$$\begin{aligned} \exists \hat{\rho}_e = |\psi\rangle\langle\psi| \in \mathcal{S}(\mathcal{H}_{AB}): \quad \mathrm{Tr}(\hat{W}|\psi\rangle\langle\psi|) &= \sum_{jk} \langle jk|\hat{W}|\psi\rangle\langle\psi|jk\rangle \\ &= \langle\psi|\hat{W}|\psi\rangle \\ &< 0 \end{aligned} \tag{2.6}$$

so from Eq. (2.5), $\hat{\rho}_e$ has to be entangled. □

Definition 14 makes it clear that, as stated above, entanglement witnesses refer to operators that can detect entangled states. Indeed, if there is an entanglement witness $\hat{W}$ such that $\mathrm{Tr}(\hat{W}\hat{\rho}) < 0$, then the state $\hat{\rho}$ is entangled. The strength of entanglement witnesses comes from the following theorem:

Theorem 3 (Ref. [10])**.** *A bipartite state $\hat{\rho}$ acting on $\mathcal{H}_{AB}$ is entangled if and only if there exists an entanglement witness $\hat{W}$ such that* $\mathrm{Tr}(\hat{W}\hat{\rho}) < 0$.

In other words, this necessary and sufficient criterion means that each entangled state can be detected by an entanglement witness. We recall that Hermitian operators acting on $\mathcal{H}_{AB}$ form a vector space, the Hilbert-Schmidt space $\mathcal{HS}(\mathcal{H}_{AB})$, provided with the Hilbert-Schmidt inner product $\langle\hat{A}|\hat{B}\rangle = \mathrm{Tr}(\hat{A}\hat{B})$, $\hat{A}, \hat{B} \in \mathcal{HS}(\mathcal{H}_{AB})$. Each Hermitian operator can be represented by a point in this vector space. The set of density operators $\mathcal{S}(\mathcal{H}_{AB}) \subset \mathcal{HS}(\mathcal{H}_{AB})$ in a convex hull in $\mathcal{HS}(\mathcal{H}_{AB})$, which means that for a given set of density operators $\{\hat{\rho}_i\}$ and convex weights p_is, the operator

$$\hat{\rho} = \sum_i p_i \hat{\rho}_i \tag{2.7}$$

is still a density operator. Then, the set of *separable* density operators is a convex hull in the set of density operators *itself*, as is represented in Figure 2.1. Therefore, $\mathrm{Tr}(\hat{W}\hat{\rho}) = \langle\hat{W}|\hat{\rho}\rangle = 0$ describes a hyperplane in $\mathcal{HS}(\mathcal{H}_{AB})$ [11]. With this representation, it is clear that for a certain entangled state, there is always a hyperplane (i.e. an entanglement witness) containing it. Of course, finding the witness that detect entanglement for a certain entangled state is not an easy task (otherwise the separability problem would be solved). Therefore, the separability problem may also be reduced to constructing the witnesses. Most witnesses are built using known entanglement criteria [11]. Indeed, if an entangled state is detected by a separability criterion, an entanglement witness also detecting it can usually easily be found (see e.g. Section 2.5). There are several advantages to having an entanglement witness for a given criterion. The main one is that, as we have seen before, detection of entanglement through an entanglement witness does not require the knowledge of the full density operator even if the criterion does.

Types of entanglement witnesses

Some widely encountered entanglement witnesses and their properties deserves to be introduced.

Definition 15 (Ref. [22])**.** An entanglement witness $\hat{W}$ is called *decomposable* if it can be expressed in the form

$$\hat{W} = \hat{X} + \hat{Y}^{T_B}$$

with $\hat{X}$ and $\hat{Y}$ being positive semi-definite operators.

Definition 16 (Ref. [20])**.** An entanglement witness $\hat{W}$ is called a k-*Schmidt witness* if

$$\begin{aligned} \mathrm{Tr}(\hat{W}\hat{\rho}_{k-1}) \geq 0 &\quad \text{for all } \hat{\rho}_{k-1} \in S_{k-1} \\ \mathrm{Tr}(\hat{W}\hat{\rho}_{k}) < 0 &\quad \text{for at least one } \hat{\rho}_{k} \in S_{k} \end{aligned}$$

where S_k is the set of states whose Schmidt numbers are smaller or equal to k, $\forall k \in \{1, \ldots, r_{max} = \min(d_A, d_B)\}$, with $S_1 \subset S_2 \subset \ldots \subset S_{r_{max}}$.

Using these entanglement witnesses, we have the following results [11]:

- a state $\hat{\rho}$ acting on $\mathcal{H}_{AB}$ has a Schmidt number greater or equal to 0 if and only if there exists a k-Schmidt witness $\hat{W}$ such that $\mathrm{Tr}(\hat{W}\hat{\rho}) < 0$. Since we know that Schmidt numbers are linked to entanglement, this is an interesting property;
- $\hat{W}$ is a decomposable entanglement witness if and only if $\mathrm{Tr}(\hat{W}\hat{\rho}) \geq 0$ for all PPT states $\hat{\rho}$;
- if $\hat{W}$ is an entanglement witness and if $\mathrm{Tr}(\hat{W}\hat{\rho}) < 0$ with $\hat{\rho}$ being a PPT state, then $\hat{W}$ is non-decomposable and $\hat{\rho}$ is an entangled PPT state;
- if $d_A d_B \leq 6$, then all entanglement witnesses are decomposable.

The second property implies that decomposable witnesses can only detect NPT states (but not all NPT states, which means that decomposable witnesses are weaker than the PPT criterion for the detection of entanglement). However, from the third property we can clearly see that non-decomposable entanglement witnesses are useful to detect PPT entangled states, thus states not detected by the PPT criterion. The last property is a consequence of the PPT criterion being necessary and sufficient for 2×2 and 2×3 systems, that is there exists no PPT entangled states for these systems. Indeed, if all entanglement witnesses are decomposable, then there exists no PPT state such that $\mathrm{Tr}(\hat{W}\hat{\rho}) < 0$, i.e. all entangled states are NPT.

Optimal witnesses

The role of entanglement witnesses is to detect entangled states. As we can see in Figure 2.1, $\hat{W}_1$ seems to be a 'better' witness than $\hat{W}_2$. This leads to the following definition:

Definition 17. An entanglement witness $\hat{W}_1$ is called *finer* than an entanglement witness $\hat{W}_2$ if $\hat{W}_1$ detects all entangled states detected by $\hat{W}_2$ and if it detects at least one more than $\hat{W}_2$.

Therefore, if

$$D_{\hat{W}_i} = \{\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB}) : \mathrm{Tr}(\hat{W}_i \hat{\rho}) < 0\} \quad i = 1, 2 \tag{2.8}$$

is the set of entangled states detected by $\hat{W}_1$ (respectively $\hat{W}_2$) and if $\hat{W}_1$ is finer than $\hat{W}_2$, then one has $D_{\hat{W}_2} \subset D_{\hat{W}_1}$. Moreover, $\hat{W}_1$ can be written as

$$\hat{W}_1 = \hat{W}_2 + \hat{P}, \tag{2.9}$$

where $\hat{P}$ is a positive operator. Eq. (2.9) leads to $\mathrm{Tr}(\hat{W}_1 \hat{\rho}) \leq \mathrm{Tr}(\hat{W}_2 \hat{\rho})$ and from that we can define an *optimal* entanglement witness.

Definition 18. An entanglement witness $\hat{W}$ is called *optimal* if there is no entanglement witness that is finer than $\hat{W}$, i.e. $\hat{X} = \hat{W} + \hat{P}$ is no longer a witness, for all positive operators $\hat{P}$.

Physically, optimal witnesses are 'the best detectors', i.e. there is no other witness that detect more entangled states [20]. Indeed, optimal witnesses are the closest to the set of separable states. Hence all entangled states can be detected only using optimal entanglement witnesses. From this definition, we see that for an optimal witness $\hat{W}$, one has [20]

$$\langle \phi \otimes \chi | \hat{W} | \phi \otimes \chi \rangle = 0 \quad \forall \, |\phi\rangle \otimes |\chi\rangle \in \mathcal{H}_{AB} \tag{2.10}$$

or equivalently that the set of product states $\{|\phi\rangle \otimes |\chi\rangle \in \mathcal{H}_{AB} : \langle \phi \otimes \chi | \hat{W} | \phi \otimes \chi \rangle = 0\}$ span the whole Hilbert space [11]. Eq. (2.10) is a necessary and sufficient condition for $\hat{W}$ to be an optimal witness [11]. From Eq. (2.10), one gets that for an optimal witness $\hat{W}$ there exist separable states $\hat{\rho}_s$ such that [20]

$$\mathrm{Tr}(\hat{W} \hat{\rho}_s) = 0. \tag{2.11}$$

However, this is obviously not a sufficient condition for $\hat{W}$ to be an optimal witness. Witnesses that satisfy Eq. (2.11) but are not optimal witnesses are called *weakly optimal* [20]. We note that given an entanglement witness, finding the corresponding optimal witness is not an easy task as it is shown in Ref. [11]. It has even been shown that it is equivalent to the separability problem: determining whether a given state is separable or not is equivalent to determining whether some entanglement witness in a higher-dimensional space is weakly optimal or not [11].

2.2.2 Link with positive maps

Entanglement witnesses have close relations with *positive* (but not *completely positive*) *linear maps*. Positive maps are also used to detect entanglement, but we do not investigate this in detail here. First, let us recall that a linear map

$$\Lambda : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B) \tag{2.12}$$

is called positive if it preserves positivity, i.e. $\hat{A} \geq 0 \Rightarrow \Lambda(\hat{A}) \geq 0$ for all $\hat{A} \in \mathcal{L}(\mathcal{H}_A)$.

Remark 2. A positive linear map Λ is called completely positive if the induced map $\Lambda \otimes \mathbb{1}_n : \mathcal{L}(\mathcal{H}_A) \otimes \mathcal{M}_n \to \mathcal{L}(\mathcal{H}_B) \otimes \mathcal{M}_n$ is positive for all n, with $\mathcal{M}_n$ being the set of $n \times n$ complex matrices [10].

The relation between entanglement witnesses and positive maps is given by the *Choi–Jamiołkowski isomorphism*, an isomorphism between linear maps from $\mathcal{L}(\mathcal{H}_A)$ to $\mathcal{L}(\mathcal{H}_B)$ and bipartite operators of $\mathcal{L}(\mathcal{H}_{AB})$. This means that for each operator $\hat{O}$ in $\mathcal{L}(\mathcal{H}_{AB})$, there exists a corresponding positive linear map Λ and vice versa (for more details, see Ref. [20]). The Choi–Jamiołkowski isomorphism is interesting for many reasons that are not discussed here but can be found in Refs. [20, 11]. Among these are the following properties [11], linked to entanglement witnesses:

- the map Λ is positive but not completely positive if an only if $\hat{L}$ is an entanglement witness,
- the map Λ is a decomposable map if and only if $\hat{L}$ is a decomposable entanglement witness.

Using these properties, entanglement witnesses can be used to construct positive maps. But if one takes an entanglement witness $\hat{W}$ and its corresponding positive map $\Lambda(\hat{W})$, it has been shown [11] that $\hat{W}$ detects a larger set of entangled states than $\Lambda(\hat{W})$.

2.2.3 Geometric entanglement witnesses

Entanglement witnesses that are of great use are the so-called *geometric entanglement witnesses*. They are defined as follows

Definition 19 (Geometric entanglement witnesses [23])**.** A geometric entanglement witness $\hat{W}_{\mathcal{G}}$ is an entanglement witness of the form

$$\hat{W}_{\mathcal{G}}(\hat{\rho}_1, \hat{\rho}_2) \equiv \hat{\rho}_1 - \hat{\rho}_2 - \mathrm{Tr}\big(\hat{\rho}_1(\hat{\rho}_1 - \hat{\rho}_2)\big)\mathbb{1},$$

with $\hat{\rho}_1, \hat{\rho}_2 \in \mathcal{S}(\mathcal{H}_{AB})$ and $\hat{\rho}_1 \neq \hat{\rho}_2$.

From this definition, one gets

$$\begin{aligned} \mathrm{Tr}(\hat{W}_{\mathcal{G}}(\hat{\rho}_1, \hat{\rho}_2)\,\hat{\rho}_1) &= \mathrm{Tr}(\hat{\rho}_1^2) - \mathrm{Tr}(\hat{\rho}_2\hat{\rho}_1) - \mathrm{Tr}\big(\hat{\rho}_1(\hat{\rho}_1 - \hat{\rho}_2)\big) \\ &= 0 \end{aligned} \tag{2.13}$$

and

$$
\begin{aligned}
\mathrm{Tr}(\hat{W}_{\mathcal{G}}(\hat{\rho}_1,\hat{\rho}_2)\,\hat{\rho}_2) &= \mathrm{Tr}(\hat{\rho}_1\hat{\rho}_2) - \mathrm{Tr}(\hat{\rho}_2^2) - \mathrm{Tr}\big(\hat{\rho}_1(\hat{\rho}_1-\hat{\rho}_2)\big) \\
&= 2\mathrm{Tr}(\hat{\rho}_1\hat{\rho}_2) - \mathrm{Tr}(\hat{\rho}_2^2) - \mathrm{Tr}(\hat{\rho}_1^2) \\
&= -\mathrm{Tr}\big((\hat{\rho}_1-\hat{\rho}_2)^2\big) \\
&= -||\hat{\rho}_1-\hat{\rho}_2||^2_{\mathrm{HS}} \\
&< 0.
\end{aligned} \tag{2.14}
$$

This means that $\hat{\rho}_2$ has to be entangled to assure that $\hat{W}_{\mathcal{G}}$ is an engagement witness. We note that it is not trivial to find a $\hat{\rho}_1$ such that $\mathrm{Tr}(\hat{W}_{\mathcal{G}}(\hat{\rho}_1,\hat{\rho}_2)\,\hat{\rho}_s) \geq 0$ for all separable states $\hat{\rho}_s \in \mathcal{S}(\mathcal{H}_{AB})$. Since $\hat{W}_{\mathcal{G}}$ is an entanglement witness, the hyperplane in $\mathcal{HS}(\mathcal{H}_{AB})$ given by $\mathrm{Tr}(\hat{W}_{\mathcal{G}}\hat{\rho}) = 0$ divides $\mathcal{S}(\mathcal{H}_{AB})$ into two regions, one with $\mathrm{Tr}(\hat{W}_{\mathcal{G}}\hat{\rho}) \geq 0$ that contains all separable states and one with $\mathrm{Tr}(\hat{W}_{\mathcal{G}}\hat{\rho}) < 0$ where all states are entangled. The point in geometric entanglement witnesses comes from the following theorems:

Theorem 4 (Ref. [23])**.** *All entangled states can be detected by a geometric entanglement witness.*

Theorem 5 (Ref. [23])**.** *If*

$$\hat{W}^{\lambda}_{\mathcal{G}} \equiv \hat{\rho}_\lambda - \hat{\rho} - \mathrm{Tr}\big(\hat{\rho}_\lambda(\hat{\rho}_\lambda - \hat{\rho})\mathbb{1}$$

is a geometric entanglement witness with a parametrized family of states

$$\hat{\rho}_\lambda \equiv \lambda\hat{\rho} - (1-\lambda)\hat{\rho}', \quad \lambda \in [\lambda_i, 1[\,(\lambda_i \geq 0),\ \hat{\rho},\hat{\rho}' \in \mathcal{S}(\mathcal{H}_{AB}),$$

then state $\hat{\rho}_\lambda$ is entangled for $\lambda \in\,]\lambda_i, 1]$.

The latter theorem is particularly efficient when used in the following way: taking a known separable state and a known entangled state as $\hat{\rho}'$ and $\hat{\rho}$ of Theorem 5 respectively. Then, it is possible to detect entangled states along the line in $\mathcal{HS}(\mathcal{H}_{AB})$ between $\hat{\rho}$ and $\hat{\rho}'$ by 'shifting' the geometric entanglement witness $\hat{W}^{\lambda}_{\mathcal{G}}$, as we can see on Figure 2.2. Other methods build on geometric entanglement witness exist, see e.g. [23] for examples. By choosing a PPT entangled state for $\hat{\rho}$, this method allows to find PPT entangled states, which is of great interest [23].

2.2.4 Multipartite entanglement

Entanglement witnesses can be generalised to the multipartite scenario (with states belonging to a Hilbert space $\mathcal{H}_{tot} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$). Of course, since there are different types of multipartite entanglements, there are different types of entanglement witnesses. The natural generalisation of Definition 13 using Definition 10 of multipartite entanglement is

Definition 20 (Ref. [20])**.** An operator $\hat{W}$ acting on $\mathcal{H}_{tot}$ is called a multipartite entanglement witness for a partition $\{I_1, \ldots, I_m\}$ if

$$\langle \phi_{I_1} \otimes \cdots \otimes \phi_{I_m} | \hat{W} | \phi_{I_1} \otimes \cdots \otimes \phi_{I_m} \rangle \geq 0 \quad \forall\, |\phi_{I_l}\rangle \in \mathcal{H}_{I_l},$$

where $\mathcal{H}_{I_l}$ are subsets of $\mathcal{H}_{AB}$.

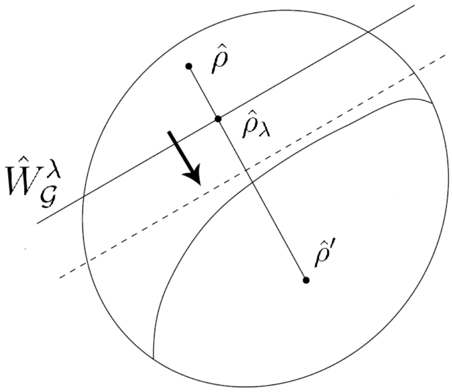

Figure 2.2: Illustration of a detection method in the set of all bipartite states $\mathcal{S}(\mathcal{H}_{AB})$ [23].

Remark 3. A state $\hat{\rho}_e$ such that $\mathrm{Tr}(\hat{W}\hat{\rho}_e) < 0$ cannot be written as $\hat{\rho}_e = \sum_i p_i \left|\phi_{I_1}^{(i)} \otimes \cdots \otimes \phi_{I_m}^{(i)}\right\rangle \left\langle\phi_{I_1}^{(i)} \otimes \cdots \otimes \phi_{I_m}^{(i)}\right|$ and therefore is not m-separable with respect to the partition $\{I_1, \ldots, I_m\}$.

Witnesses that detect all non-fully separable operators satisfy

$$\langle \phi_1 \otimes \cdots \otimes \phi_N | \hat{W} | \phi_1 \otimes \cdots \otimes \phi_N \rangle \geq 0 \quad \forall \, |\phi_j\rangle \in \mathcal{H}_j, j = 1, \ldots, N. \tag{2.15}$$

We notice that the definition of entanglement witnesses generalises quite naturally and therefore the rest of the theory also does. More details can be found in Refs. [20, 11, 12].

2.3 Entanglement measures

Entanglement measures are real functions of states that quantify 'how much entanglement' is contained in a given state [12]. In order to be an entanglement measure for bipartite states, a function $E : \mathcal{S}(\mathcal{H}_{AB}) \to \mathbb{R} : \hat{\rho} \mapsto E(\hat{\rho})$ has to verify some properties.

Vanishing on separable states

Since entanglement measures have to quantify entanglement of a given state, it seems natural to impose that an entanglement measure vanishes for separable states, i.e. that $E(\hat{\rho}_s) = 0$ for all separable $\hat{\rho}_s$ acting on $\mathcal{H}_{AB}$. This means that entanglement measures lead to necessary criteria for separability.

Monotonicity

Then, it is known that entanglement cannot be created or increased by any LOCC procedure [15]. In particular, starting with two separated states, one can never obtain an entangled states via a LOCC procedure. Therefore, we do not allow $E(\hat{\rho})$ to increase if a LOCC operation $\hat{\Lambda}$ is applied to $\hat{\rho}$, i.e. $E\left(\hat{\Lambda}(\hat{\rho})\right) \leq E(\hat{\rho})$ for all LOCC operations $\hat{\Lambda}$ acting on $\mathcal{H}_{AB}$ [12]. This property is called *monotonicity under LOCC*. Usually, entanglement measures satisfy an even stronger condition, that is

$$\sum_i p_i E(\hat{\rho}_i) \leq E(\hat{\rho}), \tag{2.16}$$

for all sets $\{p_i, \hat{\rho}_i\}$ such that $\sum_i p_i \hat{\rho}_i = \hat{\rho}$, with $\hat{\rho}_i$ being density operators of pure states [12]. Eq. (2.16) means that entanglement measures do not increase on average.

These two properties are the ones that are necessarily required, but other properties may also be verified by some entanglement measures [11, 12]. For instance, there is the property of *convexity*: most known entanglement measures are convex [11], which means that they verify

$$E\left(\sum_i p_i \hat{\rho}_i\right) \leq \sum_i p_i E(\hat{\rho}_i), \tag{2.17}$$

with p_is being convex weights and $\hat{\rho}_i \in \mathcal{S}(\mathcal{H}_{AB})$, $\forall i$.

2.3.1 Entropy of entanglement

A widely used entanglement measure in quantum information is *entropy of entanglement*, which is defined for pure bipartite states. It is defined using the *von Neumann entropy*, which is, for state $\hat{\rho} \in \mathcal{S}(\mathcal{H})$ [15]

$$S(\hat{\rho}) \equiv -\mathrm{Tr}\left(\hat{\rho}\log_2(\hat{\rho})\right) = -\sum_i \lambda_i \log_2(\lambda_i), \tag{2.18}$$

where $\hat{\rho} = \sum_i \lambda_i \left|\phi_i\right\rangle \left\langle\phi_i\right|$ is the spectral decomposition of $\hat{\rho}$, with λ_is being its eigenvalues and $\left|\phi_i\right\rangle$s being its eigenvectors. The von Neumann entropy has the following properties:

1. S is zero only for pure states,
2. S is maximal for maximally mixed states, i.e. for $\hat{\rho} = \frac{1}{d}\mathbb{1} \Rightarrow S(\hat{\rho}) = -\sum_i \frac{1}{d}\log_2\left(\frac{1}{d}\right) = \log_2(d)$,
3. S is concave, that is for a set of density operator $\{\hat{\rho}_i\}$ and convex weights p_is one has $S\left(\sum_i p_i \hat{\rho}_i\right) \geq \sum_i p_i S(\hat{\rho}_i)$,
4. $S(\hat{\rho}^{(A)} \otimes \hat{\rho}^{(B)}) = S(\hat{\rho}^{(A)}) + S(\hat{\rho}^{(B)})$, with $\hat{\rho}^{(A,B)} \in \mathcal{S}(\mathcal{H}_{A,B})$,
5. $S(\hat{\rho}) \leq S(\hat{\rho}^{(A)}) + S(\hat{\rho}^{(B)})$ with $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB})$ and $\hat{\rho}^{(A,B)} \in \mathcal{S}(\mathcal{H}_{A,B})$, the partial traces of $\hat{\rho}$.

The last property means that the entropy of the subsystems is greater than the entropy of the entire system, this may be seen as a signature of entanglement [15]. We can now define the entropy of entanglement for any pure state $\hat{\rho} = \left|\Psi\right\rangle \left\langle\Psi\right| \in \mathcal{S}(\mathcal{H}_{AB})$

$$E_{\mathcal{E}}(\hat{\rho}) \equiv S(\hat{\rho}^{(A)}) = S(\hat{\rho}^{(B)}), \tag{2.19}$$

with $\hat{\rho}^{(A)} \in \mathcal{S}(\mathcal{H}_A)$ ($\hat{\rho}^{(B)} \in \mathcal{S}(\mathcal{H}_B)$), the partial trace $\hat{\rho}^{T_B}$ ($\hat{\rho}^{T_A}$) of $\hat{\rho}$. The equality between $S(\hat{\rho}^{(A)})$ and $S(\hat{\rho}^{(B)})$ comes from the fact that $\hat{\rho}^{(A)}$ and $\hat{\rho}^{(B)}$ have the same eigenvalues [16]. The entropy of entanglement cancels for separable states.

Proof. Let $\hat{\rho}_s \in \mathcal{S}(\mathcal{H})$ be a pure separable state. It can be written as $\hat{\rho}_s = |\phi, \chi\rangle\langle\phi, \chi|$ and one has

$$E_{\mathcal{E}}(\hat{\rho}_s) = S(|\phi\rangle\langle\phi|) = S(|\chi\rangle\langle\chi|) = 0, \tag{2.20}$$

due to the first property of S. $\square$

Many entanglement measures reduce to the entropy of entanglement when they are computed for pure states.

Remark 4. The von Neumann entropy does not verify the conditions to be an entanglement measure (it does not vanish for separable states), it may however be directly used to detect entanglement. Indeed, if

$$S(\hat{\rho}) < S(\hat{\rho}^{(A)}), \tag{2.21}$$

holds with $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB})$ and $\hat{\rho}^{(A)} = \hat{\rho}^{T_B}$, then the state $\hat{\rho}$ in entangled [15].

2.3.2 Negativity

An interesting entanglement measure is the *negativity* of a state. Indeed, this measure is easily computed for any mixed state, which is not always the case for entanglement measures. It is defined as [24]

$$\mathcal{N}(\hat{\rho}) \equiv \frac{||\hat{\rho}^{T_A}||_{\mathrm{Tr}} - 1}{2} = \frac{\sum_i |\lambda_i| - 1}{2} \quad \forall \hat{\rho} \in \mathcal{H}_{AB} \tag{2.22}$$

with $\{\lambda_i\}$ being the eigenvalues of $\hat{\rho}^{T_A}$. We recall that $||\cdot||_{\mathrm{Tr}}$ is the trace norm, which is equal to the sum of the singular values and equal to the sum of the absolute values of the eigenvalues for Hermitian operators. This quantity is equal to the absolute value of sum of the negative eigenvalues of $\hat{\rho}^{T_A}$, i.e.

$$\mathcal{N}(\hat{\rho}) = \left| \sum_{i:\lambda_i<0} \lambda_i \right| = \sum_{i:\lambda_i<0} |\lambda_i|. \tag{2.23}$$

Indeed, since $\mathrm{Tr}(\hat{\rho}^{T_A}) = 1$, the trace norm of $\hat{\rho}^{T_A}$ reads

$$||\hat{\rho}^{T_A}||_{\mathrm{Tr}} = \sum_i |\lambda_i| = \underbrace{\sum_i \lambda_i}_{\mathrm{Tr}(\hat{\rho}^{T_A})} + 2 \sum_{i:\lambda_i<0} |\lambda_i| = 1 + 2 \sum_{i:\lambda_i<0} |\lambda_i| \tag{2.24}$$

and therefore one has Eq. (2.23). It is clear that for separable states, the negativity is zero. It has been proven in Ref. [24] that negativity also verifies monotonicity under LOCC. The negativity can be seen as a measure of 'how much a given state is NPT', i.e. a quantification of the PPT criterion. Let us also notice that $\mathcal{N}$ verifies the convexity condition, due to the fact that it is defined using the trace norm which is a convex function [24].

Remark 5. Negativity can also be used to characterise entanglement of multipartite systems. For a three-party system, there are 6 quantities that can be defined on the basis of bipartite negativity. For a four-party system, there are 26, and so one (see Ref. [24]).

2.3.3 Convex roof construction

Some entanglement measures can be build using a *convex roof construction.* One starts by defining the measure for pure states, i.e. $E(\phi)$ for all pure states $|\phi\rangle \in \mathcal{H}_{AB}$, and then expands it to mixed states as follows

$$E(\hat{\rho}) \equiv \inf_{\{p_i, |\phi_i\rangle\}} \left\{ \sum_i p_i E(\phi_i) \right\} \tag{2.25}$$

where the infimum is taken over all sets $\{p_i, |\phi_i\rangle\}$ such that $\hat{\rho} = \sum_i p_i |\phi_i\rangle \langle\phi_i|$. It is obvious that a convex roof measure vanishes for separable states. Indeed, $E(\phi_s)$ vanishes for all separable pure states, which implies that for a separable mixed state $\hat{\rho}$ (i.e. a state that can be written as $\hat{\rho} = \sum_i p_i |\phi_s^{(i)}\rangle \langle\phi_s^{(i)}|$, with separable states $|\phi_s^{(i)}\rangle$), the entanglement measure $E(\hat{\rho})$ vanishes. It has also been proven that all convex roof measures verify monotonicity. Due to the infinitum, convex roof constructions are not always computable [12]. An example of a *computable* convex roof measure is *concurrence*, which is developed in the next section.

2.4 Concurrence criterion

As stated before, concurrence is an entanglement measure defined using convex roof construction. This measure was first introduced by Wootters in 1998 [25] for two-qubit systems, in order to prove a formula for *entanglement of formation*, $E_{\mathcal{F}}$, which is also built using a convex roof construction with entropy of entanglement

$$E_{\mathcal{F}}(\hat{\rho}) \equiv \inf_{\{p_i, |\phi_i\rangle\}} \left\{ \sum_i p_i E_{\mathcal{E}}\big(|\phi_i\rangle \langle\phi_i|\big) \right\}. \tag{2.26}$$

The rough physical interpretation of entanglement of formation $E_{\mathcal{F}}$ is, for bipartite pure states, the number of qubits that must have been exchanged by two observers in order to obtain a given state. In the same paper, Wootters derived a necessary and sufficient criterion for the separability of two qubits based on the concurrence. Therefore, concurrence turned out to be interesting on its own and has been expanded to multipartite systems by introducing *generalised concurrences.* This leads to a necessary and sufficient criterion for the separability of multipartite pure states, and to a necessary criterion for the mixed state scenario.

2.4.1 Two-qubit concurrence

First, let us focus on Wootters's two-qubit concurrence. This entanglement measure is defined for two-qubit systems, i.e. 2×2 systems. The Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ is of dimension 4, with $\mathcal{H}_A$ and $\mathcal{H}_B$ both of dimension 2. Let us introduce the *spin flip* operation for pure states of single qubits, which is

$$|\tilde{\Psi}\rangle \equiv \hat{\sigma}_2 |\Psi^*\rangle\,, \quad |\Psi\rangle \in \mathcal{H}_A \tag{2.27}$$

where $|\Psi\rangle$ is expressed in the computational basis and where $\hat{\sigma}_2$ is the second Pauli operator(see Appendix A.2.1). This operation is generalisable to N-partite systems of qubits, by applying it to each individual qubit. Thus, for bipartite systems of qubits the spin-flipped state reads

$$\hat{\tilde{\rho}} = (\hat{\sigma}_2 \otimes \hat{\sigma}_2)\hat{\rho}^*(\hat{\sigma}_2 \otimes \hat{\sigma}_2), \quad \hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB}). \tag{2.28}$$

The two-qubit concurrence for bipartite pure states is defined as

$$C(|\Psi\rangle) \equiv |\langle\Psi|\tilde{\Psi}\rangle| = |\langle\Psi|\hat{\sigma}_2 \otimes \hat{\sigma}_2|\Psi^*\rangle|, \quad \forall\, |\Psi\rangle \in \mathcal{H}_{AB} \tag{2.29}$$

and for mixed states, it is defined using the convex roof construction. Interestingly, this entanglement measure can be analytically computed, even for mixed states, which is not always the case for convex roof constructions, as stated in the previous section. Here, the convex roof construction simplifies to

$$C(\hat{\rho}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \quad \forall \hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB}), \tag{2.30}$$

where $\{\lambda_i\}$ is the set of the square roots of the eigenvalues of $\hat{\rho}\hat{\tilde{\rho}}$, sorted in the decreasing order[3], all non-negative [25]. With the two-qubit concurrence comes the following necessary and sufficient criterion for separability:

Theorem 6 (Two-qubit concurrence criterion [25])**.** *A 2×2 system represented by a density operator $\hat{\rho}$ acting on a Hilbert space $\mathcal{H}_{AB}$ is separable if and only if its concurrence is zero, i.e.*

$$\hat{\rho} \textit{ is separable} \Leftrightarrow C(\hat{\rho}) = 0.$$

Remark 6. The two-qubit concurrence for pure states can also be defined as $C(\psi) \equiv \sqrt{2\,(1 - \mathrm{Tr}(\hat{\rho}_A^2))}$, where $\hat{\rho}_A$ is the partial trace of $\hat{\rho}$ over the second subsystem, and then again using a convex roof construction for mixed states [11]. This definition holds for all bipartite systems, not only 2×2 systems.

2.4.2 Generalised concurrences

When it comes to general bipartite or multipartite systems, one needs to use a set of *generalised concurrences* C_α ($\alpha \in \mathbb{N}_0$). These concurrences are defined for pure states as in Eq. (2.29) but with an appropriated generalised spin flip operation for each α. Then, again, this definition is expanded for mixed states using a convex roof construction. Using this set of generalised concurrences, one can obtain a separability criterion for both pure and mixed states. We first focus on pure states and then expand our analyse to mixed states.

Pure states

Let $|\Psi\rangle$ be some state in the Hilbert space $\mathcal{H}_{tot} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$. This state can be written in the computational basis as

$$|\Psi\rangle = \sum_{\mathbf{i}} a_{\mathbf{i}}\, |\mathbf{i}\rangle = \sum_{i_1=0}^{d_1-1} \cdots \sum_{i_N=0}^{d_N-1} a_{i_1,\ldots,i_N}\, |i_1, \ldots, i_N\rangle\,. \tag{2.31}$$

[3]The λ_i's can also be seen as the eigenvalues of $\sqrt{\sqrt{\hat{\rho}}\hat{\tilde{\rho}}\sqrt{\hat{\rho}}}$.

The generalised concurrences for $|\Psi\rangle$ are defined as

$$C_\alpha(\Psi) \equiv | \langle \Psi|\hat{S}_\alpha|\Psi^*\rangle |, \quad \forall \, |\Psi\rangle \in \mathcal{H}_{tot}, \tag{2.32}$$

where each $\hat{S}_\alpha$ corresponds to a generalised spin flip operator of the form

$$\hat{S}_\alpha = \hat{S}_{\mathbf{i},\mathbf{i}',\mathbf{j},\mathbf{j}'} = |\mathbf{i}\rangle \langle \mathbf{i}'| - |\mathbf{j}\rangle \langle \mathbf{j}'| + \text{h.c.}, \tag{2.33}$$

where h.c. stands for the Hermitian conjugate of the previous terms. Hence, to each index α corresponds a quadruplet $(\mathbf{i},\mathbf{i}',\mathbf{j},\mathbf{j}')$ (see Ref. [9] for a detailed description). For two qubits, there is only one concurrence and the generalised spin flip operator is $\hat{S}_1 = |00\rangle \langle 11| - |10\rangle \langle 01| + \text{h.c.} = \hat{\sigma}_2 \otimes \hat{\sigma}_2$, which is the two-qubit concurrence defined in the previous subsection [9]. The generalisation of Theorem 6 for multipartite pure states is

Theorem 7 (Concurrence criterion for pure states [9])**.** *A pure state $|\Psi\rangle \in \mathcal{H}_{tot}$ is separable if and only if all its concurrences are zero, i.e.*

$$|\Psi\rangle \;\textit{ is separable} \Leftrightarrow C_\alpha(|\Psi\rangle) = 0 \quad \forall \alpha = 1, \ldots, Q,$$

with Q being the number of generalised concurrences for a given system.

For a $d^{\times N}$ system[4], the number of generalised concurrences is [9]

$$Q = d^{N+1}\frac{d-1}{4}\left(1 - 2\left(1+\frac{1}{d}\right)^N + \left(1+\frac{2}{d}\right)^N\right). \tag{2.34}$$

Theorem 7 gives rise to a system of Q *independent* equations which are all computable and therefore gives rise to a *practical* necessary and sufficient separability criterion for multipartite pure states. From that, we can consider that generalised concurrences solve the separability problem for multipartite pure states.

Mixed states

Now, let us focus on multipartite mixed states. The set of generalised concurrences for mixed states is defined using the convex roof construction, that is

$$C_\alpha(\hat{\rho}) \equiv \inf_{\{p_i,|\psi_i\rangle\}} \left\{ \sum_i p_i C_\alpha(\psi_i) \right\}, \tag{2.35}$$

where the infimum is taken over all sets $\{p_i, |\psi_i\rangle\}$ such that $\hat{\rho} = \sum_i p_i |\psi_i\rangle \langle\psi_i|$. We note that the generalised concurrences can also be computed through Eq. (2.30) with the appropriate $\hat{\tilde{\rho}}$. For the mixed state scenario, the separability criterion based on the set of generalised concurrences is no longer necessary and sufficient, but only necessary. Indeed, for a separable state $\hat{\rho}_s$ acting on $\mathcal{H}_{tot}$, there exists a decomposition of the form $\sum_i p_i |\psi_i^{(s)}\rangle \langle\psi_i^{(s)}|$, with $|\psi_i^{(s)}\rangle \in \mathcal{H}_{tot}$ being a separable state $\forall i$. Hence, since $C_\alpha(\psi_i^{(s)}) = 0$, $\forall \alpha, i$, we have that $C_\alpha(\hat{\rho}) = 0$, $\forall\alpha$. Conversely, if $C_\alpha = 0$, $\forall\alpha$, that means that for all α, there exists a decomposition of $\hat{\rho}$ such that $C_\alpha(\hat{\rho}) = 0$, but the decomposition for which this occurs may be different for each α. We can therefore write

[4]The number of equations for general multipartite systems is given in Ref. [9].

Theorem 8 (Concurrence criterion for mixed states [9]). *If a state $\hat{\rho}$ defined on $\mathcal{H}_{tot}$ is separable, then all its concurrences are zero, i.e.*

$$\hat{\rho} \text{ is separable} \Rightarrow C_\alpha(\hat{\rho}) = 0 \quad \forall \alpha.$$

Of course, the fact that there is no necessary and sufficient criterion based on the generalised concurrences was predictable since the separability problem for mixed states is still an open problem, as seen before.

2.5 Computable cross-norm or realignment criterion

The *computable cross norm* or *realignment* (CCNR) criterion is a necessary separability criterion that was discovered in two different forms, by Rudolph [26] and by Chen and Wu [27], both in 2003. The separability criterion is given either by defining a new norm (a *cross norm*) or by 'realigning' the density operator and then taking the usual trace norm of the realigned matrix. Then, an enhanced CCNR criterion was proposed by Zhang *et al.* [28] in 2008, with a generalisation to the multipartite scenario. The latter criterion is discussed in Section 3.3. In this section, we first investigate the criterion via the cross norm and then via the realignment of the density operator.

Cross norm criterion

First, we define a new norm, the *greatest cross norm.*

Definition 21 (Ref. [26]). The greatest cross norm of a density operator $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB})$ is defined by

$$||\hat{\rho}||_\gamma \equiv \inf_{\{\hat{u}_i, \hat{v}_i\}} \left\{ \sum_i ||\hat{u}_i||_{\mathrm{Tr}} \cdot ||\hat{v}_i||_{\mathrm{Tr}} \right\},$$

where the infimum is taken over all sets $\{\hat{u}_i, \hat{v}_i\}$ such that $\hat{\rho} = \sum_i \hat{u}_i \otimes \hat{v}_i$ is a finite decomposition of $\hat{\rho}$ into elementary tensors.

With this definition, we can obtain the following criterion:

Theorem 9 (Ref. [26]). *A state $\hat{\rho}$ acting on $\mathcal{H}_{AB}$ is separable if and only if its greatest cross norm is equal to one, i.e.*

$$\hat{\rho} \text{ is separable} \Leftrightarrow ||\hat{\rho}||_\gamma = 1.$$

This is a necessary and sufficient criterion for the separability of quantum states. As stated in the introduction of Chapter 2, this means that so far, the criterion is not computable. Indeed, computing the greatest cross norm is not an easy task, due to the infimum taken over all finite decompositions of $\hat{\rho}$. A *computable* criterion based on this was found by Rudoph, but is not necessary and sufficient any more. In order to obtain this criterion, we need to use the Schmidt decomposition of a density operator $\hat{\rho}$, which is given by Eq. (1.75) that we recall to be

$$\hat{\rho} = \sum_i \sigma_i \hat{G}_i^{(A)} \otimes \hat{G}_i^{(B)}. \tag{2.36}$$

The necessary criterion is the following:

Theorem 10 (Computable cross norm criterion [26]). *If a state $\hat{\rho}$ acting on $\mathcal{H}_{AB}$ is separable and has the Schmidt decomposition (see Eq. (1.74))*

$$\hat{\rho} = \sum_i \sigma_i \hat{G}_i^{(A)} \otimes \hat{G}_i^{(B)},$$

then

$$\sum_i \sigma_i \leq 1$$

must hold.

It means that if $\sum_i \sigma_i > 1$, then the state $\hat{\rho}$ is entangled. Since the decomposition of $\hat{\rho}$ in Eq. (2.36) is computable (see Section 1.2.3), it follows that Theorem 10 also is.

Remark 7. Recall that the coefficients σ_is are the singular values of the matrix $\mathcal{C}$ of Eq. (1.77), which means that the computable cross norm criterion criterion can be written as

$$||\mathcal{C}||_{\mathrm{Tr}} \leq 1.$$

Realignment criterion

Then, the CCNR criterion can also be formulated in terms of the trace norm of a realigned matrix, computed from the density operator. Let us introduce this realigned matrix $\mathcal{R}(\hat{\rho})$ whose matrix elements are

$$\langle a_i, b_l|\mathcal{R}(\hat{\rho})|b_k, a_j\rangle = \langle a_i, b_k|\hat{\rho}|a_j, b_l\rangle\,, \tag{2.37}$$

where $\{|a_i, b_k\rangle\,, i = 1, \ldots, d_A, k = 1, \ldots, d_B\}$ is a basis of $\mathcal{H}_{AB}$. Equivalently, if $\hat{\rho}$ is written as in Eq. (1.50), i.e. as

$$\hat{\rho} = \sum_{i,j} \sum_{k,l} c_{ijkl} \, |a_i, b_k\rangle \, \langle a_j, b_l|\,, \tag{2.38}$$

then

$$\mathcal{R}(\hat{\rho}) = \sum_{i,j} \sum_{k,l} c_{ijkl} \, |a_i, b_l\rangle \, \langle b_k, a_j|\,. \tag{2.39}$$

The realigned matrix is also of dimension $d_A d_B \times d_A d_B$. The realignment criterion based on the matrix $\mathcal{R}(\hat{\rho})$ reads

Theorem 11 (Realignment criterion [12]). *Consider a bipartite state $\hat{\rho}$ acting on $\mathcal{H}_{AB}$ as in Eq. (2.38) and its realigned matrix $\mathcal{R}(\hat{\rho})$ as in Eq. (2.39). If the state is separable, then*

$$||\mathcal{R}(\hat{\rho})||_{\mathrm{Tr}} \leq 1,$$

must hold.

These two criteria (Theorems 10 and 11) are equivalent and known under computable cross norm or realignment criterion. Both of them are operational and simple to compute, which makes the CCNR criterion interesting. The CCNR criterion is proved to be non equivalent to the PPT criterion and neither weaker nor stronger, but complementary [26]. Indeed, the criterion can detect some entangled PPT states but cannot detect all states detected by the PPT criterion (see Refs. [27, 26] for examples).

CCNR and entanglement witnesses

The CCNR criterion also gives rise to an entanglement witness. One can construct the following operator acting on $\mathcal{H}_{AB}$:

$$\hat{W}^{(CCNR)} = \mathbb{1} \otimes \mathbb{1} - \sum_i \hat{G}_i^{(A)} \otimes \hat{G}_i^{(B)}. \tag{2.40}$$

where $\{\hat{G}_i^{(A)}\}$ and $\{\hat{G}_k^{(B)}\}$ are the orthonormal basis that lead to the Schmidt decomposition of a given state $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB})$. The operator $\hat{W}^{(CCNR)}$ is an entanglement witness for the state $\hat{\rho}$. Indeed, it is block positive (see Ref. [20]) and if the state $\hat{\rho}$ is detected to be entangled by the CCNR criterion, then

$$\begin{aligned} \mathrm{Tr}\left(\hat{W}^{(CCNR)}\hat{\rho}\right) &= 1 - \sum_i \sigma_i \\ &< 0, \end{aligned} \tag{2.41}$$

which means that any entangled state $\hat{\rho}$ detected by the CCNR criterion is detected by the entanglement witness $\hat{W}^{(CCNR)}$, defined with the appropriate basis.

Proof. Let us prove Eq. (2.41). Let $\hat{\rho}$ be an entangled state detected by the CCNR criterion whose Schmidt basis are $\{\hat{G}_i^{(A)}\}$ and $\{\hat{G}_i^{(B)}\}$. One has

$$\begin{aligned} \hat{W}^{CCNR}\hat{\rho} &= \sum_i \sigma_i \hat{G}_i^{(A)} \otimes \hat{G}_i^{(B)} - \sum_i (\hat{G}_i^{(A)} \otimes \hat{G}_i^{(B)}) \sum_j (\sigma_j \hat{G}_j^{(A)} \otimes \hat{G}_j^{(B)}) \\ &= \hat{\rho} - \sum_{ij} \sigma_j (\hat{G}_i^{(A)} \otimes \hat{G}_i^{(B)})(\hat{G}_j^{(A)} \otimes \hat{G}_j^{(B)}). \end{aligned} \tag{2.42}$$

Then,

$$\begin{aligned} \mathrm{Tr}(\hat{W}^{CCNR}\hat{\rho}) &= \mathrm{Tr}(\hat{\rho}) - \sum_{ij} \sigma_j \underbrace{\mathrm{Tr}\left((\hat{G}_i^{(A)} \otimes \hat{G}_i^{(B)})(\hat{G}_j^{(A)} \otimes \hat{G}_j^{(B)})\right)}_{=\delta_{ij}} \\ &= 1 - \sum_i \sigma_i. \end{aligned} \tag{2.43}$$

since

$$\begin{aligned} \mathrm{Tr}\left((\hat{G}_i^{(A)} \otimes \hat{G}_i^{(B)})(\hat{G}_j^{(A)} \otimes \hat{G}_j^{(B)})\right) &= \mathrm{Tr}\left((\hat{G}_i^{(A)}\hat{G}_j^{(A)}) \otimes (\hat{G}_j^{(B)}\hat{G}_i^{(B)})\right) \\ &= \mathrm{Tr}\left(\hat{G}_i^{(A)}\hat{G}_j^{(A)}\right) \mathrm{Tr}\left(\hat{G}_j^{(B)}\hat{G}_i^{(B)}\right) \\ &= \delta_{ij}\delta_{ij} = \delta_{ij}, \end{aligned} \tag{2.44}$$

from the relation between two orthonormal basis elements. □

Chapter 3

Recent separability criteria

In this chapter, we introduce more recently developed criteria that appear to be promising in the search for a necessary and sufficient criterion for the separability of quantum states. We end the chapter by comparing the criteria exposed in this chapter, but also in the previous one. The notations of Chapter 2 hold in this chapter as well.

3.1 Correlation matrix criterion

The *correlation matrix* (or *de Vicente*) criterion is a surprisingly simple criterion involving the Bloch representation of density operators, which was introduced in Section 1.1.3. The criterion, developed in 2007, is a necessary condition and can detect PPT entangled states [29]. First, it is useful to express the definition of entanglement in terms of the Bloch representation of states through the following theorem:

Theorem 12 (Ref. [29]). *A bipartite state with Bloch representation*

$$\hat{\rho} = \frac{1}{d_A d_B}\mathbb{1} \otimes \mathbb{1} + \frac{1}{2d_B}\sum_{i=1}^{d_A^2-1} r_i \hat{\lambda}_i \otimes \mathbb{1} + \frac{1}{2d_A}\sum_{j=1}^{d_B^2-1} s_j \mathbb{1} \otimes \hat{\sigma}_j + \frac{1}{4}\sum_{i=1}^{d_A^2-1}\sum_{j=1}^{d_B^2-1} \mathcal{T}_{ij}\hat{\lambda}_i \otimes \hat{\sigma}_j$$

(see Eq. (1.59)) is separable if and only if there exist Bloch vectors $\mathbf{u}_i \in \mathbb{R}^{d_A^2-1}$ and $\mathbf{v}_i \in \mathbb{R}^{d_B^2-1}$ and if there exist convex weights p_is such that

$$\mathbf{r} = \sum_i p_i \mathbf{u}_i, \quad \mathbf{s} = \sum_i p_i \mathbf{v}_i, \quad \mathcal{T} = \sum_i p_i \mathbf{u}_i \mathbf{v}_i^T.$$

where $\mathbf{r} = (r_1, \ldots, r_{d_A^2-1})$, $\mathbf{s} = (s_1, \ldots, s_{d_B^2-1})$ and $\mathcal{T}$ is the matrix with entries $\mathcal{T}_{ij}$.

Proof. Consider the bipartite Bloch representation of a state $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB})$. Now, the state $\hat{\rho}$ is separable if and only if there exist sets of pure states $\{\hat{\rho}_i^{(A)}\}$ and $\{\hat{\rho}_i^{(B)}\}$ and convex weights p_is such that $\hat{\rho} = \sum_i p_i \hat{\rho}_i^{(A)} \otimes \hat{\rho}_i^{(B)}$. The states $\hat{\rho}_i^{(A)}$ and $\hat{\rho}_i^{(B)}$ can be written as

$$\hat{\rho}_i^{(A)} = \frac{1}{d_A}\mathbb{1} + \frac{1}{2}\sum_k (u_i)_k \hat{\lambda}_k = \frac{1}{d_A}\mathbb{1} + \frac{1}{2}\mathbf{u}_i \cdot \hat{\lambda}, \tag{3.1}$$

and

$$\hat{\rho}_i^{(B)} = \frac{1}{d_B}\mathbb{1} + \frac{1}{2}\sum_k (v_i)_k \hat{\sigma}_k = \frac{1}{d_B}\mathbb{1} + \frac{1}{2}\mathbf{v}_i \cdot \hat{\sigma}. \tag{3.2}$$

Then, putting Eqs. (3.1) to (3.2) in $\sum_i p_i \hat{\rho}_i^{(A)} \otimes \hat{\rho}_i^{(B)}$, one has: $\hat{\rho}$ is separable if and only if

$$\hat{\rho} = \sum_i p_i \mathbb{1} \otimes \mathbb{1} + \frac{1}{2d_B}\sum_{i,k} p_i (u_i)_k \hat{\lambda}_k \otimes \mathbb{1} + \frac{1}{2d_A}\sum_{i,l} p_i (v_i)_l \mathbb{1} \otimes \hat{\sigma}_l + \frac{1}{4}\sum_{i,k,l} p_i (u_i)_k (v_i)_l \hat{\lambda}_k \otimes \hat{\sigma}_l. \tag{3.3}$$

Comparing with its Bloch representation, this means that $\hat{\rho}$ is separable if and only if

$$r_k = \sum_i p_i (u_i)_k, \quad s_l = \sum_i p_i (v_i)_l, \quad \mathcal{T}_{kl} = \sum_i p_i (u_i)_k (v_i)_l \tag{3.4}$$

$$\Leftrightarrow \quad \mathbf{r} = \sum_i p_i \mathbf{u}_i, \quad \mathbf{s} = \sum_i p_i \mathbf{v}_i, \quad \mathcal{T} = \sum_i p_i \mathbf{u}_i \mathbf{v}_i^T. \tag{3.5}$$

□

Again, this redefinition of separability can be seen as a necessary and sufficient condition for the separability of bipartite states although non-computable. From this definition, de Vicente was able to prove the following criterion:

Theorem 13 (Correlation matrix criterion [29])**.** *Consider a bipartite state of dimension $d_A \times d_B$ with Bloch representation*

$$\hat{\rho} = \frac{1}{d_A d_B}\mathbb{1} \otimes \mathbb{1} + \frac{1}{2d_B}\sum_{i=1}^{d_A^2-1} r_i \hat{\lambda}_i \otimes \mathbb{1} + \frac{1}{2d_A}\sum_{j=1}^{d_B^2-1} s_j \mathbb{1} \otimes \hat{\sigma}_j + \frac{1}{4}\sum_{i=1}^{d_A^2-1}\sum_{j=1}^{d_B^2-1} \mathcal{T}_{ij} \hat{\lambda}_i \otimes \hat{\sigma}_j$$

(see Eq. (1.59)). If the state is separable, then the inequality

$$||\mathcal{T}||_{\mathrm{Tr}} \leq \sqrt{\frac{4(d_A-1)(d_B-1)}{d_A d_B}}$$

must hold, with $\mathcal{T}$ the matrix of matrix elements $\mathcal{T}_{ij}$.

Hence, if $||\mathcal{T}||_{\mathrm{Tr}} > \sqrt{4(d_A-1)(d_B-1)/(d_A d_B)}$, the state is entangled. Physically, this necessary condition means that there is an upper bound to the amount of correlation contained in a separable state[1], i.e. that the correlations in separable states cannot be 'too large'. From Definition 12, it is possible to obtain several sufficient criteria for separability. The most relevant one is the following:

Theorem 14 (Ref. [29])**.** *Consider a bipartite state of dimension $d_A \times d_B$ in its normal form*

$$\hat{\rho} = \frac{1}{d_A d_B}\mathbb{1} + \frac{1}{4}\sum_{i=1}^{d_A^2-1}\sum_{j=1}^{d_B^2-1} \mathcal{T}_{ij}(\hat{\lambda}_i \otimes \hat{\sigma}_j).$$

[1] We recall that all the information about the correlations is contained in $\mathcal{T}$ and that $||\mathcal{T}||_{\mathrm{Tr}}$ is invariant under local unitaries.

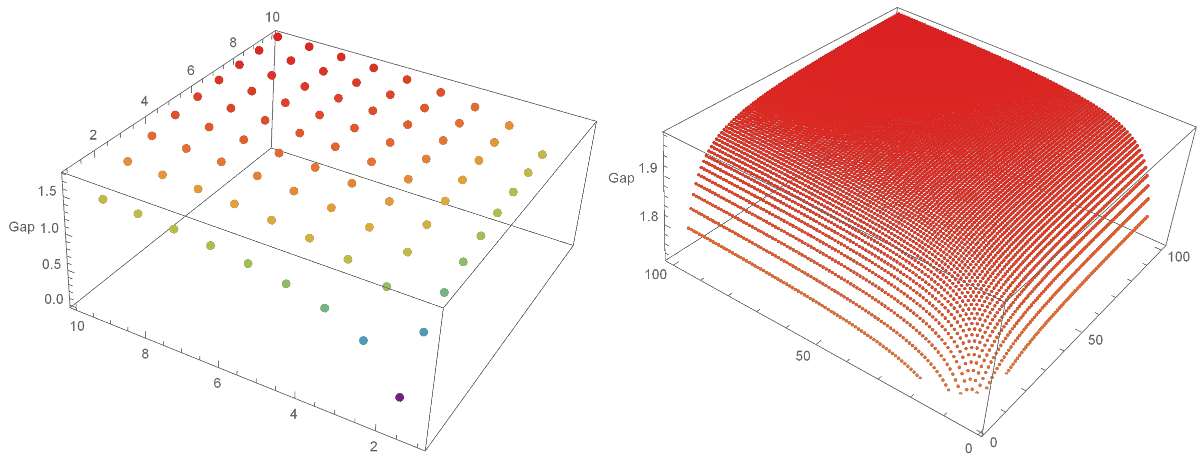

Figure 3.1: Gap as a function of d_A and d_B.

If the state satisfies

$$||\tilde{\mathcal{T}}||_{\mathrm{Tr}} \leq \sqrt{\frac{4}{d_A d_A (d_A - 1)(d_B - 1)}},$$

then it is separable.

Here, the physical interpretation in the opposite, if the correlations are low enough, then the state is automatically separable[2]. Combining Theorems 13 and 14 for a two-qubit system in its normal form (Theorem 13 obviously also holds for states in their normal forms), one gets a necessary and sufficient criterion, which is that such a state is separable if and only if

$$||\tilde{\mathcal{T}}||_{\mathrm{Tr}} \leq 1. \tag{3.6}$$

Then, for a 2×3 system, one has

$$\begin{aligned} &\text{if } ||\tilde{\mathcal{T}}||_{\mathrm{Tr}} \geq \frac{2\sqrt{3}}{3} \Rightarrow \text{ the state is entangled,} \\ &\text{if } ||\tilde{\mathcal{T}}||_{\mathrm{Tr}} \leq \frac{\sqrt{3}}{3} \Rightarrow \text{ the state is separable,} \end{aligned} \tag{3.7}$$

which means that in the region $2\sqrt{3}/3 \leq ||\tilde{\mathcal{T}}||_{\mathrm{Tr}} \leq \sqrt{3}/3$, the combined correlation matrix criteria cannot tell whether the state is entangled or separable. The "gap" between the upper bound and the lower bound is zero for 2×2 systems, which is why one can obtain a necessary and sufficient criterion for these systems. It gets bigger with d_A and d_B, as we can see on the left figure of Figure 3.1. It saturates at 2,

$$\lim_{d_A, d_B \to \infty} \left(\sqrt{\frac{4(d_A - 1)(d_B - 1)}{d_A d_B}} - \sqrt{\frac{4}{d_A d_A (d_A - 1)(d_B - 1)}} \right) = 2, \tag{3.8}$$

as shown on the right figure of Figure 3.1. The correlation matrix criterion detects states that are detected neither by PPT nor by CCNR, which makes it complementary to the duo PPT / CCNR and hence improves the entanglement detectability [30].

[2] The generalisation of Theorem 14 for non-normal form states can be found in Ref. [30].

3.2 Covariance matrix criterion

Another criterion complementary to the PPT criterion has been proposed in 2007 by Gühne *et al.*, namely the *covariance matrix* (CM) criterion [31]. This criterion allows to detect many PPT entangled states and can be connected to several other criteria, e.g. to the CCNR criterion or to the correlation matrix criterion exposed in the previous section [31]. First, let us define the covariance matrices. Let $\hat{\rho}$ be a state defined on $\mathcal{H}_{AB}$ and let $\{\hat{M}_i : i = 1, \dots n\}$ be a set of observables on $\mathcal{H}_{AB}$. We define the covariance matrix γ by its matrix elements γ_{ij}

$$\gamma_{ij}(\hat{\rho}) \equiv \frac{\langle \hat{M}_i \hat{M}_j \rangle_\rho + \langle \hat{M}_j \hat{M}_i \rangle_\rho}{2} - \langle \hat{M}_i \rangle_\rho \langle \hat{M}_j \rangle_\rho . \tag{3.9}$$

This matrix is real, positive definite, symmetric and has a concavity property [32], which is

$$\hat{\rho} = \sum_i p_i \hat{\rho}_i \Rightarrow \gamma(\hat{\rho}) \geq \sum_i p_i \gamma(\hat{\rho}_i). \tag{3.10}$$

Let $\{\hat{A}_i\}$ ($\{\hat{B}_i\}$) be a set of d_A^2 (d_B^2) observables that form an orthonormal basis on $\mathcal{HS}(\mathcal{H}_A)$ (on $\mathcal{HS}(\mathcal{H}_B)$). From these operators, one can construct the set $\{\hat{M}_i\} = \{\hat{A}_i \otimes \mathbb{1}, \mathbb{1} \otimes \hat{B}_i\}$ of $d_A^2 + d_B^2$ observables. With this set, the covariance matrix then reads

$$\gamma\left(\hat{\rho}, \{\hat{M}_i\}\right) = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \tag{3.11}$$

where $A = \gamma\left(\hat{\rho}^{(A)}, \{\hat{A}_i\}\right)$ and $B = \gamma\left(\hat{\rho}^{(B)}, \{\hat{B}_i\}\right)$ are the covariance matrices of the reduced states and C is defined by its matrix elements $C_{ij} = \mathrm{Tr}(\hat{A}_i \otimes \hat{B}_j \hat{\rho}) - \mathrm{Tr}(\hat{A}_i \hat{\rho})\mathrm{Tr}(\hat{B}_j \hat{\rho})$ [31]. From this, the definition of entanglement can be reformulated through the following theorem:

Theorem 15 (Covariance matrix criterion [31])**.** *Let $\gamma(\hat{\rho})$ be the covariance matrix of a state $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB})$ as in Eq. (3.11). If the state $\hat{\rho}$ is separable, then there exist pure states $\hat{\rho}_i^{(A)} \in \mathcal{S}(\mathcal{H}_A)$ and $\hat{\rho}_i^{(B)} \in \mathcal{S}(\mathcal{H}_B)$ and convex weights p_i such that*

$$\gamma\left(\hat{\rho}, \{\hat{M}_i\}\right) = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \geq \begin{pmatrix} \sum_i p_i \gamma\left(\hat{\rho}_i^{(A)}\right) & 0 \\ 0 & \sum_i p_i \gamma\left(\hat{\rho}_i^{(B)}\right) \end{pmatrix}$$

holds. If no such decomposition exists, $\hat{\rho}$ must be entangled.

This criterion does not depend on the choice of the sets $\{\hat{A}_i\}$ and $\{\hat{B}_i\}$, but can be more efficient with some suitable basis, e.g. the Schmidt basis[3] [31]. The covariance matrix criterion is not directly computable, but it leads to several other criteria that are computable. First, a consequence of the covariance matrix criterion involving the singular values of the matrix C can be found. Indeed, one has

[3]The Schmidt basis of a state $\hat{\rho}$ is the basis that leads to the Schmidt decomposition (see Eq. (1.74)) of the state.

Theorem 16 (CM and singular values criterion [32]). *Let $\gamma(\hat{\rho})$ be the covariance matrix of a state $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB})$ as in Eq. (3.11). If the state $\hat{\rho}$ is separable, then*

$$||C||_{\mathrm{Tr}} \leq \sqrt{\left(1 - \mathrm{Tr}\Big((\hat{\rho}^{(A)})^2\Big)\right)\left(1 - \mathrm{Tr}\Big((\hat{\rho}^{(B)})^2\Big)\right)}$$

must hold, with $\hat{\rho}^{(A)}$ and $\hat{\rho}^{(B)}$ being the reduced states of $\hat{\rho}$. Otherwise, the state is entangled.

Then, in Ref. [32], it has been shown that Theorem 15 leads to the correlation matrix criterion. From $\hat{A}_i$ and $\hat{B}_i$ we build the matrix $\mathfrak{C}$ with entries $\{ij\}$ equal to $\mathrm{Tr}\left((\hat{A}_i \otimes \hat{B}_j)\hat{\rho}\right)$, then we take $\mathfrak{C}$, omit its first row and first column and from that we build the matrix $\mathfrak{C}^{\mathrm{red}}$. A corollary from the covariance matrix criterion reads

Theorem 17 (CM and correlation matrix criterion [32]). *Let $\gamma(\hat{\rho})$ be the covariance matrix of a state $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB})$ as in Eq. (3.11). If the state $\hat{\rho}$ is separable, then*

$$||\mathfrak{C}^{red}||_{\mathrm{Tr}} \leq \sqrt{\frac{(d_A - 1)(d_B - 1)}{d_A d_B}}$$

must hold.

The matrix $\mathfrak{C}^{\mathrm{red}}$ is nothing but the correlation matrix $\mathcal{T}$ (up to a factor 2) with observables $\hat{A}_i = \hat{\lambda}_i$ and $\hat{B}_i = \hat{\sigma}_i$, as in the previous section[4]. So, Theorem 17 is in fact the correlation matrix criterion.

The covariance matrix criterion also leads to a criterion involving the traces of the matrices A, B and C (*CM and traces criterion*) which is however strictly weaker than the CM and singular values criterion [32] and thus not worth explicitly writing down.

Using the Schmidt decomposition in operator space, one obtains the following theorem as a consequence of the covariance matrix criterion:

Theorem 18 (CM and Schmidt decomposition criterion [32]). *Let $\gamma(\hat{\rho})$ be the covariance matrix of a state $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB})$ as in Eq. (3.11) and let $\hat{\rho}$ be in its Schmidt decomposition*

$$\hat{\rho} = \sum_{i=1}^{(d_A)^2} \sigma_i \hat{G}_i^{(A)} \otimes \hat{G}_i^{(B)}$$

(see Eq. (1.74)). If the state $\hat{\rho}$ is separable, then

$$2\sum_i \left|\sigma_i - \sigma_i^2 \mathrm{Tr}\left(\hat{G}_i^{(A)}\right)\mathrm{Tr}\left(\hat{G}_i^{(B)}\right)\right| \leq 2 - \sum_i \sigma_i^2 \left(\mathrm{Tr}^2\left(\hat{G}_i^{(A)}\right) + \mathrm{Tr}^2\left(\hat{G}_i^{(B)}\right)\right)$$

must hold, with $\mathrm{Tr}^2(\cdot) \equiv \big(\mathrm{Tr}(\cdot)\big)^2$.

As mentioned before, the CCNR criterion can be deduced from the covariance matrix criterion. Indeed, from Theorem 18, one can easily obtain Theorem 10.

[4] The factor 2 comes from the orthogonality relation of $SU(d)$ generators.

Proof. Using

$$a^2 + b^2 \geq 2|ab| \quad \text{and} \quad |a-b| \geq |a| - |b|, \tag{3.12}$$

and Theorem 18, we get

$$\begin{aligned}
&2\sum_i \left|\sigma_i - \sigma_i^2 \mathrm{Tr}\left(\hat{G}_i^{(A)}\right)\mathrm{Tr}\left(\hat{G}_i^{(B)}\right)\right| \leq 2 - \sum_i \sigma_i^2 \left(\mathrm{Tr}^2\left(\hat{G}_i^{(A)}\right) + \mathrm{Tr}^2\left(\hat{G}_i^{(B)}\right)\right) \\
\Leftrightarrow &2\sum_i \left|\sigma_i - \sigma_i^2 \mathrm{Tr}\left(\hat{G}_i^{(A)}\right)\mathrm{Tr}\left(\hat{G}_i^{(B)}\right)\right| \leq 2 - \sum_i \sigma_i^2 2\left|\mathrm{Tr}\left(\hat{G}_i^{(A)}\right)\mathrm{Tr}\left(\hat{G}_i^{(B)}\right)\right| \\
\Leftrightarrow &\sum_i \left|\sigma_i - \sigma_i^2 \mathrm{Tr}\left(\hat{G}_i^{(A)}\right)\mathrm{Tr}\left(\hat{G}_i^{(B)}\right)\right| + \sum_i \sigma_i^2 \left|\mathrm{Tr}\left(\hat{G}_i^{(A)}\right)\mathrm{Tr}\left(\hat{G}_i^{(B)}\right)\right| \leq 1 \qquad (3.13) \\
\Leftrightarrow &\sum_i \sigma_i - \sigma_i^2 \left|\mathrm{Tr}\left(\hat{G}_i^{(A)}\right)\mathrm{Tr}\left(\hat{G}_i^{(B)}\right)\right| + \sigma_i^2 \left|\mathrm{Tr}\left(\hat{G}_i^{(A)}\right)\mathrm{Tr}\left(\hat{G}_i^{(B)}\right)\right| \leq 1 \\
\Leftrightarrow &\sum_i \sigma_i \leq 1,
\end{aligned}$$

keeping in mind that the σ_i's are positive. □

Finally, one can obtain an interesting criterion involving the normal form of the density operator. By applying the CM and traces criterion to a state in its normal form, one gets the following criterion:

Theorem 19 (Filter CM criterion [32]). *Let $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB})$ be a state with $d_A < d_B$ whose filter normal form is*

$$\hat{\rho} = \frac{1}{d_A d_B}\mathbb{1} + \frac{1}{4}\sum_{i=1}^{d_A^2-1}\sum_{j=1}^{d_B^2-1} \mathcal{T}_{ij}(\hat{\lambda}_i \otimes \hat{\sigma}_j)$$

(see Eq. (1.63)). If the state $\hat{\rho}$ is separable, then

$$||\mathcal{T}||_{\mathrm{Tr}} \leq 1 - \frac{1}{d_A} + \frac{d_A^2 - 1}{d_B} + \min\left[0,\, 1 - d_B - \frac{d_A^2 - d_B^2}{d_B}\right]$$

must hold.

Remark 8. Hereafter and in general, when the covariance matrix criterion is mentioned, it implies the filter covariance matrix criterion.

As stated in Section 1.2.2, normal forms are very useful in separability criteria. For instance, the filter CM criterion is strictly stronger than the criterion it is deduced from, i.e. the CM and traces criterion [32]. It has been shown [30, 31] that the filter CM criterion is stronger than the correlation matrix criterion for states in their normal form when $d_A << d_B$, but weaker when $d_A \sim d_B$. We note that for $d_A = d_B$, the criteria are equivalent. So, the filter CM criterion and the correlations matrix criterion of previous section are complementary. Note that the correlation matrix criterion is a corollary of Theorem 15, which is not directly computable, thus the correlation matrix criterion is not strictly weaker than the *practical* CM criteria.

Physically, all these criteria, consequences of the covariance matrix criterion, express the same idea: if the correlations between the two subsystems are 'too large', then the global state is entangled. The correlations are represented in the different theorems by the matrices C, $\mathfrak{C}^{\mathrm{red}}$ and $\tilde{\mathcal{T}}$. This was already the idea in the correlation matrix criterion.

3.3 Enhanced CCNR criterion

In 2008, Zhang *et al.* proposed a necessary condition for the separability of bipartite systems that is quite similar to the CCNR criterion in its structure, although stronger. Hence it is called *enhanced CCNR* criterion. Their criterion also generalises to the multipartite scenario. First, we note that for the regular CCNR criterion, a family of *non-linear* entanglement witnesses has been developed. These are strictly stronger than the original CCNR criterion and are of the form [28]

$$\mathcal{F}(\hat{\rho}) = 1 - ||T||_{\mathrm{Tr}} - \frac{1}{2}\left(\mathrm{Tr}\left((\hat{\rho}^{(A)})^2\right) - \mathrm{Tr}\left((\hat{\rho}^{(B)})^2\right)\right), \tag{3.14}$$

where T is a $d_A^2 \times d_B^2$ matrix with matrix elements $T_{ij} = \mathrm{Tr}((\hat{\rho} - \hat{\rho}^{(A)} \otimes \hat{\rho}^{(B)})\hat{G}_i^{(A)} \otimes \hat{G}_j^{(B)})$ and where $\hat{\rho}^{(A)}$ and $\hat{\rho}^{(B)}$ are the reduced density operators. This is a family of (non-linear) entanglement witnesses, which means that

$$\begin{aligned} \forall \text{ separable state } \hat{\rho}_s \in \mathcal{S}(\mathcal{H}_{AB}) : \mathcal{F}(\hat{\rho}_s) \geq 0 \\ \text{and } \exists \text{ an entangled state } \hat{\rho}_e \in \mathcal{S}(\mathcal{H}_{AB}) : \mathcal{F}(\hat{\rho}_e) < 0. \end{aligned} \tag{3.15}$$

The enhanced CCNR criterion goes as follows

Theorem 20 (Enhanced CCNR criterion [28])**.** *Consider a bipartite density operator $\hat{\rho}$ and the realignment operator $\mathcal{R}$ as in Eq. 2.37. If the state $\hat{\rho}$ is separable, then*

$$\left|\left|\mathcal{R}\left(\hat{\rho} - \hat{\rho}^{(A)} \otimes \hat{\rho}^{(B)}\right)\right|\right|_{\mathrm{Tr}} \leq \sqrt{\left(1 - \mathrm{Tr}\left(\hat{\rho}^{(A)}\right)\right)\left(1 - \mathrm{Tr}\left(\hat{\rho}^{(B)}\right)\right)}$$

must hold.

Clearly, we recognise here the structure of the CCNR criterion. As stated above, the enhanced CCNR criterion is stronger than the regular CCNR criterion and its non-linear witnesses, which was proven in Ref. [28]. Moreover, the authors of Ref. [28] have proven that Theorem 20 is stronger than the correlation matrix criterion, although equivalent in the case of states in their normal form. The enhanced CCNR criterion is completely analytical (unlike normal forms that are computed numerically), which makes it still useful. Then, we note than Theorem 20 is complementary to the filter CM criterion, and is a corollary of the regular CM criterion (which, we recall, is not directly computable). Indeed, if $|d_A - d_B|$ is large, then Theorem 19 is better than Theorem 20, but if $|d_A - d_B|$ is small, it is the opposite. The enhanced CCNR criterion has been generalised to $2N$-partite systems but the notations used are somewhat laborious so we are not going into greater details. More information can be found in Ref. [28].

3.4 LWFL family of criteria

We present in this section a family of criteria derived in 2014 by Li *et al.* [33] that we refer to as the *LWFL* family of criteria. Again, it is based on the Bloch representation of quantum states. The criterion states that:

Theorem 21 (LWFL family of criteria [33])**.** *Consider a bipartite state $\hat{\rho}$ and its Bloch representation*

$$\hat{\rho} = \frac{1}{d_A d_B}\mathbb{1} \otimes \mathbb{1} + \frac{1}{2d_B}\sum_{i=1}^{d_A^2-1} r_i \hat{\lambda}_i \otimes \mathbb{1} + \frac{1}{2d_A}\sum_{j=1}^{d_B^2-1} s_j \mathbb{1} \otimes \hat{\sigma}_j + \frac{1}{4}\sum_{i=1}^{d_A^2-1}\sum_{j=1}^{d_B^2-1} \mathcal{T}_{ij}\hat{\lambda}_i \otimes \hat{\sigma}_j.$$

(see Eq. (1.59)). If the state is separable, then for any $d_A^2 \times d_B^2$ matrix $\mathcal{X}$ and any $(d_A^2-1) \times (d_B^2-1)$ matrix $\mathcal{Y}$ with real matrix elements $\mathcal{X}_{ij}$ and $\mathcal{Y}_{ij}$ respectively, the following inequalities must hold:

$$\left|\frac{1}{d_A d_B}\mathcal{X}_{00} + \frac{1}{2d_B}\sum_i r_i \mathcal{X}_{i0} + \frac{1}{d_A}\sum_j s_j \mathcal{X}_{0j} + \frac{1}{4}\sum_{ij}\mathcal{T}_{ij}\mathcal{X}_{ij}\right| \leq \frac{\sqrt{(d_A^2 - d_A + 2)\,(d_B^2 - d_B + 2)}}{2d_A d_B}\sigma_{\max}(\mathcal{X}),$$

$$\left|\sum_{ij}\mathcal{T}_{ij}\mathcal{Y}_{ij}\right| \leq \sqrt{\frac{4(d_A-1)(d_B-1)}{d_A d_B}}\sigma_{\max}(\mathcal{Y})$$

where $\sigma_{\max}(\mathcal{X})$ and $\sigma_{\max}(\mathcal{Y})$ are the maximal singular values of the matrices $\mathcal{X}$ and $\mathcal{Y}$ respectively.

For each matrices $\mathcal{X}$ and $\mathcal{Y}$, one gets two conditions, which is why Theorem 21 is called a *family of criteria*. By using a certain matrix $\mathcal{Y}$ based on the singular value decomposition of $\mathcal{T}$, one can prove that the second inequality of Theorem 21 implies the correlation matrix criterion [33]. This means that one of the many criteria that comes with the LFWL family of criteria is the correlation matrix criterion. Moreover, in Ref. [33], the authors have shown through an example that a specific LWFL criterion can detect states not detected by the correlation matrix criterion. Then, through the same example, they have shown that this LWFL criterion is able to detect entanglement for states that are neither detected by the PPT criterion nor by the CCNR criterion, which makes it complementary to the duo PPT/CCNR. With these criteria arises the question of how to find the matrices $\mathcal{X}$ and $\mathcal{Y}$ that lead to a good entanglement detection. The LWFL family of criteria can further be extended to the multipartite case, which gives a necessary criterion for the full separability of multipartite states[5].

3.5 Li-Qiao criterion

In 2018, Li and Qiao published a paper about a new necessary and sufficient criterion for the separability of quantum states [34]. Consider a density operator of a $d_A \times d_B$ bipartite system with full local ranks. It can be written in its normal form, that we recall to be

$$\hat{\rho} = \frac{1}{d_A d_B}\mathbb{1} + \frac{1}{4}\sum_{i=1}^{d_A^2-1}\sum_{j=1}^{d_B^2-1}\mathcal{T}_{ij}\hat{\lambda}_i \otimes \hat{\sigma}_j \tag{3.16}$$

[5] Again, this criterion is not explicitly exposed since it requires a lot of new notation.

(see Eq. (1.63)). Let

$$\mathcal{T} = (\mathbf{u}_1, ..., \mathbf{u}_{N^2-1})\Lambda_\tau(\mathbf{v}_1, ..., \mathbf{v}_{M^2-1})^T = \sum_{i=1}^{l} \tau_i \mathbf{u}_i \mathbf{v}_i^T \tag{3.17}$$

be the singular value decomposition of $\mathcal{T}$, with $l = \text{rank}(\mathcal{T})$[6] (see Appendix A.1). In the basis of $\mathbf{u_i}$ and $\mathbf{v_i}$, the matrix $\mathcal{T}$ is diagonal and its diagonal elements are its singular values. As we recall from Definition 5, if the state $\hat{\rho}$ is separable, then there exists a number L such that $\hat{\rho}$ can be decomposed as follows

$$\hat{\rho} = \sum_{i=1}^{L} p_i \hat{\rho}_i^{(A)} \otimes \hat{\rho}_i^{(B)} \tag{3.18}$$

with

$$\hat{\rho}_i^{(A)} = \frac{1}{d_A}\mathbb{1} + \frac{1}{2}\mathbf{r}_i\hat{\lambda}, \tag{3.19}$$

$$\hat{\rho}_i^{(B)} = \frac{1}{d_B}\mathbb{1} + \frac{1}{2}\mathbf{s}_i\hat{\lambda}, \tag{3.20}$$

and p_is being convex weights. The operators $\hat{\rho}_i^{(A)}$ and $\hat{\rho}_i^{(B)}$ represent physical systems in their respective Hilbert spaces and therefore are positive semi-definite operators with unit trace. Using the Bloch vectors of both subsystems and the convex weights of Eq. (3.18), we build three matrices

$$M_r \equiv \{\mathbf{r}_1, \ldots, \mathbf{r}_L\}, \quad M_s \equiv \{\mathbf{s}_1, \ldots, \mathbf{s}_L\} \quad \text{and} \quad D_p \equiv \text{diag}\{p_1, \ldots, p_L\}, \tag{3.21}$$

and from these, we build the following matrices:

$$M_{rp} \equiv M_r\sqrt{D_p} \quad \text{and} \quad M_{sp} \equiv M_s\sqrt{D_p}. \tag{3.22}$$

We can now reformulate the definition of entanglement through the following theorem:

Theorem 22 (Ref. [34])**.** *Consider a bipartite state with matrices M_{rp} and M_{sp} as in Eq. (3.22). The state is separable if and only if there exists a number L such that the matrix $\mathcal{T}$ can be decomposed as*

$$\mathcal{T} = M_{rp}M_{sp}^T$$

with the following conditions:

$$\sum_{i=1}^{L} p_i\mathbf{r}_i = 0 \quad \textit{and} \quad \sum_{i=1}^{L} p_i\mathbf{s}_i = 0.$$

This theorem is nothing but Theorem 12 for states in their normal forms. As proven in Ref. [34], this decomposition exists if and only if the following equations hold:

$$M_{rp} = (\mathbf{u}_1, \ldots, \mathbf{u}_L)XD_\alpha Q^{(1)}, \quad M_{sp} = (\mathbf{v}_1, \ldots, \mathbf{v}_L)YD_\beta Q^{(2)} \tag{3.23}$$

[6]Note that for states in their normal forms, $\text{rank}(\hat{\rho}) \equiv r = l + 1$, see Appendix A.3

and

$$D_\tau = X D_\alpha Q^{(1)} Q^{(2)T} D_\beta Y^T \tag{3.24}$$

where X, Y, $Q^{(1)}$ and $Q^{(2)}$ are $L \times L$ orthogonal matrices with determinant 1, $D_\alpha = \mathrm{diag}\{\alpha_1, \dots \alpha_L\}$, $D_\beta = \mathrm{diag}\{\beta_1, \dots \beta_L\}$, $D_\tau = \mathrm{diag}\{\tau_1, \dots \tau_L\}$, are diagonal matrices with as diagonal elements, the singular values of M_{rp}, M_{sp} and $\mathcal{T}$ respectively, arranged in decreasing order. Furthermore, the condition expressed by Eq. (3.24) is satisfied if and only if the following set of inequalities is satisfied:

$$\prod_{k\in K} \tau_k \leq \prod_{i\in I} \alpha_i \prod_{j\in J} \beta_j \quad \forall\ (I, J, K) \in T_q^L, \quad q < L. \tag{3.25}$$

These inequalities are called *Horn inequalities*[7].

Remark 9 (The sets I, J and K [34]). The sets I, J and K are subsets of the set of the n first non-zero natural numbers, i.e. subsets of $\{1, \dots, n\}$, of the form $I = \{i_1, \dots, i_q\}$, $J = \{j_1, \dots, j_q\}$ and $K = \{k_1, \dots, k_q\}$ where the elements are arranged in increasing order. Then, we define $\mathcal{F}(I) \equiv \big(i_q - q, i_{q-1} - (q-1), \dots, i_1 - 1\big)$ and build the triplet $(\lambda, \mu, \nu) \equiv \big(\mathcal{F}(I), \mathcal{F}(J), \mathcal{F}(J)\big)$. Finally, we define the set of triplets $T_q^n \equiv \{(I, J, K)\}$ by saying that a triplet (I, J, K) is in T_q^n if and only if the corresponding triplet (λ, μ, ν) occurs as eigenvalues of the triplet of $q \times q$ Hermitian matrices, with the third being the sum of the first two.

Again, the separability condition states that the correlations in separable states "cannot be too large". Note that the α_is and β_js cannot be too large either, or they would correspond to non-physical Bloch vectors. We synthesise all the above conditions in the following theorem:

Theorem 23 (Li-Qiao criterion [34]). *Consider a bipartite state with matrices M_{rp} and M_{sp} as in Eq. (3.22). The state is separable if and only if*

- *there exists a number L such that M_{rp} and M_{sp} can be decomposed as in Eq. (3.23), i.e.*

$$M_{rp} = (\mathbf{u}_1, \dots, \mathbf{u}_L) X D_\alpha Q^{(1)}, \quad M_{sp} = (\mathbf{v}_1, \dots, \mathbf{v}_L) Y D_\beta Q^{(2)},$$

- *Eq. (3.25), i.e.*

$$\prod_{k\in K} \tau_k \leq \prod_{i\in I} \alpha_i \prod_{j\in J} \beta_j \quad \forall\ (I, J, K) \in T_q^L, \quad q < L$$

  is satisfied,

- *the conditions $\sum_i p_i \mathbf{r}_i = 0$ and $\sum_i p_i \mathbf{s}_i = 0$ are satisfied.*

Let us try to arrive to a practical criterion from Theorem 23. Note that the first condition of Theorem 23 is trivially satisfied when working in the $\{\mathbf{u}_i, \mathbf{v}_j\}$ basis[8], which leaves us with the second and third conditions. Eq. (3.25) gives us a set of inequalities on the singular values of M_{rp} and M_{sp}. We generated these inequalities and we got, for

[7] This comes from the fact that they are the solutions to the Horn problem, see Ref. [34] for more details.

[8] Indeed, this leaves us with the singular value decomposition of M_{rp} and M_{sp}.

$d \times d$ systems with d from two to ten, 3, 12, 41, 142, 522, 2062, 8752, 39716 and 191353 inequalities respectively. The number of inequalities is increased by a factor of ~ 4 each time d becomes bigger. For 2×2 and 3×3 systems, the set of inequalities are

$$\begin{cases} \tau_1 \leq \alpha_1\beta_1 \\ \tau_2 \leq \alpha_1\beta_2 \\ \tau_2 \leq \alpha_2\beta_1 \end{cases} \tag{3.26}$$

and

$$\begin{cases} \tau_1 \leq \alpha_1\beta_1 \\ \tau_2 \leq \alpha_1\beta_2 \\ \tau_3 \leq \alpha_1\beta_3 \\ \tau_2 \leq \alpha_2\beta_1 \\ \tau_3 \leq \alpha_2\beta_2 \\ \tau_3 \leq \alpha_3\beta_1 \\ \tau_1\tau_2 \leq \alpha_1\alpha_2\beta_1\beta_2 \\ \tau_1\tau_3 \leq \alpha_1\alpha_2\beta_1\beta_3 \\ \tau_2\tau_3 \leq \alpha_1\alpha_2\beta_2\beta_3 \\ \tau_1\tau_3 \leq \alpha_1\alpha_3\beta_1\beta_2 \\ \tau_2\tau_3 \leq \alpha_1\alpha_3\beta_1\beta_3 \\ \tau_2\tau_3 \leq \alpha_2\alpha_3\beta_1\beta_2 \end{cases} \tag{3.27}$$

We recall that the τ_is are known, since they are the singular values of $\mathcal{T}$. So, given two sets of L singular values $\{\alpha_i\}$ and $\{\beta_i\}$ satisfying Eq. (3.25), we can construct the matrices M_{rp} and M_{sp} and we have, in the $\{\mathbf{u}_i, \mathbf{v}_j\}$ basis,

$$\begin{aligned} (M_r)_{ij} &= \sum_{k=1}^{L}\sum_{l=1}^{L}\sum_{m=1}^{L} X_{ik}(D_\alpha)_{kl}Q^{(1)}_{lm}(D_p^{-1/2})_{mj} \\ &= \sum_{k=1}^{L} X_{ik} \cdot \alpha_k \cdot p_j^{-1/2} \cdot Q^{(1)}_{kj} \end{aligned} \tag{3.28}$$

and analogously

$$(M_s)_{ij} = \sum_{k=1}^{L} Y_{ik} \cdot \beta_k \cdot p_j^{-1/2} \cdot Q^{(2)}_{kj} \tag{3.29}$$

Knowing that $M_r = (\mathbf{r}_1, ..., \mathbf{r}_L)$ and $M_s = (\mathbf{s}_1, ..., \mathbf{s}_L)$, we have

$$\mathbf{r}_i = \frac{1}{\sqrt{p_i}} \begin{pmatrix} \sum_{k=1}^{L} X_{1k} \cdot \alpha_k \cdot Q^{(1)}_{ki} \\ \vdots \\ \sum_{k=1}^{L} X_{N^2-1,k} \cdot \alpha_k \cdot Q^{(1)}_{ki} \end{pmatrix} \tag{3.30}$$

and

$$\mathbf{s}_i = \frac{1}{\sqrt{p_i}} \begin{pmatrix} \sum_{k=1}^{L} Y_{1k} \cdot \beta_k \cdot Q^{(2)}_{ki} \\ \vdots \\ \sum_{k=1}^{L} Y_{M^2-1,k} \cdot \beta_k \cdot Q^{(2)}_{ki} \end{pmatrix}. \tag{3.31}$$

The two sets $\{\mathbf{r}_i\}$ and $\{\mathbf{s}_i\}$ are sets of L vectors of size $d_A^2 - 1$ and $d_B^2 - 1$ respectively. To make $\hat{\rho}_i^{(A)}$ and $\hat{\rho}_i^{(B)}$ of Eqs. (3.19)–(3.20) represent physical systems, the vectors $\mathbf{r}_i$ and $\mathbf{s}_i$ of Eqs. (3.19)–(3.20) have to verify a set of d positivity conditions, exposed in Section 1.1.3, and the last conditions of Theorem 23. If the conditions are satisfied by the vectors $\mathbf{r}_i$ and $\mathbf{s}_i$ of Eqs. (3.30)–(3.31), then these vectors belong to Bloch-vector spaces and hence the state $\hat{\rho}$ is separable since it can be written in the form of Eq. (3.18), with acceptable Bloch vectors. Otherwise, the hypothesis of $\hat{\rho}$ being a separable state does not hold and therefore $\hat{\rho}$ is entangled.

Remark 10. Note that we worked in the $\{\mathbf{u}_i, \mathbf{v}_j\}$ basis. Therefore, in the case of $\hat{\rho}$ being separable, we need to perform a change of basis in order to obtain the separable decomposition in a more common basis.

Unfortunately, Theorem 23 does not say how to determine all the parameters involved in the separable decomposition of $\hat{\rho}$. So, to the best of our knowledge, the Li and Qiao criterion is rather a reformulation of the definition of entanglement, as stated in the introduction of Chapter 2. This comes from the fact that there are too many degrees of freedom, hence too many decompositions to test. For instance, the theorem does not give any constrains on the number L and on the corresponding set of convex weights p_is, which leaves us with an infinity of combinations to test.

In practice, one could test if a separable decomposition is obtained for a number $L \in [r, r^2]$. If not, the state is entangled. Indeed, it is known that $L \geq r$ [35] and that, for a separable state, there always exists a separable decomposition with $L \in [r, r^2]$ [36]. So this drastically constrains the number probabilistic distributions to test even if it is still infinite.

Moreover, the inequalities of Eq. (3.25) allow many different sets of singular values $\{\alpha_i\}$, $\{\beta_j\}$ and $\{\tau_k\}$. By gathering the inequalities given in Eqs. (1.43) to (1.47) and in Eq. (3.25), one might obtain stronger conditions for the values of the different singular values.

Remark 11. We note from Theorem 22 that the authors of Ref. [34] where able to obtain the PPT criterion and both correlation matrix criteria, i.e. Theorems 2, 13 and 14.

3.6 Symmetric informationally complete measures criterion

The next criterion involves a special kind of measurement, the *symmetric informationally complete positive operator-valued measures* (SIC POVMs), which are defined hereafter. The criterion was proposed in 2018 and has been proven to be stronger than the CCNR criterion [37].

Symmetric informationally complete positive operator-valued measures

First, let us define *positive operator-valued measures* (POVMs).

Definition 22 (POVM [16])**.** A set of operators $\{\hat{E}_i\}$ is called a POVM if

- each operator $\hat{E}_i$ is positive, which also means each operator is Hermitian;

– the completeness relation $\sum_i \hat{E}_i = \mathbb{1}$ if verified.

Operators $\hat{E}_i$ are called POVM elements.

A POVM applied to a state $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB})$ gives the outcome i with probability $p_i = \mathrm{Tr}(\hat{E}_i\hat{\rho})$ and thus a given POVM is sufficient to determine the probabilities of the different measurement outcomes. The POVM does not determine the measurement operators associated to it uniquely [16].

A POVM in a d-dimensional Hilbert space is called *informationally complete* if the probabilities p_i uniquely determine the density operator [15], which means that it must contain at least d^2 elements in order to span $\mathcal{HS}(\mathcal{H})$, where d is the dimension of $\mathcal{H}$, as usual. Now, a POVM is called *symmetric informationally complete* (SIC) if it is composed of d^2 elements $\hat{\Pi}_i = \frac{1}{d} |\psi_i\rangle \langle\psi_i|$ with

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{d\delta_{ij}+1}{d+1}, \quad i,j = 1,\ldots,d^2. \tag{3.32}$$

It is conjectured that SIC POVMs exist in all finite dimensions. Indeed, analytical proofs and strong numerical evidence that SIC POVMs exist for some specific dimensions[9] have been found, but a general proof is still lacking [37]. The density operator $\hat{\rho}$ can effectively be determined using the probabilities $p_i = \mathrm{Tr}(\hat{\Pi}_i\hat{\rho})$, [37]

$$\begin{aligned} \hat{\rho} &= \sum_{i=1}^{d^2} \big(d(d+1)p_i - 1\big)\hat{\Pi}_i \\ &= d(d+1)\sum_{i=1}^{d^2} p_i\hat{\Pi}_i - \mathbb{1}. \end{aligned} \tag{3.33}$$

Moreover, one has

$$\sum_{i=1}^{d^2} p_i^2 = \frac{1+\mathrm{Tr}(\hat{\rho}^2)}{d(d+1)} \leq \frac{2}{d(d+1)} \tag{3.34}$$

where the equality holds if and only if $\hat{\rho}$ is pure. Let us introduce the renormalized SIC POVM $\{\hat{E}_i\}$ with elements

$$\hat{E}_i \equiv \sqrt{\frac{d(d+1)}{2}}\hat{\Pi}_i = \sqrt{\frac{d+1}{2d}}\,|\psi_i\rangle\langle\psi_1|\,. \tag{3.35}$$

and 'renormalized probabilities'

$$e_i = \mathrm{Tr}(\hat{E}_i\hat{\rho}) = \sqrt{\frac{d(d+1)}{2}}p_i, \tag{3.36}$$

which leads to

$$\sum_{i=1}^{d^2} e_i^2 = \frac{1+\mathrm{Tr}(\hat{\rho}^2)}{2} \leq 1. \tag{3.37}$$

Again, the equality holds if and only if $\hat{\rho}$ is pure.

[9] Analytically for dimensions $d = 2-24, 28, 30, 31, 35, 37, 39, 43, 48, 124$, and numerically up to dimension $d = 151$.

Entanglement criterion

Now that SIC POVMs have been introduced, we can expose the entanglement criterion based on them, called the *ESIC* criterion. Let $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB})$ be the density operator of a bipartite system AB, of dimension $d_{AB} = d_A \times d_B$ and let $\{\hat{E}_i^{(A)}, i = 1, \ldots, (d_A)^2\}$ and $\{\hat{E}_i^{(B)}, i = 1, \ldots, (d_B)^2\}$ be normalised SIC POVMs for the subsystems A and B respectively. The linear correlations between the two SIC POVMs are

$$\mathcal{P}_{ij}(\hat{\rho}) \equiv \mathrm{Tr}\left((\hat{E}_i^{(A)} \otimes \hat{E}_j^{(B)})\hat{\rho}\right) \tag{3.38}$$

from which one can obtain the ESIC criterion.

Theorem 24 (ESIC criterion [37])**.** *Consider a bipartite state with matrix $\mathcal{P}$ of matrix elements as in Eq. (3.38). If the state is separable, then*

$$||\mathcal{P}||_{\mathrm{Tr}} \leq 1$$

must hold. This condition is independent of the choice of the SIC POVMs.

Remark 12. Note that in terms of SIC POVMs $\{\hat{\Pi}_i^{(A)}\}$ and $\{\hat{\Pi}_i^{(B)}\}$ instead of normalised SIC POVMs, this criterion states that if a bipartite state $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB})$ is separable, then

$$\left|\left| \left[\mathrm{Tr}\left((\hat{\Pi}_i^{(A)} \otimes \hat{\Pi}_j^{(B)})\hat{\rho}\right)\right] \right|\right|_{\mathrm{Tr}} \leq \sqrt{\frac{4}{d_A d_B (d_A+1)(d_B+1)}}, \tag{3.39}$$

must hold, where $[A_{ij}]$ represents the matrix A.

3.7 SSC family of criteria

Finally, we introduce again a family of separability criteria, proposed very recently in 2020. Interestingly, these criteria are linear with respect to the density operator. This family of criteria (that we call the *SSC* family of criteria) is based on the matrix $\mathcal{C}$ of a bipartite state $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB})$ that we recall has the matrix elements $\mathcal{C}_{\alpha\beta}$

$$\mathcal{C}_{\alpha\beta} = \mathrm{Tr}\left((\hat{G}_\alpha^{(A)} \otimes \hat{G}_\beta^{(B)})\hat{\rho}\right), \tag{3.40}$$

where $\{\hat{G}_\alpha^{(A)} : \alpha = 0 \ldots, d_A^2 - 1\}$ and $\{\hat{G}_\beta^{(B)} : \beta = 0 \ldots, d_B^2 - 1\}$ are orthonormal basis of $\mathcal{HS}(\mathcal{H}_A)$ and $\mathcal{HS}(\mathcal{H}_B)$ respectively. If one chooses basis elements such that $\hat{G}_0^{(A)} = \mathbb{1}/\sqrt{d_A}$ and $\hat{G}_0^{(B)} = \mathbb{1}/\sqrt{d_B}$, one can write

$$\begin{aligned}\hat{\rho} =& \mathcal{C}_{00}\frac{\mathbb{1}}{\sqrt{d_A}} \otimes \frac{\mathbb{1}}{\sqrt{d_B}} + \sum_{i=1}^{d_A^2-1} \mathcal{C}_{i0}\hat{G}_i^{(A)} \otimes \frac{\mathbb{1}}{\sqrt{d_B}} + \sum_{j=1}^{d_B^2-1} \mathcal{C}_{0j}\frac{\mathbb{1}}{\sqrt{d_A}} \otimes \hat{G}_j^{(B)} + \sum_{i=1}^{d_A^2-1}\sum_{j=1}^{d_B^2-1} \mathcal{C}_{ij}\hat{G}_i^{(A)} \otimes \hat{G}_j^{(B)} \\ =& \sum_{\alpha=0}^{d_A^2-1}\sum_{\beta=0}^{d_B^2-1} \mathcal{C}_{\alpha\beta}^{\mathrm{can}}\hat{G}_\alpha^{(A)} \otimes \hat{G}_\beta^{(B)}.\end{aligned} \tag{3.41}$$

We call this type of basis *canonical basis.* Obviously, since the trace norm is independent of the basis choice, we have that $||\mathcal{C}^{(\mathrm{can})}||_{\mathrm{Tr}} = ||\mathcal{C}||_{\mathrm{Tr}}$. Then, we introduce two square diagonal matrices of dimensions $d_A^2 \times d_A^2$ and $d_B^2 \times d_B^2$ respectively,

$$D_x^{(A)} = \mathrm{diag}\{x, 1, \ldots, 1\} \quad \text{and} \quad D_y^{(B)} = \mathrm{diag}\{y, 1, \ldots, 1\}, \tag{3.42}$$

with x and y being real positive parameters. We now obtain the family of separability criteria

Theorem 25 (SSC family of criteria [38])**.** *Consider a bipartite state as in Eq. (3.41) and matrices as in Eq. (3.42). If the state is separable, then*

$$||D_x^{(A)}\mathcal{C}^{\mathrm{can}}D_y^{(B)}||_{\mathrm{Tr}} \leq \sqrt{\frac{d_A - 1 + x^2}{d_A}}\sqrt{\frac{d_B - 1 + y^2}{d_B}}$$

must hold, for arbitrary $x, y \geq 0$.

Link with other entanglement criteria

This criterion is a set of criteria, as for each pair of (x, y) one gets a different separability criterion. For the pair $(1, 1)$, one gets

$$||\mathcal{C}^{\mathrm{can}}||_{\mathrm{Tr}} = ||\mathcal{C}||_{\mathrm{Tr}} \leq \sqrt{\frac{d_A}{d_A}}\sqrt{\frac{d_B}{d_B}} = 1, \tag{3.43}$$

which is the CCNR criterion. Then, the pair $(0, 0)$ leads to

$$\begin{aligned} ||D_0^{(A)}\mathcal{C}^{\mathrm{can}}D_0^{(B)}||_{\mathrm{Tr}} = \frac{1}{2}||\mathcal{T}||_{\mathrm{Tr}} &\leq \sqrt{\frac{d_A - 1}{d_A}}\sqrt{\frac{d_B - 1}{d_B}} \\ \Leftrightarrow ||\mathcal{T}||_{\mathrm{Tr}} &\leq \sqrt{\frac{4(d_A - 1)(d_B - 1)}{d_A d_B}}, \end{aligned} \tag{3.44}$$

which is the correlation matrix criterion. The pair $\left(\sqrt{\frac{2}{d_A}}, \sqrt{\frac{2}{d_B}}\right)$ gives the LWFL family of criteria and finally, the authors of Ref. [38] have shown that the pair $(\sqrt{d_A + 1}, \sqrt{d_B + 1})$ leads to the ESIC criterion. We note that the SSC family of criteria is not stronger than the filter CM criterion but that it detects states note detected by the filter CM criterion and thus these criteria are complementary.

In a recent preprint, Sarbicki *et al.* have proven that the enhanced CCNR criterion is perfectly equivalent to the SSC family of criteria [39]. First, the authors proved that a state satisfying the enhanced CCNR criterion satisfies Theorem 25 for all $x, y \geq 0$. Then, they proved that an entangled state detected by the enhanced CCNR is also detected by Theorem 25 for some $x, y \geq 0$. So, this means all states detected by enhanced CCNR criterion are detected by Theorem 25 and states not detected by enhanced CCNR criterion are not detected by Theorem 25, which makes both criteria equivalent. However, as we see below, Theorem 25 gives rise to a class of entanglement witness, whereas enhanced CCNR criterion is clearly non-linear in $\hat{\rho}$, which makes Theorem 25 definitively interesting.

Entanglement witnesses

Sarbicki *et al.* have proven in Ref. [38] that Theorem 25 gives rise to a class of witnesses parametrised by $d_A^2 \times d_B^2$ isometry[10] matrices O and by two positive numbers x and y. They have proven that Theorem 25 is equivalent to

$$\mathrm{Tr}(\hat{W}_O^{(x,y)}\hat{\rho}) \geq 0 \tag{3.45}$$

where the entanglement witness $\hat{W}_O^{(x,y)}$ has the structure

$$\hat{W}_O^{(x,y)} = \sum_{\alpha,\beta} w_{\alpha\beta}\hat{G}_\alpha^{(A)} \otimes \hat{G}_\beta^B \tag{3.46}$$

with

$$w_{00} = \sqrt{(d_A - 1 + x^2)(d_B - 1 + y^2)} + xyO_{00},$$

$$w_{0j} = xO_{0j}, \quad w_{i0} = yO_{i0}, \quad w_{ij} = O_{ij}, \tag{3.47}$$

for $i = 1, \ldots, d_A^2 - 1$ and $j = 1, \ldots, d_B^2 - 1$.

Proof. First, we note that the definition of the trace norm of any $m \times n$ matrix X can be given by [38]

$$||X||_{\mathrm{Tr}} = \max_{O\in\mathcal{O}(m,n)} \left\{\mathrm{Tr}(X^\dagger O)\right\} \tag{3.48}$$

where the maximum is taken over all isometry $m \times n$ matrices O. Then, from Theorem 25, one has for a separable state $\hat{\rho}_s$ and with x, y two positive parameters,

$$\begin{aligned} &\sqrt{\frac{d_A - 1 + x^2}{d_A}}\sqrt{\frac{d_B - 1 + y^2}{d_B}} - ||D_x^{(A)}C^{\mathrm{can}}D_y^{(B)}||_{\mathrm{Tr}} \geq 0 \\ \Leftrightarrow &\sqrt{\frac{d_A - 1 + x^2}{d_A}}\sqrt{\frac{d_B - 1 + y^2}{d_B}} \underbrace{- \max_{O\in\mathcal{O}(d_A^2,d_B^2)} \left\{\mathrm{Tr}\left((D_x^{(A)}C^{\mathrm{can}}D_y^{(B)})^\dagger O\right)\right\}}_{+\min_{O\in\mathcal{O}(d_A^2,d_B^2)}\left\{\mathrm{Tr}\left((D_x^{(A)}C^{\mathrm{can}}D_y^{(B)})^\dagger O\right)\right\}} \geq 0, \end{aligned} \tag{3.49}$$

since for $O \in \mathcal{O}(m,n)$ such that $\mathrm{Tr}(X^\dagger O)$ is maximal, $\mathrm{Tr}(X^\dagger(-O))$ is minimal. Then,

$$\begin{aligned} &\sqrt{\frac{d_A - 1 + x^2}{d_A}}\sqrt{\frac{d_B - 1 + y^2}{d_B}} + \min_{O\in\mathcal{O}(d_A^2,d_B^2)} \left\{\mathrm{Tr}\left((D_x^{(A)}C^{\mathrm{can}}D_y^{(B)})^\dagger O\right)\right\} \geq 0 \\ \Leftrightarrow &\sqrt{\frac{d_A - 1 + x^2}{d_A}}\sqrt{\frac{d_B - 1 + y^2}{d_B}} + \mathrm{Tr}\left((D_x^{(A)}C^{\mathrm{can}}D_y^{(B)})^\dagger O\right) \geq 0 \quad \forall O \in \mathcal{O}(d_A^2, d_B^2) \\ \Leftrightarrow &\sqrt{\frac{d_A - 1 + x^2}{d_A}}\sqrt{\frac{d_B - 1 + y^2}{d_B}}\mathrm{Tr}(\mathbb{1} \otimes \mathbb{1} \cdot \hat{\rho}_s) + \underbrace{\mathrm{Tr}\left((D_x^{(A)}C^{\mathrm{can}}D_y^{(B)})^\dagger O\right)}_{(*)} \geq 0 \quad \forall O \in \mathcal{O}(d_A^2, d_B^2) \end{aligned} \tag{3.50}$$

[10] Distance preserving matrices.

$$
\begin{aligned}
(*) &= \mathrm{Tr}\left((D_y^{(B)})^\dagger (\mathcal{C}^{\mathrm{can}})^\dagger (D_x^{(A)})^\dagger O\right) \\
&= \sum_\alpha \left((D_y^{(B)})^\dagger (\mathcal{C}^{\mathrm{can}})^\dagger (D_x^{(A)})^\dagger O\right)_{\alpha\alpha} \\
&= \sum_{\alpha,\beta,\gamma,\sigma} (D_y^{(B)})^*_{\beta\alpha} (C^{\mathrm{can}})^*_{\gamma\beta} (D_x^{(A)})^*_{\sigma\gamma} O_{\sigma\alpha} \\
&= \sum_{\alpha,\beta,\gamma,\sigma} (D_y^{(B)})_{\beta\alpha} \underbrace{(C^{\mathrm{can}})_{\gamma\beta}}_{\mathrm{Tr}(\hat{G}_\gamma^{(A)} \otimes \hat{G}_\beta^B \cdot \hat{\rho}_s)} (D_x^{(A)})_{\sigma\gamma} O_{\sigma\alpha} \\
&= \mathrm{Tr}\left[\left(\sum_{\beta,\gamma}\sum_{\alpha,\sigma} (D_x^{(A)})_{\sigma\gamma} O_{\sigma\alpha} (D_y^{(B)})_{\beta\alpha} \hat{G}_\gamma^{(A)} \otimes \hat{G}_\beta^{(B)}\right)\hat{\rho}_s\right] \qquad (3.51) \\
&= \mathrm{Tr}\left[\left(\sum_{\beta,\gamma} (D_x^{(A)})_{\gamma\gamma} O_{\gamma\beta} (D_y^{(B)})_{\beta\beta} \hat{G}_\gamma^{(A)} \otimes \hat{G}_\beta^{(B)}\right)\hat{\rho}_s\right] \\
&= \mathrm{Tr}\Bigg[\Bigg(xyO_{00}\hat{G}_0^{(A)} \otimes \hat{G}_0^{(B)} + \sum_{j>0} xO_{0j}\hat{G}_0^{(A)} \otimes \hat{G}_j^{(B)} \\
&\qquad + \sum_{i>0} yO_{i0}\hat{G}_i^{(A)} \otimes \hat{G}_0^{(B)} + \sum_{i,j>0} O_{ij}\hat{G}_i^{(A)} \otimes \hat{G}_j^{(B)}\Bigg)\hat{\rho}_s\Bigg],
\end{aligned}
$$

using the fact that $D_x^{(A)}$, $D_y^{(B)}$ and C^{can} are real matrices. So Eq. (3.49) reads

$$
\begin{aligned}
&\mathrm{Tr}\Bigg[\Bigg(\sqrt{d_A - 1 + x^2}\sqrt{d_B - 1 + y^2}\frac{\mathbb{1}}{\sqrt{d_A}} \otimes \frac{\mathbb{1}}{\sqrt{d_B}} + xyO_{00}\frac{\mathbb{1}}{\sqrt{d_A}} \otimes \frac{\mathbb{1}}{\sqrt{d_B}} + \sum_{j>0} xO_{0j}\frac{\mathbb{1}}{\sqrt{d_A}} \otimes \hat{G}_j^{(B)} \\
&\qquad + \sum_{i>0} yO_{i0}\hat{G}_i^{(A)} \otimes \frac{\mathbb{1}}{\sqrt{d_B}} + \sum_{i,j>0} O_{ij}\hat{G}_i^{(A)} \otimes \hat{G}_j^B\Bigg)\hat{\rho}_s\Bigg] \geq 0 \quad \forall O \in \mathcal{O}(d_A^2, d_B^2) \\
&\Leftrightarrow \mathrm{Tr}\left(\hat{W}_O^{(x,y)}\hat{\rho}_s\right) \geq 0 \quad \forall O \in \mathcal{O}(d_A^2, d_B^2)
\end{aligned}
\tag{3.52}
$$

□

Then, in Ref. [39], the authors have shown that for each entangled state detected by the enhanced CCNR criterion, it is possible to construct an entanglement witness that will also detect the state. This class of entanglement witnesses is characterised by

$$
\hat{W}^\infty = \sum_{\alpha\beta} w_{\alpha\beta}^\infty \hat{G}_\alpha^{(A)} \otimes \hat{G}_\beta^B, \tag{3.53}
$$

with

$$
w_{00}^\infty = \frac{1}{2}\left((d_B - 1)\mathrm{cotg}(\theta) + (d_A - 1)\mathrm{tg}(\theta) + \eta^2 \sin(\theta)\cos(\theta)\right),
$$

$$
w_{0j}^\infty = \eta\cos(\theta)v_j, \quad w_{i0}^\infty = \eta\sin(\theta)u_i, \quad w_{ij}^\infty = O_{ij}, \tag{3.54}
$$

for $i = 1, \ldots, d_A^2 - 1$ and $j = 1, \ldots, d_B^2 - 1$, with $\eta \geq 0$ and θ, two real parameters. The matrix O is a $(d_A^2 - 1) \times (d_B^2 - 1)$ isometry. The vectors $\mathbf{u}$ and $\mathbf{v}$ are normalised and satisfy $\mathbf{u} = O\mathbf{v}$. We note that the subscript '∞' on $\hat{W}^{\infty}$ is there to remind of the fact that $\hat{W}^{\infty}$ was found by taking the limit of $\hat{W}_O^{(x,y)}$ with $x, y \to \infty$.

Multipartite entanglement

The SSC family of criteria generalises to a necessary condition in the multipartite case. For this purpose, we need to generalise some quantities. The hypermatrix $\mathcal{C}$ of a N-partite state $\hat{\rho}$ acting on $\mathcal{H}_{tot} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ reads

$$\mathcal{C}^{(N)}_{\alpha_1 \ldots \alpha_N} = \langle \hat{G}^{(1)}_{\alpha_1} \otimes \cdots \otimes \hat{G}^{(N)}_{\alpha_1} \rangle_{\hat{\rho}} \,. \tag{3.55}$$

The two diagonal matrices used above are generalised to N diagonal matrices of dimensions $d_i^2 \times d_i^2$

$$D^{(i)}_{x_i} = \mathrm{diag}\{x_i, 1, \ldots, 1\} \tag{3.56}$$

Then, one has

Theorem 26. *Consider a N-partite state with hypermatrix $\mathcal{C}$ as in Eq. (3.55) and matrices as in Eq. (3.56). If the state is fully separable, then*

$$\left|\left|\mathcal{C}^{(N)}_{\alpha_1 \ldots \alpha_N} (D^{(1)}_{x_1})_{i_1 i_1} \ldots (D^{(N)}_{x_N})_{i_N i_N}\right|\right|_{\mathrm{Tr}} \leq \sqrt{\frac{d_1 - 1 + x_1^2}{d_1}} \cdots \sqrt{\frac{d_N - 1 + x_N^2}{d_N}}$$

must hold, for arbitrary $x_i \geq 0$ $(i = 1, \ldots, N)$.

We notice that the SSC family of criteria unifies a lot of criteria described in the previous sections, that is the CCNR criterion of Section 2.5, the correlation matrix criterion of Section 3.1, the enhanced CCNR criterion of Section 3.3, the LWFL family of criteria of Section 3.4 and the ESIC criterion of Section 3.6.

3.8 Comparing the criteria

In this section, we try to give an overview of the previous sections. The following table summarises the relative strength of various *computable* criteria encountered throughout the manuscript:

PPT	/	compl.	compl.	compl.	compl.	compl.	compl.	compl.
CCNR	compl.	/	compl.	?	weaker	compl.	?	weaker
Corr. M[11]	compl.	compl.	/	compl.	weaker	weaker	?	weaker
filter CM	compl.	?	compl.	/	compl.	?	?	compl.
eCCNR[12]	compl.	stronger	stronger	compl.	/	stronger	stronger	equiv.
LWFL	compl.	compl.	stronger	?	weaker	/	?	weaker
ESIC	compl.	?	?	?	weaker	?	/	weaker
SSC	compl.	stronger	stronger	compl.	equiv.	stronger	stronger.	/
×	PPT	CCNR	Corr. M	filter CM	eCCNR	LWFL	ESIC	SSC

[11]Stands for correlation matrix criterion.
[12]Stands for enhanced CCNR criterion.

– equiv.: line criterion is equivalent to row criterion i.e. line criterion and row criterion detect exactly the same states;

– compl.: line criterion is complementary to row criterion i.e. line criterion can detect states not detected by row criterion and vice versa;

– stronger: line criterion is stronger than row criterion i.e. line criterion can detect all states detected by row criterion and at least one more;

– weaker: line criterion is weaker than row criterion i.e. all states detected by line criterion are detected by row criterion and the latter can detect at least one more.

First, we notice the strength of the PPT criterion, which is to date still not supplanted. Indeed, there is no known to us criterion that is strictly stronger than the PPT criterion. It is also worth recalling the simplicity of this necessary criterion, and the fact that is solves the separability problem for 2×2 and 2×3 systems. Then, this table also highlights the strength of the SSC family of criteria. It has been proven to recover four other criteria by means of simple inequalities. We note however that these inequities require the knowledge of the whole density operator. The whole SSC family is equivalent to the enhanced CCNR criterion. This is somehow surprising since the enhanced CCNR criterion does not seem to have been strongly investigated by the entanglement theory community. Indeed, literature is scarce on the subject. From this equivalence, one can deduce that the enhanced CCNR criterion is stronger than the LWFL family of criteria and than the ESIC criterion.

It seems that the trio of criteria PPT / filter CM / enhanced CCNR is the most efficient to detect entanglement. We favour the enhanced CCNR to the SSC family since applying the latter to a state for all $x, y \geq 0$ is hardly doable whereas the former only gives one inequality to check. However, we note that the SSC family gives rise to a new class of entanglement witnesses and thus expands the entanglement witness theory. We recall that the great benefit of entanglement witnesses is that they do not require the knowledge of the whole density operator contrary to most of the criteria. Finally, we must note that if we allow ourselves to consider states in their filter normal form, the correlation matrix criterion is equivalent to the enhanced CCNR criterion, which means it is also equivalent to the SSC family of criteria. This emphasises again the enhancement capability of filter normal forms, and thus the trio of criteria PPT / filter CM / filter correlation matrix may also be considered.

Conclusion

The theory of quantum entanglement is still under construction and has generated great interest over the last thirty years. The aim of the present manuscript was to review several relevant existing separability criteria, with an emphasis on criteria developed in the last ten years. We analysed these criteria following chronological order of appearance in literature and finally compared and contrasted them in the last section of this work.

In the first chapter, we summarised the notions of quantum mechanics required to enable the full understanding of the separability criteria developed in the next chapters. In Section 1.1, we defined quantum states and their representations on Hilbert spaces, which are density operators. We distinguished pure states from mixed states, the former being a special case of the latter. We introduced the Bloch representation of quantum states, a representation that is involved in many different criteria, as we saw in Chapter 3. Section 1.2 was devoted to bipartite systems and their particularities, the most important one being the heart of this manuscript, quantum entanglement. In the same section, the operation of partial trace was introduced, such as the bipartite Bloch representation and the Schmidt decomposition of quantum states. The last section, was dedicated to the generalization of the definitions of entanglement to multipartite systems. This chapter was essential to the good understanding of the following chapters.

Chapter 2 was devoted to the analysis of separability criteria developed from 1996 to 2003. We began the chapter with the most celebrated one, namely the positive partial transpose criterion. We recall that this criterion is necessary *and* sufficient for 2×2 and 2×3 systems and is still not, to our knowledge, supplanted by any other criterion. Section 2.2 was devoted to entanglement witnesses, powerful tools to detect entangled states, which does not require the full knowledge of the density operator. Then, in Section 2.3, we defined entanglement measures and presented the most relevant ones in this context, namely entropy of entanglement, negativity and in Section 2.4, concurrences. Concurrences allowed us to consider the separability problem for pure states solved. The second chapter was ended by the computable cross norm or realignment criterion, which is complementary to the PPT criterion and is able to detect many PPT entangled states.

In the last chapter, we developed and analysed seven more recently developed separability criteria. In Section 3.1, two criteria based on the correlation matrix are presented. Together, they are necessary and sufficient for the separability of 2×2 systems, however not of 2×3 systems. We also noted that the correlation matrix criterion is complementary to the duo PPT / CCNR. Section 3.2 was dedicated to the covariance matrix criterion, a necessary and sufficient but non-computable criterion that led to several computable criteria. The third criterion that was exposed in this chapter is an enhanced version of the CCNR criterion. Section 3.4 was dedicated to a family of criteria, which recovered the correlation matrix criterion. Then, in Section 3.5, we analysed a necessary and sufficient but (to our knowledge) non-computable criterion, the Li-Qiao criterion. In Section 3.6, we presented the ESIC criterion. The last criterion introduced was the SSC family of criteria, an impressively strong family of criteria. Indeed, it led to the recovering of several strong criteria, namely the CCNR criterion, the correlation matrix criterion, the LWFL family of criteria and the ESIC criterion. It has been shown that the SSC family of criteria is equivalent to the enhanced CCNR criterion. Moreover, it has been proven that the family of criteria lead to a new class of entanglement witnesses.

The last section of Chapter 3, that is to say Section 3.8, ended the present manuscript with an original comparison of several computable criteria previously analysed. In this section, based on the relative strengths of the criteria, we reached the conclusion that the trio of criteria PPT / filter CM / enhanced CCNR is the most efficient in this manuscript to detect entanglement. When considering normal forms, this trio is equivalent to PPT / filter CM / filter correlation matrix criteria. Lastly, we want to stress that a further analysis of the Li-Qiao criterion could lead to a better characterization of the set of separable states. Indeed, the link between the positivity condition of Section 1.1.3 and the Horn inequalities of Section 3.5 has not been analysed in depth and could reveal a bit more on the structure of the set of separable states.

Appendix A

Appendix

A.1 Singular value decomposition

The following theorem is completely taken from Ref. [40], with slightly adapted notations:

Theorem 27 (Singular value decomposition [40]). *Let* $A \in \mathcal{M}_{n,m}(\mathbb{C})$ *be given, let* $q = \min\{m, n\}$*, and suppose that* $\mathrm{rank}(A) = r$.

1. *There are unitary matrices* $V \in \mathcal{M}_n(\mathbb{C})$ *and* $W \in \mathcal{M}_m(\mathbb{C})$*, and a square diagonal matrix*

$$\Sigma_q = \begin{pmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_q \end{pmatrix}$$

such that $\sigma_1 \geq \sigma_2 \geq \dots \sigma_r > 0 = \sigma_{r+1} = \dots = \sigma_q$ *and* $A = V\Sigma W^\dagger$*, in which*

$$\begin{aligned} &\Sigma = \Sigma_q \textit{ if } m = n, \\ &\Sigma = \begin{pmatrix} \Sigma_q & 0 \end{pmatrix} \in \mathcal{M}_{n,m}(\mathbb{R}) \textit{ if } m > n \textit{ and} \\ &\Sigma = \begin{pmatrix} \Sigma_q \\ 0 \end{pmatrix} \in \mathcal{M}_{n,m}(\mathbb{R}) \textit{ if } n > m. \end{aligned}$$

The decomposition $A = V\Sigma W^\dagger$ *is called the* singular value decomposition *of the matrix* A.

2. *The parameters* σ_1, ..., σ_r *are the positive square roots of the decreasingly ordered non-zero eigenvalues of* $AA^\dagger$*, which are the same as the decreasingly ordered non-zero eigenvalues of* $A^\dagger A$*. They are called the* singular values *of the matrix* A.

Note that $\mathrm{rank}(A)$ is equal to the number of its non-zero singular values whereas it is smaller or equal to the number of its non-zero eigenvalues. If $\sigma_1, \dots, \sigma_n$ are the singular values of $A \in \mathcal{M}_{n,m}(\mathbb{C})$ with $n \leq m$, then

$$|\det(A)| = \prod_{i=1}^{n} \sigma_i, \quad \sqrt{\mathrm{Tr}(A^\dagger A)} = \sqrt{\sum_{i=1}^{n} \sigma_i^2} \quad \text{and} \quad \mathrm{Tr}(\sqrt{A^\dagger A}) = \sum_{i=1}^{n} \sigma_i. \tag{A.1}$$

We recognize in Eq. (A.1) the definitions of the Hilbert-Schmidt and trace norms. We also note that in the case of Hermitian matrices, the absolute value of the eigenvalues are the singular values [40].

A.2 Special unitary group

In this section, we give some basic notions about the special unitary group $SU(d)$ $(d \geq 1)$ in order to understand this work. First, we recall the definition of a group, a Lie group and a Lie algebra.

Definition 23 (Group)**.** A group G is a set together with a map

$$\cdot : G \times G \to G : (a,b) \mapsto a \cdot b$$

that satisfies the following properties:

- associativity : $\forall a,b,c \in G : (a \cdot b) \cdot c = a \cdot (b \cdot c)$;
- existence of a neutral element : $\exists\, e \in g : e \cdot a = a = a \cdot e, \forall a \in G$;
- existence of an inverse : $\forall a \in G, \exists\, b \in G : a \cdot b = e = b \cdot a$.

Definition 24 (Lie group)**.** A Lie group G is a group that is also a differential manifold, such that the multiplication and inversion operations are differential maps.

Definition 25 (Lie algebra)**.** A $\mathbb{K}$-Lie algebra $\mathfrak{g}$ of a Lie group G is the tangent $\mathbb{K}$-vector space at the unit element of the group together with a bilinear, anti-symmetric map

$$\left[\cdot,\cdot\right]_{\mathcal{L}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (x,y) \mapsto \left[x,y\right]_{\mathcal{L}}$$

called the *Lie bracket* that verifies the Jacobi identity

$$\left[x, \left[y,z\right]_{\mathcal{L}}\right]_{\mathcal{L}} + \left[y, \left[z,x\right]_{\mathcal{L}}\right]_{\mathcal{L}} + \left[z, \left[x,y\right]_{\mathcal{L}}\right]_{\mathcal{L}} = 0 \quad \forall x,y,z \in \mathfrak{g}.$$

As mentioned, the special unitary group of degree d $SU(d)$ is the Lie group of $d \times d$ unitary complex matrices of unit determinant. In other words, a matrix $A \in \mathcal{M}_d(\mathbb{C})$ belongs to $SU(d)$ if and only if

$$A^\dagger A = \mathbb{1} = AA^\dagger, \quad \det(A) = 1. \tag{A.2}$$

The Lie algebra of $SU(d)$, denoted $\mathfrak{su}(d)$, can be identified with the real vector space of traceless anti-Hermitian[1] $d \times d$ complex matrices, with the regular commutator $[\cdot,\cdot]$ as Lie bracket. However, $\mathfrak{su}(d)$ may also be isomorphically identified with the real vector space of traceless, *Hermitian* $d \times d$ complex matrices, with $1/2i$ times the regular commutator as Lie bracket. In physics, this is the usual convention and hence this convention is adopted here. The Lie algebra $\mathfrak{su}(d)$ is a real vector space of dimension $d^2 - 1$ equipped with the inner product $\langle x|y\rangle = \mathrm{Tr}(xy)$. Conventionally, an orthogonal basis of this vector space $\{\lambda_i : i = 1, \ldots, d^2 - 1\}$ is chosen such that

$$\langle \lambda_i|\lambda_j\rangle = \mathrm{Tr}(\lambda_i \lambda_j) = 2\delta_{ij} \quad \forall i,j = 1,\ldots,d^2-1. \tag{A.3}$$

[1] A matrix A is called anti-Hermitian if $A^\dagger = -A$.

The product of the basis elements is given by, for all $i, j = 1, \ldots, d^2 - 1$, [15]

$$\lambda_i \lambda_j = \frac{2}{d}\delta_{ij} + \sum_k g_{ijk}\lambda_k + i\sum_k f_{ijk}\lambda_k, \tag{A.4}$$

where f_{ijk} is totally anti-symmetric in its indices and g_{ijk} is totally symmetric and traceless (i.e. $g_{iik} = 0$), for all $i, j, k = 1, \ldots, d^2 - 1$. The coefficients f_{ijk} are the *structure constants* of $\mathfrak{su}(d)$ with respect to the chosen basis. They can be computed through [15]

$$[\lambda_i, \lambda_j] \equiv \lambda_i\lambda_j - \lambda_j\lambda_i = 2i\sum_k f_{ijk}\lambda_k \tag{A.5}$$

$$[\lambda_i, \lambda_j]_+ \equiv \lambda_i\lambda_j + \lambda_j\lambda_i = \frac{4}{d}\delta_{ij} + 2\sum_k g_{ijk}\lambda_k, \tag{A.6}$$

for all $i, j = 1, \ldots, d^2 - 1$.

The basis elements λ_i ($i = 1, \ldots, d^2 - 1$) are called *generators of the $SU(d)$ group*. Together with the identity matrix, they form the following set of $d \times d$ matrices

$$\{\mathbb{1}, \hat{\lambda}_i : i = 1, \ldots, d^2 - 1\}, \tag{A.7}$$

which forms a basis of the real vector space of $d \times d$ Hermitian matrices and of the complex vector space of $d \times d$ matrices.

Every element A of $SU(d)$ can be generated by an element $\lambda = \sum_i c_i\lambda_i$ of $\mathfrak{su}(d)$ through

$$A = e^{i\lambda} = e^{i\sum_i c_i\lambda_i}, \tag{A.8}$$

which justifies their name. Indeed, $\det(A) = e^{i\mathrm{Tr}(\lambda)} = 1$ for any traceless λ.

A.2.1 $SU(2)$ group

The $SU(2)$ group is the group of unitary 2×2 matrices with determinant equal to one. The conventionally chosen generators of $SU(2)$ are the $2^2 - 1 = 3$ Pauli matrices, $\sigma = (\sigma_i, \sigma_2, \sigma_3)$, given by [15]

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.9}$$

They form an orthogonal basis of the Lie algebra $\mathfrak{su}(d)$, since they are of the number of the dimension of the vector space, linearly independent and verify

$$\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}. \tag{A.10}$$

Together with the 2×2 identity matrix,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.11}$$

they form a basis for the real vector space of 2×2 Hermitian matrices, and of the complex vector space of 2×2 matrices. The constants f_{ijk} and g_{ijk} are given by [17]

$$\begin{aligned} f_{ijk} &= \epsilon_{ijk}, \\ g_{ijk} &= 0 \end{aligned} \tag{A.12}$$

for all $i, j, k = 1, 2, 3$, where ϵ_{ijk} is the Levi-Civita symbol.

A.2.2 $SU(3)$ group

The conventionally chosen generators of the $SU(3)$ group are the $3^2 - 1 = 8$ Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \tag{A.13}$$
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

that can be seen as a generalisation of the Pauli matrices. One can easily check that they verify $\mathrm{Tr}(\lambda_i \lambda_j) = 2\delta_{ij}$. The non-vanishing constants f_{ijk} and g_{ijk} are given by [17]

$$\begin{aligned} &f_{123} = 1, \\ &f_{458} = f_{678} = \frac{\sqrt{3}}{2}, \\ &f_{147} = f_{246} = f_{257} = f_{345} = -f_{156} = -f_{367} = \frac{1}{2}, \\ &g_{118} = g_{228} = g_{338} = -g_{888} = \frac{\sqrt{3}}{3}, \\ &g_{448} = g_{558} = g_{668} = g_{778} = -\frac{\sqrt{3}}{6}, \\ &g_{146} = g_{157} = g_{256} = g_{344} = g_{355} = -g_{247} = -g_{366} = -g_{377} = \frac{1}{2}. \end{aligned} \tag{A.14}$$

A.2.3 $SU(d)$ group ($d \geq 1$)

For arbitrary d, the conventionally chosen set of $SU(d)$ generators $\{\lambda_i : i = 1, \ldots, d^2 - 1\}$ is given by the set of matrices $\{A_{kl}, B_{kl}, C_k : k = 1, \ldots, d-1; l = k+1, \ldots, d-1\}$ with, $\forall m, n = 1, \ldots, d$,

$$\begin{cases} (A_{kl})_{mn} = \delta_{m,l}\delta_{n,k} + \delta_{m,k}\delta_{n,l} \\ (B_{kl})_{mn} = i\left(\delta_{m,l}\delta_{n,k} - \delta_{m,k}\delta_{n,l}\right) \\ (C_k)_{mn} = \sqrt{\frac{2}{k(k+1)}} \left(\sum_{k'=1}^{k} \delta_{m,k'} - k\delta_{m,k+1}\right) \delta_{m,n}. \end{cases} \tag{A.15}$$

The number of A_{kl} matrices is given by $\sum_{k'=1}^{d-1}(d-k') = \sum_{k'=1}^{d-1} k' = d(d-1)/2$, which is identical to the number of B_{kl} matrices. The number of C_k matrices is equal to $d-1$, hence the total number of A_{kl}, B_{kl} and C_k matrices is equal to $d^2 - 1$. One easily checks that, $\forall k, k' = 1, \ldots, d-1; l = k+1, \ldots, d-1; l' = k'+1, \ldots, d-1$,

$$\mathrm{Tr}(A_{kl}A_{k'l'}) = \mathrm{Tr}(B_{kl}B_{k'l'}) = 2\delta_{k,k'}\delta_{l,l'}, \tag{A.16}$$
$$\mathrm{Tr}(C_k C_{k'}) = 2\delta_{k,k'}, \tag{A.17}$$
$$\mathrm{Tr}(A_{kl}B_{k'l'}) = \mathrm{Tr}(A_{kl}C_{k'}) = \mathrm{Tr}(B_{kl}C_{k'}) = 0 \tag{A.18}$$

Eqs. (A.16)–(A.18) imply that $\mathrm{Tr}(\lambda_i\lambda_j) = 2\delta_{i,j}$.

Proof. We consider hereafter arbitrary $k, k' = 1, \dots, d-1; l = k+1, \dots, d-1; l' = k'+1, \dots, d-1$. First, let us prove Eq. (A.16):

$$\begin{aligned}
\mathrm{Tr}(A_{kl}A_{k'l'}) &= \sum_m (A_{kl}A_{k'l'})_{mm} = \sum_{m,n} (A_{kl})_{mn}(A_{k'l'})_{nm} \\
&= \sum_{mn} (\delta_{m,l}\delta_{n,k} + \delta_{m,k}\delta_{n,l})(\delta_{n,l'}\delta_{m,k'} + \delta_{n,k'}\delta_{m,l'}) \\
&= 2(\delta_{l,k'}\delta_{k,l'} + \delta_{k,k'}\delta_{l,l'}) \\
&= 2\delta_{k,k'}\delta_{l,l'}
\end{aligned} \tag{A.19}$$

The proof of $\mathrm{Tr}(B_{kl}B_{k'l'}) = 2\delta_{k,k'}\delta_{l,l'}$ goes analogously. Then, we prove Eq. (A.17):

$$\begin{aligned}
\mathrm{Tr}(C_kC_{k'}) =& \sum_m (C_kC_{k'})_{mm} = \sum_m (C_k)_{mm}(C_{k'})_{mm} \\
=& \frac{2}{\sqrt{k(k+1)k'(k'+1)}} \sum_m \left[\left(\sum_{l=1}^{k} \delta_{m,l} - k\delta_{m,k+1} \right) \left(\sum_{l'=1}^{k'} \delta_{m,l'} - k'\delta_{m,k'+1} \right) \right] \\
=& \frac{2}{\sqrt{k(k+1)k'(k'+1)}} \sum_m \left[\sum_{l=1}^{k} \sum_{l'=1}^{k'} \delta_{m,l}\delta_{m,l'} - k' \sum_{l=1}^{k} \delta_{m,l}\delta_{m,k'+1} - k \sum_{l'=1}^{k'} \delta_{m,k+1}\delta_{m,l'} \right. \\
& \left. + kk'\delta_{m,k+1}\delta_{m,k'+1} \right] \\
=& \frac{2}{\sqrt{k(k+1)k'(k'+1)}} \Bigg[\underbrace{\underbrace{\sum_{l=1}^{k}\sum_{l'=1}^{k'} \delta_{l,l'}}_{=\min(k,k')} \underbrace{- k' \underbrace{\sum_{l=1}^{k} \delta_{l,k'+1}}_{=1 \text{ if } k<k', =0 \text{ otherwise}} - k \underbrace{\sum_{l'=1}^{k'} \delta_{l',k+1}}_{=1 \text{ if } k'<k, =0 \text{ otherwise}}}_{=-\min(k,k')(1-\delta_{k,k'})}}_{=\min(k,k')\delta_{k,k'}=k\delta_{k,k'}} + kk'\delta_{k,k'} \Bigg] \\
=& \frac{2(k+k^2)}{k(k+1)}\delta_{k,k'} \\
=& 2\delta_{k,k'}
\end{aligned} \tag{A.20}$$

We now prove Eq. (A.18) by three calculations:

$$\begin{aligned}
\mathrm{Tr}(A_{kl}B_{k'l'}) &= \sum_m (A_{kl}B_{k'l'})_{mm} = \sum_{m,n} (A_{kl})_{mn}(B_{k'l'})_{nm} \\
&= \sum_{mn} (\delta_{m,l}\delta_{n,k} + \delta_{m,k}\delta_{n,l}) i (\delta_{n,l'}\delta_{m,k'} - \delta_{n,k'}\delta_{m,l'}) \\
&= 0
\end{aligned} \tag{A.21}$$

$$
\begin{aligned}
\mathrm{Tr}(A_{kl}C_{k'}) &= \sum_m (A_{kl}C_{k'})_{mm} = \sum_{m,n} (A_{kl})_{mn}(C_{k'})_{nm} \\
&= \sqrt{\frac{2}{k'(k'+1)}} \sum_{mn} (\delta_{m,l}\delta_{n,k} + \delta_{m,k}\delta_{n,l}) \left(\sum_{k''=1}^{k'} \delta_{n,k''} - k'\delta_{n,k'+1} \right) \delta_{m,n} \\
&= \sqrt{\frac{2}{k'(k'+1)}} \sum_m 2\delta_{m,l}\delta_{m,k} \left(\sum_{k''=1}^{k'} \delta_{m,k''} - k'\delta_{m,k'+1} \right) \\
&= \sqrt{\frac{2}{k'(k'+1)}} 2\delta_{k,l} \left(\sum_{k''=1}^{k'} \delta_{k,k''} - k'\delta_{k,k'+1} \right) \\
&= 0
\end{aligned} \tag{A.22}
$$

and finally

$$
\begin{aligned}
\mathrm{Tr}(B_{kl}C_{k'}) &= \sum_m (B_{kl}C_{k'})_{mm} = \sum_{m,n} (B_{kl})_{mn}(C_{k'})_{nm} \\
&= \sqrt{\frac{2}{k'(k'+1)}} \sum_{m,n} i\left(\delta_{m,l}\delta_{n,k} - \delta_{m,k}\delta_{n,l}\right) \left(\sum_{k''=1}^{k'} \delta_{n,k''} - k'\delta_{n,k'+1} \right) \delta_{m,n} \\
&= \sqrt{\frac{2}{k'(k'+1)}} \sum_m i\left(\delta_{m,l}\delta_{m,k} - \delta_{m,k}\delta_{m,l}\right) \left(\sum_{k'=1}^{k} \delta_{m,k'} - k\delta_{m,k+1} \right) \\
&= 0
\end{aligned} \tag{A.23}
$$

which ends the proof. □

Explicitly, the generators of Eq. (A.15) are given by

$$
\begin{pmatrix}0&1&0&\dots&0&0\\1&0&0&\dots&0&0\\0&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&0&0\\0&0&0&\dots&0&0\end{pmatrix},\quad
\begin{pmatrix}0&0&1&\dots&0&0\\0&0&0&\dots&0&0\\1&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&0&0\\0&0&0&\dots&0&0\end{pmatrix},\quad\dots,\quad
\begin{pmatrix}0&0&0&\dots&0&1\\0&0&0&\dots&0&0\\0&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&0&0\\1&0&0&\dots&0&0\end{pmatrix},
$$

$$
\begin{pmatrix}0&0&0&\dots&0&0\\0&0&1&\dots&0&0\\0&1&0&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&0&0\\0&0&0&\dots&0&0\end{pmatrix},\quad\dots,\quad
\begin{pmatrix}0&0&0&\dots&0&0\\0&0&0&\dots&1&0\\0&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&1&0&\dots&0&0\\0&0&0&\dots&0&0\end{pmatrix},\quad
\begin{pmatrix}0&0&0&\dots&0&0\\0&0&0&\dots&0&1\\0&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&0&0\\0&1&0&\dots&0&0\end{pmatrix},
$$

$$
\dots,
$$

$$
\begin{pmatrix}0&0&0&\dots&0&0\\0&0&0&\dots&0&0\\0&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&0&1\\0&0&0&\dots&1&0\end{pmatrix},
$$

$$
\begin{pmatrix}0&-i&0&\dots&0&0\\i&0&0&\dots&0&0\\0&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&0&0\\0&0&0&\dots&0&0\end{pmatrix},\quad
\begin{pmatrix}0&0&-i&\dots&0&0\\0&0&0&\dots&0&0\\i&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&0&0\\0&0&0&\dots&0&0\end{pmatrix},\quad\dots,\quad
\begin{pmatrix}0&0&0&\dots&0&-i\\0&0&0&\dots&0&0\\0&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&0&0\\i&0&0&\dots&0&0\end{pmatrix},
$$

$$
\begin{pmatrix}0&0&0&\dots&0&0\\0&0&-i&\dots&0&0\\0&i&0&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&0&0\\0&0&0&\dots&0&0\end{pmatrix},\quad\dots,\quad
\begin{pmatrix}0&0&0&\dots&0&0\\0&0&0&\dots&-i&0\\0&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&i&0&\dots&0&0\\0&0&0&\dots&0&0\end{pmatrix},\quad
\begin{pmatrix}0&0&0&\dots&0&0\\0&0&0&\dots&0&-i\\0&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&0&0\\0&i&0&\dots&0&0\end{pmatrix},
$$

$$
\dots,
$$

$$
\begin{pmatrix}0&0&0&\dots&0&0\\0&0&0&\dots&0&0\\0&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&0&-i\\0&0&0&\dots&i&0\end{pmatrix},
$$

$$
\begin{pmatrix}1&0&0&0&\dots&0\\0&-1&0&0&\dots&0\\0&0&0&0&\dots&0\\0&0&0&0&\dots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&\dots&0\end{pmatrix},\quad
\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0&0&\dots&0\\0&1&0&0&\dots&0\\0&0&-2&0&\dots&0\\0&0&0&0&\dots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&\dots&0\end{pmatrix},\quad\dots,\quad
\sqrt{\frac{2}{d(d-1)}}\begin{pmatrix}1&0&0&0&\dots&0\\0&1&0&0&\dots&0\\0&0&1&0&\dots&0\\0&0&0&1&\dots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&\dots&-d+1\end{pmatrix}.
\tag{A.24}
$$

A.3 From the basis decomposition to the Bloch representation

Let $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{AB})$ be a state, and let $\{\hat{G}_\alpha^{(A)}, \alpha = 0, \dots, d_A^2-1\}$ and $\{\hat{G}_\alpha^{(B)}, \alpha = 0, \dots, d_B^2-1\}$ be orthonormal basis of $\mathcal{HS}(\mathcal{H}_A)$ and $\mathcal{HS}(\mathcal{H}_B)$ respectively that verify the orthonormality relation

$$\mathrm{Tr}(\hat{G}_\alpha^{(A)}\hat{G}_\beta^{(A)}) = \mathrm{Tr}(\hat{G}_\alpha^{(B)}\hat{G}_\beta^{(B)}) = \delta_{\alpha\beta}. \tag{A.25}$$

The state $\hat{\rho}$ can be written as

$$\begin{aligned}\hat{\rho} = & \sum_{\alpha=0}^{d_A^2-1}\sum_{\beta=0}^{d_B^2-1} \mathcal{C}_{\alpha\beta}\hat{G}_\alpha^{(A)} \otimes \hat{G}_\beta^{(B)} \\ = & \mathcal{C}_{00}\hat{G}_0^{(A)} \otimes \hat{G}_0^{(B)} + \sum_{i=1}^{d_A^2-1} \mathcal{C}_{i0}\hat{G}_i^{(A)} \otimes \hat{G}_0^{(B)} \\ & + \sum_{j=1}^{d_B^2-1} \mathcal{C}_{0j}\hat{G}_0^{(A)} \otimes \hat{G}_j^{(B)} + \sum_{i=1}^{d_A^2-1}\sum_{j=1}^{d_B^2-1} \mathcal{C}_{ij}\hat{G}_i^{(A)} \otimes \hat{G}_j^{(B)},\end{aligned} \tag{A.26}$$

with $\mathcal{C}_{\alpha\beta} = \mathrm{Tr}\left((\hat{G}_\alpha^{(A)} \otimes \hat{G}_\beta^{(B)})\hat{\rho}\right)$. Then, we take the following orthonormal basis of $\mathcal{HS}(\mathcal{H}_A)$ and $\mathcal{HS}(\mathcal{H}_B)$ respectively:

$$\left\{\frac{1}{\sqrt{d_A}}\mathbb{1}, \frac{\hat{\lambda}_i}{\sqrt{2}} : i = 1, \dots, d_A^2 - 1\right\} \quad \text{and} \quad \left\{\frac{1}{\sqrt{d_B}}\mathbb{1}, \frac{\hat{\sigma}_i}{\sqrt{2}} : i = 1, \dots, d_B^2 - 1\right\}, \tag{A.27}$$

where the $\hat{\lambda}_i$s and $\hat{\sigma}_i$s are the generators of the $SU(d_A)$ and $SU(d_B)$ groups respectively that verify the orthogonality relation

$$\mathrm{Tr}(\hat{\lambda}_i\hat{\lambda}_j) = \mathrm{Tr}(\hat{\sigma}_i\hat{\sigma}_j) = 2\delta_{ij}. \tag{A.28}$$

So the Eq. (A.26) reads

$$\begin{aligned}\hat{\rho} = & \mathcal{C}_{00}\frac{\mathbb{1}}{\sqrt{d_A}} \otimes \frac{\mathbb{1}}{\sqrt{d_B}} + \sum_{i=1}^{d_A^2-1} \mathcal{C}_{i0}\frac{\hat{\lambda}_i}{\sqrt{2}} \otimes \frac{\mathbb{1}}{\sqrt{d_B}} + \sum_{j=1}^{d_B^2-1} \mathcal{C}_{0j}\frac{\mathbb{1}}{\sqrt{d_A}} \otimes \frac{\hat{\sigma}_j}{\sqrt{2}} + \sum_{i=1}^{d_A^2-1}\sum_{j=1}^{d_B^2-1} \mathcal{C}_{ij}\frac{\hat{\lambda}_i}{\sqrt{2}} \otimes \frac{\hat{\sigma}_j}{\sqrt{2}} \\ = & \frac{1}{d_A d_B}\mathbb{1} \otimes \mathbb{1} + \frac{1}{2d_B}\sum_{i=1}^{d_A^2-1} r_i\hat{\lambda}_i \otimes \mathbb{1} + \frac{1}{2d_A}\sum_{j=1}^{d_B^2-1} s_j\mathbb{1} \otimes \hat{\sigma}_j + \frac{1}{4}\sum_{i=1}^{d_A^2-1}\sum_{j=1}^{d_B^2-1} \mathcal{T}_{ij}\hat{\lambda}_i \otimes \hat{\sigma}_j,\end{aligned} \tag{A.29}$$

where

$$\mathcal{C}_{00} = \frac{1}{\sqrt{d_A d_B}}, \quad \frac{\mathcal{C}_{i0}}{\sqrt{2d_B}} = \frac{1}{2d_B}r_i, \quad \frac{\mathcal{C}_{0j}}{\sqrt{2d_A}} = \frac{1}{2d_A}s_j \quad \text{and} \quad \frac{\mathcal{C}_{ij}}{2} = \frac{1}{4}\mathcal{T}_{ij}. \tag{A.30}$$

Remark 13. The coefficient $\mathcal{C}_{00}$ has been chosen such that $\hat{\rho}$ remains of trace 1, since generators of $SU(d)$ groups are traceless.

Remark 14. For states in their normal form, the matrix $\mathcal{C}$ reads, in the basis of Eq. (A.27),

$$\mathcal{C} = \begin{pmatrix} \frac{1}{\sqrt{d_A d_B}} & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \mathcal{T} & \\ 0 & & & \end{pmatrix} \tag{A.31}$$

which means that $\text{rank}(\hat{\rho}) = \text{rank}(\mathcal{C}) = \text{rank}(\mathcal{T}) + 1$.

Bibliography

[1] A. Einstein, B. Podolsky, and N. Rosen. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.*, 47(777), 1935.

[2] E. Schrödinger. Discussion of probability relations between separated systems. *Proc. Camb. Phil. Soc.*, 31(555), 1935.

[3] Einstein attacks quantum theory. *New York Times*, May 4th, 1935.

[4] J. S. Bell. On the Einstein Podolsky Rosen paradox. *Physics*, 1(3), 1964.

[5] A. Aspect, P. Grangier, and G. Roger. Experimental realization of einstein-podolsky-rosen-bohm gedankenexperiment: A new violation of bell's inequalities. *Phys. Rev. Lett.*, 49(91), 1982.

[6] C. H. Bennett and G. Brassard. Quantum cryptography: Public key distribution and coin tossing. *Proceedings of IEEE International Conference on Computers Systems and Signal Processing*, 175, 1984.

[7] C. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. Wootters. Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels. *Phys. Rev. Lett.*, 70, 1993.

[8] R. Jozsa and N. Linden. On the role of entanglement in quantum-computational speed-up. *Proc. R. Soc. Lond. A*, 459(2036), 2003.

[9] A. Neven and T. Bastin. The quantum separability problem is a simultaneous hollowisation matrix analysis problem. *J. Phys. A: Math. Theor.*, 51(315305), 2018.

[10] M. Horodecki, P. Horodecki, and R. Horodecki. Separability of mixed states: necessary and sufficient condition. *Phys. Lett. A*, 223(1), 1996.

[11] O. Gühne and G. Tóth. Entanglement detection. *Phys. Rep.*, 474(1-6), 2009.

[12] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki. Quantum entanglement. *Rev. of Mod. Phys.*, 81(2), 2009.

[13] C. Cohen-Tannoudji, B. Diu, and F. Laloë. *Mécanique quantique, vol. I.* Hermann, Paris, France, 1997.

[14] O. Gamel and D. James. Measures of quantum state purity and classical degree of polarization. *Phys. Rev. A*, 86(033830), 2012.

[15] I. Bengtsson and K. Zyczkowski. *Geometry of quantum states: an introduction to quantum entanglement.* Cambridge University Press, Cambridge, UK, 2006.

[16] M. Nielsen and I. Chuang. *Quantum Computation and Quantum Information.* Cambridge University Press, Cambridge, UK, 2010.

[17] G. Kimura. The bloch vector for n-level systems. *Phys. Lett. A*, 314(5-6), 2003.

[18] F. Verstraete, J. Dehaene, and B. De Moor. Normal forms and entanglement measures for multipartite quantum states. *Phys. Rev. A*, 68(012103), 2003.

[19] J.-L. Li and C.-F. Qiao. Separable decompositions of bipartie mixed states. *Quant. Inf. Proc.*, 17 (92), 2018.

[20] D. Chruciski and G. Sarbicki. Entanglement witnesses: construction, analysis and classification. *J. Phys. A: Math. Theor.*, 47(483001), 2014.

[21] A. Peres. Separability criterion for density matrices. *Phys. Rev. Lett.*, 77(8), 1996.

[22] N. Ganguly, S. Adhikari, and A. Majumdar. Common entanglement witnesses and their characteristics. *Quant. Inf. Pro.*, 12(425-436), 2013.

[23] P. Krammer. Characterizing entanglement with geometric entanglement witnesses. *J. Phys. A: Math. Theor.*, 42(065305), 2009.

[24] G. Vidal and R. Werner. Computable measure of entanglement. *Phys. Rev. A*, 65(032314), 2002.

[25] W. Wootters. Entanglement formation of an arbitrary state of two qubits. *Phys. Rev. Lett*, 80(10), 1998.

[26] O. Rudolph. Further results on the cross norm criterion for separability. *Quant. Inf. Pro.*, 4(3), 2003.

[27] K. Chen and L.-A. Wu. A matrix realignment method for recognizing entanglement. *Quant. Inf. Comput.*, 3(3), 2003.

[28] C.-J. Zhang, Y.-S. Zhang, S. Zhang, and G.-C. Guo. Entanglement detection beyond the computable cross-norm or realignment criterion. *Phys. Rev. A*, 77(060301(R)), 2008.

[29] J. de Vicente. Separability criteria based on the bloch representation of density matrices. *Quant. Inf. Comput.*, 7(624), 2007.

[30] J. de Vicente. Further results on entanglement detection and quantification from the correlation matrix criterion. *J. Phys. A: Math. Theor.*, 41(065309), 2008.

[31] O. Gühne, P. Hyllus, O. Gittsovich, and J. Eisert. Covariance matrices and the separability problem. *Phys. Rev. Lett.*, 99(130504), 2007.

[32] O. Gittsovich, O. Gühne, P. Hyllus, and J. Eisert. Unifying several separability conditions using the covariance matrix criterion. *Phys. Rev. A*, 78(052319), 2008.

[33] M. Li, J. Wang, S.-M. Fei, and X. Li-Jost. Quantum separability criteria for arbitrary-dimensional multipartite states. *Phys. Rev. A*, 89(022325), 2014.

[34] J.-L. Li and C.-F. Qiao. A necessary and sufficient criterion for the separability of quantum states. *Sci. Rep.*, 8(1442), 2018.

[35] L. Hughston, R. Jozsa, and W. Wootters. A complete classification of quantum ensembles having a given density matrix. *Phys. Lett. A*, 182, 1993.

[36] A. Neven. *Development of new tools to detect, characterize and quantify quantum entanglement in multipartite systems.* PhD thesis, University of Liège, 2018.

[37] J. Shang, A. Asadian, H. Zhu, and O. Gühne. Enhanced entanglement criterion via symmetric informationally complete measurement. *Phys. Rev. A*, 98(022309), 2018.

[38] G. Sarbicki, G. Scala, and D. Chruscinski. Familiy of multipartite separability criteria based on a correlation tensor. *Phys. Rev. A*, 101(012341), 2020.

[39] G. Sarbicki, G. Scala, and D. Chruscinski. Enhanced realignment criterion vs. linear entanglement witnesses. arXiv:2002.00646, 2020.

[40] R. Horn and C. Johnson. *Matrix analysis (second edition)*. Cambridge University Press, Cambridge, UK, 2013.